Representation Theory

Basic idea:

$$G \xrightarrow{\pi} GL(V)$$

$$g \longmapsto \left(V \xrightarrow{\pi(g)} V\right)$$

$$\begin{array}{cc} \cap & \cap \\ \text{element of group (complicated)} & \text{linear operator on vector sp. (easier)} \end{array}$$

Textbook:

E. B. Vinberg: Linear representations of groups.

Today: Chapter 0.

k : field ($k = \mathbb{R}$ or $k = \mathbb{C}$)

V : k-vector space

$$GL(V) = \{ f : V \to V \mid f \text{ } k\text{-linear isomorphism} \}$$

with group structure

$$GL(V) \times GL(V) \xrightarrow{\circ} GL(V)$$

$$(f, g) \longmapsto f \circ g$$

Def A k-linear representation of a group G is a pair (V, π) of ~~a~~ k-vector space V and a group homomorphism

$$G \xrightarrow{\pi} GL(V).$$

So elements $g \in G$ are represented by k-linear operators $\pi(g) : V \to V$ s.t.

$$\pi(g \cdot h) = \pi(g) \circ \pi(h)$$

and

$$\pi(e) = id_V.$$

$e \in G$ identity element.

Def A matrix representation of a group G over a field k is a group homomorphism

$$G \xrightarrow{\pi} GL_n(k)$$

$GL_n(k)$ = group of invertible $n \times n$-matrices with entries in k

If $\dim_k(V) = n < \infty$, and if

$$(e_1, e_2, \ldots, e_n)$$

is a basis of V, then we have an isomorphism of grps

$$GL(V) \simeq GL_n(k)$$

$$V \xrightarrow{f} V \longleftrightarrow A = (a_{ij})$$

where

$$f(e_j) = e_1 a_{1j} + e_2 a_{2j} + \cdots + e_n a_{nj}$$

$$\begin{tikzcd} & GL(V) \arrow[d, phantom, "\wr"] \\ G \arrow[ur, "\pi"] \arrow[dr, "\pi'"'] & GL_n(k) \end{tikzcd}$$

depends on the choice of basis!

Let (V_1, π_1) and (V_2, π_2) be two k-linear representations of G. A morphism

$$(V_1, \pi_1) \xrightarrow{f} (V_2, \pi_2)$$

is a map

$$V_1 \xrightarrow{f} V_2$$

is k-linear and for all $g \in G$ and $x \in V_1$,

$$f(\pi_1(g)(x)) = \pi_2(g)(f(x))$$

we say that f is an intertwining (or equivariant) operator

Often we simply write

$$gx = \pi(g)(x)$$

So f is intertwining if

$$f(gx) = gf(x)$$

for all $g \in G$ and $x \in V_1$.

A morphism

$$(V_1, \pi_1) \xrightarrow{f} (V_2, \pi_2)$$

is an isomorphism if the map f is a bijection.

We say that two representations (V_1, π_1) and (V_2, π_2) are isomorphic (or equivalent) if there exists an isomorphism

$$(V_1, \pi_1) \xrightarrow{f} (V_2, \pi_2),$$

and we write

$$(V_1, \pi_1) \simeq (V_2, \pi_2).$$

<u>Examples</u> Representations of

$$G = (\mathbb{R}, +).$$

The exponential function

$$G \xrightarrow{\pi_a} GL_1(\mathbb{R})$$

$$t \longmapsto e^{at}$$

for some $a \in \mathbb{R}$. This is a 1-dimensional representation of $G = (\mathbb{R}, +)$:

$$\begin{array}{ccc} \pi_a(t+u) & = & e^{a(t+u)} \\ \| & & \| \\ & & e^{at+au} \\ & & \| \\ \pi_a(t)\cdot\pi_a(u) & = & e^{at}\cdot e^{au} \end{array}$$

$$\pi_a(0) = e^{a\cdot 0} = 1$$

Question: Are there other 1-dim'l representations of $G = (\mathbb{R}, +)$?

Answer: No, provided that we require π to be continuous.

<u>Lemma</u> Let $\pi : (\mathbb{R},+) \longrightarrow GL_1(\mathbb{R})$ be a continuous group homomorphism. Then for all $t \in \mathbb{R}$,

$$\pi(t) = e^{at}$$

for some (fixed) $a \in \mathbb{R}$.

<u>Pf</u> That π is a group homom. means that $\pi(0) = 1$ and that for all $t, u \in \mathbb{R}$,

$$\pi(t+u) = \pi(t)\cdot\pi(u).$$

Assume that π is diff. and differentiate with respect to u at $u = 0$:

$$\pi'(t) = \pi(t)\cdot\pi'(0)$$

a

Every sol. to this ODE has the form

$$\pi(t) = C\cdot e^{at}.$$

The initial cond. $\pi(0) = 1$ implies that $C = 1$.

Matrix exponential:

$$L_n(k) = \{n\times n\text{-matrices} / k\}$$

For $A \in L_n(k)$ $(= M_n(k))$,

$$e^A = \sum_{n=0}^{\infty} A^n/n! \in L_n(k)$$

Sum converges in operator norm because

$$\|A^n/n!\| \le \|A\|^n/n! \le e^{\|A\|}$$

If $AB = BA$, then

$$e^{A+B} = e^A\cdot e^B,$$

but this not true in general!

In particular, the map

$$G = (\mathbb{R}, +) \xrightarrow{\pi} GL_n(k)$$

$$t \longmapsto e^{tA}$$

is a group homomorphism, and hence an "n-dimensional" (real or complex) representation of $G = (\mathbb{R}, +)$.

<u>Lemma</u> Let $\pi : (\mathbb{R}, +) \to GL_n(k)$ be a differentiable group homomorphism. Then

$$\pi(t) = e^{tA},$$

where $A = \pi'(0) \in L_n(k)$.

<u>Pf</u> Same as before:

$$\pi'(t) = \pi(t) \cdot \underbrace{\pi'(0)}_{A}$$

is an ODE with initial cond. $\pi(0) = E$. Only sol. is

$$\pi(t) = e^{tA}.$$

//

Ex $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Calc.

$$\left.\begin{aligned} A^{2n} &= (-1)^n E \\ A^{2n+1} &= (-1)^n A \end{aligned}\right\} \Longrightarrow$$

$$e^{tA} = \sum_{\substack{n\geq 0 \\ \text{even}}} t^n A^n/n! + \sum_{\substack{n\geq 0 \\ \text{odd}}} t^n A^n/n!$$

$$= \sum_{m\geq 0} \frac{(-1)^m t^{2m}}{(2m)!} E + \sum_{m\geq 0} \frac{(-1)^m t^{2m+1}}{(2m+1)!} A$$

$$= \cos t \cdot E + \sin t \cdot A$$

$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Hence, the map

$$(\mathbb{R}, +) \xrightarrow{\pi} GL_2(\mathbb{R})$$

$$t \longmapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is a 2-dimensional real representation of $G = (\mathbb{R}, +)$.

Let $\pi : G \to GL(V)$ be a repr. of G. Define

$$\ker(\pi) = \{g \in G \mid \pi(g) = id_V\}$$

$\subset G$ normal subgrp.

Recall that

$$\ker(\pi) = \{e\} \iff$$

$\pi : G \to GL(V)$ is injective

In this case, we say that π is a faithful representation of G.

<u>Ex</u> $(\mathbb{R}, +) \xrightarrow{\pi} GL_1(\mathbb{R})$

$$t \longmapsto e^{at}$$

is faithful $\iff a \neq 0$.

$(\mathbb{R}, +) \xrightarrow{\pi} GL_2(\mathbb{R})$

$$t \longmapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is not faithful, since

$$\ker(\pi) = 2\pi \cdot \mathbb{Z} < \mathbb{R}.$$

<u>Ex</u> Let $G = S_n$ be the (finite) symmetric group on n letters. Its elements are all bijections

$$\{1, 2, \ldots, n\} \xrightarrow{\sigma} \{1, 2, \ldots, n\}$$

with group structure

$$S_n \times S_n \xrightarrow{\circ} S_n .$$
$$(\sigma, \tau) \longmapsto \sigma \circ \tau$$

To $\sigma \in S_n$, we assign the permutation matrix

$$P(\sigma) = \left(e_{\sigma(1)} \; e_{\sigma(2)} \; \cdots \; e_{\sigma(n)} \right)$$
$$\in GL_n(k) .$$

By definition, we have

$$\operatorname{sgn}(\sigma) = \det(P(\sigma)) \in \{\pm 1\} .$$

Also $P(\sigma \circ \tau) = P(\sigma) \cdot P(\tau)$ and $P(id) = E$, so

$$S_n \xrightarrow{\pi} GL_n(k)$$
$$\sigma \longmapsto P(\sigma)$$

is a (faithful) repr.,

The regular representation:
If X is any set and k is a field, then we write

$$k[X] = \{\text{functions } f: X \to k\}.$$

It is a k-vector space with vector sum and scalar mult.

$$(f+g)(x) := f(x) + g(x)$$

$$(a \cdot f)(x) := a \cdot f(x)$$

The left regular representation of a group G is the repr.

$$G \xrightarrow{L} GL(k[G])$$

defined by

$$L(g)(f)(x) = f(g^{-1}x),$$

and the right regular repr. is the repr. defined by

$$G \xrightarrow{R} GL(k[G])$$

$$R(g)(f)(x) = f(xg).$$

Check that L and R are group homomorphisms:

Clearly,

$$L(e) = \mathrm{id}_{k[G]} = R(e),$$

and

$$L(g)(L(h)(f))(x)$$
$$= L(h)(f)(g^{-1}x)$$
$$= f(h^{-1}g^{-1}x)$$
$$= f((gh)^{-1}x)$$
$$= L(gh)(f)(x). \qquad \text{OK.}$$

Similarly,

$$R(g)(R(h)(f))(x)$$
$$= R(h)(f)(xg)$$
$$= f(xgh)$$
$$= R(gh)(f)(x). \qquad \text{OK.}$$

Can consider subspaces $V \subset k[G]$ provided that V is G-invariant.

<u>Ex</u> If $G = (\mathbb{R}, +)$, then

$$L(t)(f)(x) = f(x-t),$$

so can take

$$V = \{\text{polynomial fct.}\} \subset k[\mathbb{R}]$$

or

$$V = \mathrm{Span}(\cos, \sin) \subset k[\mathbb{R}].$$

In the latter case, we have

$$L(t)(\cos)(x) = \cos(x-t)$$

$$= \cos t \cos x + \sin t \sin x$$

$$L(t)(\sin)(x) = \sin(x-t)$$

$$= -\sin t \cos x + \cos t \sin x,$$

so we recover the repr.

$$(\mathbb{R}, +) \xrightarrow{\pi} GL_2(\mathbb{R})$$

$$t \longmapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Representations in function spaces:

Let X be a set, and let

$$S(X) = \{\text{bijective maps } \sigma : X \to X\}$$

with group structure

$$S(X) \times S(X) \xrightarrow{\circ} S(X)$$

$$(\sigma, \tau) \longmapsto \sigma \circ \tau$$

__Ex__ $S_n = S(\{1, 2, \dots, n\})$.

Define representation

$$S(X) \xrightarrow{\pi} GL(k[X])$$

by

$$\pi(\sigma)(f)(x) = f(\sigma^{-1}(x)).$$

(Book writes $\sigma_*(f) = \pi(\sigma)(f)$.)

Recall that a left action by a group G on a set X is a group homomorphism

$$G \xrightarrow{\rho} S(X).$$

We can compose with π to get a representation

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[rr, bend right, "\pi \circ \rho"'] & S(X) \arrow[r, "\pi"] & GL(k[X]) \end{tikzcd}.$$

We say that a representation of this form is a permutation representation.

Ex The identity map

$$S_n \to S(\{1, 2, \dots, n\})$$

is a left action, where

$$\rho(\sigma)(i) = \sigma(i).$$

So we get the permutation representation

$$S_n \xrightarrow{\pi = \pi \circ \rho} GL(k[\{1, 2, \dots, n\}]).$$

Let us calculate the corresponding matrix representation with respect to the basis

$$(e_1^*, e_2^*, \dots, e_n^*)$$

of $k[\{1, 2, \dots, n\}]$, where

$$e_i^*(j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

By the definition of π,

$$\pi(\sigma)(e_i^*)(j) = e_i^*(\sigma^{-1}(j))$$

$$= \begin{cases} 1 & \text{if } i = \sigma^{-1}(j) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

That is,

$$\pi(\sigma)(e_i^*) = e_{\sigma(i)}^*,$$

so the matrix that represents

$$k[\{1,2,\dots,n\}] \xrightarrow{\pi(\sigma)} k[\{1,2,\dots,n\}]$$

with respect to the basis

$$(e_1^*, e_2^*, \dots, e_n^*)$$

is permutation matrix

$$P(\sigma) = (e_{\sigma(1)}\ e_{\sigma(2)} \cdots e_{\sigma(n)})$$

$$\in GL_n(k).$$

So we recover the matrix repr. of $G = S_n$ that we considered on page 10. /