Report problems[1]

Problem 1. *Due*: Tuesday, April 16, 2019, in Science Building 1, Room 105.

Let A be a ring and let $f \in A$ be an element. Show that the continuous map

$$\mathrm{Spec}(A_f) \to \mathrm{Spec}(A)$$

induced by the localization map $A \to A_f$ is a homeomorphism onto

$$D(f) \subset \mathrm{Spec}(A)$$

with the subspace topology.

(You can find the proof in the Stacks Project, but you must write it out by yourself.)

Problem 2. *Due*: Tuesday, April 23, 2019, in Science Building 1, Room 105.

This problem concerns the tensor product of commutative rings.

We first recall that the tensor product of two abelian groups A and B is defined to be the initial $\mathbb{Z}$-bilinear map $u\colon A \times B \to A \otimes B$. That u is $\mathbb{Z}$-bilinear means that u is $\mathbb{Z}$-linear each factor, and that u is initial with this property means that if also $f\colon A \times B \to C$ is a $\mathbb{Z}$-bilinear map, then there exists a *unique* $\mathbb{Z}$-linear map

$$A \otimes B \xrightarrow{\tilde{f}} C$$

such that $f = \tilde{f} \circ u$. The elements of $A \otimes B$ are called tensors, and the elements of the form $a \otimes b = u(a, b)$ are called elementary tensors. (Every tensor can be written as a sum of elementary tensors, but there is no unique way to do so.) Using this notation, the bilinearity of u amounts to the identities

$$(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$$
$$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2.$$

Next, if A and B are commutative rings, then the formula

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$$

defines a multiplication on the (additive) abelian group $A \otimes B$ that makes it a commutative ring. Moreover, the maps $i_1\colon A \to A \otimes B$ and $i_2\colon B \to A \otimes B$ defined by $i_1(a) = a \otimes 1$ and $i_2(b) = 1 \otimes b$ are ring homomorphisms with respect to this ring structure on $A \otimes B$. Show that the pair

$$(A \otimes B, (A \xrightarrow{i_1} A \otimes B \xleftarrow{i_2} B))$$

is a coproduct of A and B in the category of commutative rings.

[1] Course homepage: www.math.nagoya-u.ac.jp/~larsh/teaching/S2019_A