Pespectives in Mathematical Sciences: Report Problems

Due: Tuesday, July 30, 2019, in Science Building 1, Room 105.

Problem 1. Let $\mathbb{H}$ be the division ring of quaternions defined in Example 1.3 (4), and $V = (\mathbb{R}^4, +, \cdot\,)$ be the left $\mathbb{H}$-vector space with "+" given by the usual vectorsum in $\mathbb{R}^4$ and with scalar multiplication $\cdot : \mathbb{H} \times \mathbb{R}^4 \to \mathbb{R}^4$ given by the formula

$$(a + ib + jc + kd) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{pmatrix}.$$

defines a left $\mathbb{H}$-vector space structure on $\mathbb{R}^4$.

(a) Show that the family $(\boldsymbol{v})$ consisting of the single vector

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is a linearly independent family in the left $\mathbb{H}$-vector space V.

(b) Show that the family $(\boldsymbol{v})$ generates the left $\mathbb{H}$-vector space V.

(c) Conclude that the family $(\boldsymbol{v})$ is a basis left $\mathbb{H}$-vector space V.

Problem 2. Let R be a ring, and let R^{op} be the opposite ring defined in Remark 2.6. Let $M_n(R)$ be the ring of $n \times n$-matrices with entries in R, and let

$$(-)^t \colon M_n(R)^{\mathrm{op}} \to M_n(R^{\mathrm{op}})$$

be the map that to a matrix $A = (a_{ij})$ assigns its transpose $A^t = (a_{ji})$.

(a) Show that $(-)^t$ is a ring homomorphism.

(b) Show that $(-)^t$ is a ring isomorphism.

Problem 3. Let D be a division ring, and let $R = M_n(D)$ be the matrix ring. The set $S = M_{n,1}(D)$ of column vectors has both a structure of left R-module and of right D-module with sum given by matrix sum and scalar multiplication given by matrix product. Moreover, for all $A \in R$, $\boldsymbol{x} \in S$, and $a \in D$,

$$(A \cdot \boldsymbol{x}) \cdot a = A \cdot (\boldsymbol{x} \cdot a),$$

by the associativity of matrix product.

(a) Show that the family $(\boldsymbol{v})$ consisting of the single vector

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

generates the left R-module S.

(b) Show that if $n \geq 2$, then the family $(\boldsymbol{v})$ is *not* a linearly independent family in the left R-module S.

(c) Find $P \in R$ such that $P\boldsymbol{v} = \boldsymbol{v}$ and such that $PS = \boldsymbol{v}D \subset S$.