Report problems[1]

Problem 1. *Due*: Tuesday, April 16, 2019, in Science Building 1, Room 105.

Let A be a ring and let $f \in A$ be an element. Show that the continuous map

$$\mathrm{Spec}(A_f) \to \mathrm{Spec}(A)$$

induced by the localization map $A \to A_f$ is a homeomorphism onto

$$D(f) \subset \mathrm{Spec}(A)$$

with the subspace topology.

(You can find the proof in the Stacks Project, but you must write it out by yourself.)

Problem 2. *Due*: Tuesday, April 23, 2019, in Science Building 1, Room 105.

This problem concerns the tensor product of commutative rings.

We first recall that the tensor product of two abelian groups A and B is defined to be the initial $\mathbb{Z}$-bilinear map $u\colon A \times B \to A \otimes B$. That u is $\mathbb{Z}$-bilinear means that u is $\mathbb{Z}$-linear each factor, and that u is initial with this property means that if also $f\colon A \times B \to C$ is a $\mathbb{Z}$-bilinear map, then there exists a *unique* $\mathbb{Z}$-linear map

$$A \otimes B \xrightarrow{\tilde{f}} C$$

such that $f = \tilde{f} \circ u$. The elements of $A \otimes B$ are called tensors, and the elements of the form $a \otimes b = u(a, b)$ are called elementary tensors. (Every tensor can be written as a sum of elementary tensors, but there is no unique way to do so.) Using this notation, the bilinearity of u amounts to the identities

$$(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$$
$$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2.$$

Next, if A and B are commutative rings, then the formula

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$$

defines a multiplication on the (additive) abelian group $A \otimes B$ that makes it a commutative ring. Moreover, the maps $i_1\colon A \to A \otimes B$ and $i_2\colon B \to A \otimes B$ defined by $i_1(a) = a \otimes 1$ and $i_2(b) = 1 \otimes b$ are ring homomorphisms with respect to this ring structure on $A \otimes B$. Show that the pair

$$(A \otimes B, (A \xrightarrow{i_1} A \otimes B \xleftarrow{i_2} B))$$

is a coproduct of A and B in the category of commutative rings.

Problem 3. *Due*: Tuesday, May 14, 2019, in Science Building 1, Room 105.

This problem concerns the simplest non-trivial case of *recollement*.

A *discrete valuation ring* is a ring V that is a principal ideal domain and that has exactly one non-zero maximal ideal $\mathfrak{m} \subset V$. Hence, $X = \mathrm{Spec}(V)$ has two points,

[1] Course homepage: www.math.nagoya-u.ac.jp/~larsh/teaching/S2019_A

namely, the *special point* $s \in X$ corresponding to the maximal ideal $\mathfrak{m} \subset V$ and the *generic point* $\eta \in X$ corresponding to the zero ideal $\{0\} \subset V$.

(i) Show $\{s\} \subset X$ is closed and that $\{\eta\} \subset X$ is open.
(ii) Show that the closure of $\{\eta\} \subset X$ is all of X.

Let K be the quotient field of V, let $k = V/\mathfrak{m}$ be the residue field of V, and let

$$\operatorname{Spec}(k) \xrightarrow{\;i\;} \operatorname{Spec}(V) \xleftarrow{\;j\;} \operatorname{Spec}(K)$$

be the maps induced by the projection map $V \to k$ and the localization map $V \to K$, respectively. We write $i \colon Y \to X$ and $j \colon U \to X$ for the two maps.

(iii) Show that $i \colon Y \to X$ and $j \colon U \to X$ are the closed inclusion of the special point and the open inclusion of the generic point, respectively.

We now consider sheaves on sets on Y, U, and X. A sheaf F on Y is determined, up to unique isomorphism, by its set $F_0 = \Gamma(\{s\}, F_0)$ of global sections, and every set (in our universe of discourse) can occur. More precisely, the functor

$$Y^{\sim} \longrightarrow \text{Sets}$$

that to a sheaf F on Y assigns the set $F_0 = \Gamma(\{s\}, F)$ is an equivalence of categories. In the same way, the functor

$$U^{\sim} \longrightarrow \text{Sets}$$

that to a sheaf F on U assigns the set $F_1 = \Gamma(\{\eta\}, F)$ is an equivalence of categories. A sheaf F on X is determined, up to unique isomorphism, by the map

$$F_0 = \Gamma(X, F) \xrightarrow{\;\rho^X_U\;} F_1 = \Gamma(U, F),$$

and any map (in our universe of discourse) may occur. Again, the precise statement is that the functor

$$X^{\sim} \longrightarrow \text{Ar}(\text{Sets}) = \text{Set}^{[1]}$$

that to a sheaf F on X assigns the map above is an equivalence of categories.

(iv) Identifying the respective categories of sheaves of sets as above, describe the five functors in the *recollement* diagram

$$Y^{\sim} \underset{i_*}{\overset{i^*}{\rightleftarrows}} X^{\sim} \quad X^{\sim} \xrightarrow{\;j^*\;} U^{\sim}, \quad j_! , j_* \colon U^{\sim} \to X^{\sim}.$$

(For example, $i_*(F_0) = (F_0 \to 1)$, where 1 is a terminal object.)