Pespectives in Mathematical Sciences: Report Problems

Due: Tuesday, May 29, 2018, in Science Building 1, Room 105.

Problem 1. Let R be a ring, not necessarily commutative, and recall that a left R-module M is said to be free, if it admits a basis. A left R-module P is said to be *projective*, if there exists a left R-module Q such that $P \oplus Q$ is free. We let $n \geqslant 1$ be an integer and consider the ring $R = M_n(A)$ of $n \times n$-matrices with entries in a commutative and non-zero ring A. We further consider the abelian group $P = M_{n,1}(A)$ of $n \times 1$-matrices with entries in A as a left R-module via matrix multiplication.

(i) Show that the left R-module P is not free, unless $n = 1$.
(ii) Show that the left R-module P is projective, for all $n \geqslant 1$.

[Hint: Consider the case $n = 2$ first.]

Problem 2. Let R be a commutative ring. Show that the following statements are equivalent.

(i) For all $a, b \in R$, if $ab = 0$, then $a = 0$ or $b = 0$.
(ii) The ring R is a subring of a field K.

[Hint: To prove that (i) implies (ii), observe that $S = R \smallsetminus \{0\} \subset R$ is a multiplicative subset, and let K be the localization $S^{-1}R$.] A ring R for which the equivalent statements (i)–(ii) hold is called an *integral domain*.

Problem 3. Let R be a commutative ring and let $\mathfrak{p} \subset R$ be a proper ideal. Show that the following statements are equivalent.

(i) For all elements $a, b \in R$, $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.
(ii) For all ideals $\mathfrak{a}, \mathfrak{b} \subset R$, $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ implies $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.

An ideal $\mathfrak{p} \subset R$ for which the equivalent statements (i)–(ii) hold is called a *prime ideal*. We read (ii) as "if $\mathfrak{p}$ divides $\mathfrak{a}\mathfrak{b}$, then $\mathfrak{p}$ divides $\mathfrak{a}$ or $\mathfrak{p}$ divides $\mathfrak{b}$."

Problem 4. Let $\varphi(n)$ be Euler's phi-function that counts the number of integers $0 \leqslant k < n$ such that k and n are relatively prime. Show that

$$\sum_{d \mid n} \varphi(d) = n.$$

Here the sum ranges over all positive integers d that divide n.