Today: Induced representations II

We apply the theory developed last time to the particular map of left G-sets given by the unique map

$$G/H \xrightarrow{p} G/G = \{G\},$$

where $H \subset G$ is a subgroup. (We do not assume that $H \subset G$ is normal.) Last time, we define functors

$$\begin{tikzcd} BG = [{}_G\backslash(G/G)] & & [{}_H\backslash(H/H)] = BH \arrow[ll, "f"'] \arrow[dl, "i"] \\ & {[{}_G\backslash(G/H)]} \arrow[ul, "p"] & \end{tikzcd}$$

with i an equivalence of categories, and adjoint functors

$$\begin{tikzcd} \mathrm{Rep}_k(G) \arrow[r, shift left, "\mathrm{Res}^G_H"] \arrow[d, equal] & \mathrm{Rep}_k(H) \arrow[l, shift left, "\mathrm{Ind}^G_H"] \arrow[d, equal] \\ \mathrm{QCoh}([{}_G\backslash(G/G)]) \arrow[r, shift left, "f^*"] \arrow[dr, shift right, "p^*"'] & \mathrm{QCoh}([{}_H\backslash(H/H)]) \arrow[l, shift left, "f_*"] \arrow[d, shift left, "i_*"] \\ & \mathrm{QCoh}([{}_G\backslash(G/H)]) \arrow[ul, shift right, "p_*"'] \arrow[u, shift left, "i^*"] \end{tikzcd}$$

We call $\mathrm{Res}^G_H := f^*$ the restriction from G to H and its left adjoint $\mathrm{Ind}_H^G := f_*$ the induction from H to G. Since composition of functors is (strictly) associative, we have

$$f^* = (p \circ i)^* = i^* \circ p^*,$$

but it is <u>not</u> true that

$$f_* = (p \circ i)_* = p_* \circ i_* .$$

What is true, however, is that the composite natural transformations

$$f_* \longrightarrow p_* \circ p^* \circ f_* \longrightarrow p_* \circ i_* \circ i^* \circ p^* \circ f_*$$
$$= p_* \circ i_* \circ f^* \circ f_* \longrightarrow p_* \circ i_*$$

and

$$p_* \circ i_* \longrightarrow f_* \circ f^* \circ p_* \circ i_*$$
$$= f_* \circ i^* \circ p^* \circ p_* \circ i_*$$
$$\longrightarrow f_* \circ i^* \circ i_* \longrightarrow f_*$$

defined using the counits and units

of the three adjunctions are each others inverses. In this way, the two right adjoints f_* and $p_* \circ i_*$ of f^* are uniquely naturally isomorphic. This is a general fact:

Prop Let $(f^*, f_*, \varepsilon, \eta)$ and $(f^*, \bar{f}_*, \bar{\varepsilon}, \bar{\eta})$ be two adjunctions with the same left adjoint f^*. Then

$$f_* \xrightarrow{\bar{\eta} \circ f_*} \bar{f}_* \circ f^* \circ f_* \xrightarrow{\bar{f}_* \circ \varepsilon} \bar{f}_*$$

is the unique natural transformation $\sigma : f_* \to \bar{f}_*$ s.t.

$$\begin{tikzcd} f^* \circ f_* \arrow[rr, "f^* \circ \sigma"] \arrow[dr, "\varepsilon"'] & & f^* \circ \bar{f}_* \arrow[dl, "\bar{\varepsilon}"] \\ & \mathrm{id} & \end{tikzcd} \qquad \begin{tikzcd} f_* \circ f^* \arrow[rr, "\sigma \circ f^*"] & & \bar{f}_* \circ f^* \\ & \mathrm{id} \arrow[ul, "\eta"] \arrow[ur, "\bar{\eta}"'] & \end{tikzcd}$$

commute. In particular, σ is a natural isom. with inverse

$$\bar{f}_* \xrightarrow{\eta \circ \bar{f}_*} f_* \circ f^* \circ \bar{f}_* \xrightarrow{f_* \circ \bar{\varepsilon}} f_* .$$

Pf This is not so easy to show. See Saunders MacLane, Categories for the Working Mathematician, Chapter IV, Section 7, Theorem 2. //

Here is an application:

Cor The adjunction

$$\mathrm{QCoh}([{}_G\backslash(G/H)]) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Rep}_k(H)$$

is an adjoint equivalence.

Pf In the adjunction $(i^*, i_*, \varepsilon, \eta)$, i^* and i_* are given by restriction and right Kan extension along

$$BH = [{}_H\backslash(H/H)] \xrightarrow{i} [{}_G\backslash(G/H)],$$

and we wish to prove that ε and η are isomorphisms. We proved last time that i is an equivalence of categories. So let h be a quasi-inverse of i. Then h^* is a quasi-inverse of i^*, and we can choose natural isom. $\bar{\varepsilon} : i^* \circ h^* \to \mathrm{id}$, $\bar{\eta} : \mathrm{id} \to h^* \circ i^*$ s.t. $(i^*, h^*, \bar{\varepsilon}, \bar{\eta})$ is an adjunction. By the proposition the nat'l transf. $\sigma : i_* \to h^*$ defined by

$$i_* \xrightarrow{\bar{\eta} \circ i_*} h^* \circ i^* \circ i_* \xrightarrow{h^* \circ \varepsilon} h^*$$

is an isomorphism, and the diagr.

$$
\begin{tikzcd}
i^* \circ i_* \arrow[rr, "i^* \circ \sigma"] \arrow[dr, "\varepsilon"'] & & i^* \circ h^* \arrow[dl, "\bar{\varepsilon}"] \\
& \mathrm{id} &
\end{tikzcd}
\qquad
\begin{tikzcd}
i_* \circ i^* \arrow[rr, "\sigma \circ i^*"] & & h^* \circ i^* \\
& \mathrm{id} \arrow[ul, "\eta"] \arrow[ur, "\bar{\eta}"'] &
\end{tikzcd}
$$

commute. So ε and η are isom. //

The proposition also implies that to "calculate" the induction functor

$$\mathrm{Rep}_k(H) \xrightarrow{\mathrm{Ind}_H^G} \mathrm{Rep}_k(G),$$

it suffices to produce an adjunction

$$(\mathrm{Res}_H^G, \mathrm{Ind}_H^G, \varepsilon, \eta)$$

with $\mathrm{Res}_H^G = f^*$. Indeed, the prop. will then give a unique nat'l isom. $\mathrm{Ind}_H^G \simeq f_*$ to any other right adjoint functor f_* of f^*, say, the right Kan extension along $f : BH \to BG$. Now, if

$$
\begin{tikzcd}
BH \arrow[r, "\tau"] & \mathrm{Vect}_k \\
1 \arrow[loop below, "h"] \arrow[r, mapsto] & W \arrow[loop below, "\tau(h)"]
\end{tikzcd}
$$

is a k-lin. repr. of H, then we define the induced repr.

$$\begin{array}{ccc} BG & \xrightarrow{\mathrm{Ind}_H^G(\tau)} & \mathrm{Vect}_k \\ 1 \circlearrowleft g & \longmapsto & V \circlearrowleft \pi(g) \end{array}$$

as follows: The k-vector space

$$V = \mathrm{Map}_H(G, W)$$

is the set maps $f : G \longrightarrow W$ s.t. for all $h \in H$ and $x \in G$,

$$f(h \cdot x) = \tau(h) \cdot f(x),$$

with vector sum and scalar mult. by $a \in k$ defined pointwise:

$$(f + f')(x) := f(x) + f(x')$$

$$(a \cdot f)(x) := a \cdot f(x).$$

For $g \in G$, the map $\pi(g) : V \to V$ is given by

$$\pi(g)(f)(x) := f(x \cdot g).$$

We define the counit

$$(\mathrm{Res}_H^G \circ \mathrm{Ind}_H^G)(\tau) \xrightarrow{\varepsilon_\tau} \tau$$

to be the k-linear map

$$\begin{aligned} \mathrm{Map}_H(G, W) &\xrightarrow{\varepsilon_\tau} W \\ f &\longmapsto f(e) \end{aligned}$$

It intertwines between the two representations of H, since

$$\varepsilon_\tau(h \cdot f) = (h \cdot f)(e) = f(e \cdot h)$$
$$= f(h \cdot e) = h \cdot f(e) = h \cdot \varepsilon_\tau(f).$$

Finally, we define the unit

$$\pi \xrightarrow{\eta_\pi} (\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G)(\pi)$$

as follows: If $\pi : G \to GL(V)$ be the k-lin. repr. of G, then

$$V \xrightarrow{\eta_\pi} \mathrm{Map}_H(G, V)$$

is the map given by

$$\eta_\pi(v)(x) = \pi(x)(v).$$

It is well-defined, since

$$\eta_\pi(v)(h\cdot x) = \pi(h\cdot x)(v)$$

$$= (\pi(h)\circ\pi(x))(v) = \pi(h)(\eta_\pi(v)(x)),$$

so $\eta_\pi(v) \in \mathrm{Map}_H(G, V)$. And it is intertwining, since

$$\eta_\pi(\pi(g)(v))(x) = \pi(x)(\pi(g)(v))$$

$$= \pi(x\cdot g)(v) = \eta_\pi(v)(x\cdot g)$$

$$= (g\cdot\eta_\pi(v))(x).$$

This gives the following special case of a theorem from last time.

<u>Thm</u> (Frobenius reciprocity II) Let G be a group, $H\subset G$ a subgroup, and π and τ k-lin. repr. of G and H, respectively. There is a can. isom.

$$\mathrm{Hom}(\mathrm{Res}^G_H(\pi), \tau) \simeq \mathrm{Hom}(\pi, \mathrm{Ind}^G_H(\tau)).$$

<u>Pf</u> This is the isom. of Hom-sets that we get for any adjunction. //

<u>Ex</u> Let $G = \Sigma_4$ be the symmetric group on 4 letters, and let $H < G$ be the subgroup of permutations σ s.t. $\sigma(4) = 4$. So $H \simeq \Sigma_3$. Let $\pi_1, \dots, \pi_5$ be the irred. f.-d. complex repr. of G defined in lecture 7, and let $\tau_1, \dots, \tau_3$ be the irred. f.-d. complex repr. of H. So π_1 and τ_1 are the 1-dim'l trivial repr., π_2 and τ_2 are the 1-dim'l sign repr., and π_3 and τ_3 are the standard repr. of dim. 3 and 2, respectively, $\pi_4 = \pi_2 \otimes \pi_3$ is 3-dim'l, and π_5 is 2-dim'l. We wish to understand

$$\pi = \mathrm{Ind}_H^G(\tau_1),$$

which has dimension

$$\dim_{\mathbb{C}}(\pi) = [G:H] \cdot \dim_{\mathbb{C}}(\tau_1) = 4.$$

We have the canonical isom.

$$\bigoplus_{i=1}^{5} \mathrm{Hom}(\pi_i, \pi) \otimes \pi_i \xrightarrow{\sim} \pi$$

$$f_i \otimes x_i \longmapsto f(x_i)$$

and, by Frobenius reciprocity,

$$\mathrm{Hom}(\pi_i, \pi) = \mathrm{Hom}(\pi_i, \mathrm{Ind}_H^G(\tau_1))$$

$$\simeq \mathrm{Hom}(\mathrm{Res}_H^G(\pi_i), \tau_1).$$

We calculate

$$\mathrm{Res}_H^G(\pi_1) = \tau_1$$

$$\mathrm{Res}_H^G(\pi_2) = \tau_2$$

$$\mathrm{Res}_H^G(\pi_3) \simeq \tau_1 \oplus \tau_3$$

so by Schur's lemma,

$$\begin{matrix} \mathrm{Hom}(\pi_1, \pi) \otimes \pi_1 \\ \oplus \\ \mathrm{Hom}(\pi_3, \pi) \otimes \pi_3 \end{matrix} \xrightarrow{\sim} \pi$$

is an isomorphism, or less canonically,

$$\pi \simeq \pi_1 \oplus \pi_3.$$

But let us finish and calc.

$$\mathrm{Res}_H^G(\pi_4) = \mathrm{Res}_H^G(\pi_2 \otimes \pi_3)$$

$$\simeq \mathrm{Res}_H^G(\pi_2) \otimes \mathrm{Res}_H^G(\pi_3)$$

$$\simeq \tau_2 \otimes (\tau_1 \oplus \tau_3) \simeq \tau_2 \oplus \tau_3$$

Finally, we have

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] & G \arrow[r, "q"] & H \arrow[r] & 1 \\ & & H \arrow[u, "f"] \arrow[ur, equal] & & \end{tikzcd}$$

where

$$N = \{e, (12)(34), (13)(24), (14)(23)\}$$

and where q maps $g \in G$ to the unique element

$$q(g) \in H \cap gN,$$

and $\pi_5 = q^*(\tau_3)$. So

$$\mathrm{Res}_H^G(\pi_5) = f^*(q^*(\tau_3))$$

$$= (q \circ f)^*(\tau_3) = \tau_3 .$$

As the example shows, if π is irred., then $\mathrm{Res}_H^G(\pi)$ may well not be irred. (Physicists call this "symmetry breaking.") Also, if τ is irred., then $\mathrm{Ind}_H^G(\tau)$ may well not be irred. /

Suppose that $[G:H] < \infty$. In this case, the map

$$G/H \xrightarrow{p} G/G$$

is finite, so $p_! \to p_*$ is an isom. This means that Ind_H^G is also left adjoint to Res_H^G. Let us spell out the adjunction

$$(\mathrm{Ind}_H^G, \mathrm{Res}_H^G, \varepsilon', \eta').$$

Let $g_1, \dots, g_n \in G$ be a choice of representatives of the right cosets $Hg \in H\backslash G$. Then for (V, π) a k-lin. repr. of G,

$$(\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G)(\pi) \xrightarrow{\varepsilon'_\pi} \pi$$

is the map

$$\mathrm{Map}_H(G, V) \xrightarrow{\varepsilon'_\pi} V,$$

$$f \longmapsto \sum_{i=1}^n f(g_i^{-1})$$

名古屋大学大学院多元数理科学研究科

and for (W, τ) a k-lin. repr. of H,

$$\tau \xrightarrow{\eta'_\tau} (\mathrm{Res}^G_H \circ \mathrm{Ind}^G_H)(\tau)$$

is the map

$$W \xrightarrow{\eta'_\tau} \mathrm{Map}_H(G, W)$$

given by

$$\eta'_\tau(w)(x) = \begin{cases} \tau(x)(w) & \text{if } x \in H \\ 0 & \text{if } x \notin H. \end{cases}$$

So we get the following special case of two theorems from last:

Thm (Frobenius reciprocity I) Let G be a group, and let $H \subset G$ be a subgroup with $[G:H] < \infty$. If π and τ are k-lin. repr. of G and H, resp., then there is a canonical isomorphism

$$\mathrm{Hom}(\mathrm{Ind}^G_H(\tau), \pi) \simeq \mathrm{Hom}(\tau, \mathrm{Res}^G_H(\pi)). \quad //$$

Rmk $\mathrm{Res}^G_H = f^*$ always has the left adj. $f_!$, but it only agrees with $\mathrm{Ind}^G_H = f_*$ if $[G:H] < \infty$. //

Let $H, K \subset G$ be two subgroups, and let σ and τ be k-lin. repr. of H and K, resp. We have

$$\mathrm{Hom}(\mathrm{Ind}_H^G(\sigma), \mathrm{Ind}_K^G(\tau))$$

$$\simeq \mathrm{Hom}(\mathrm{Res}_K^G(\mathrm{Ind}_H^G(\sigma)), \tau),$$

so we would like to understand the functor $\mathrm{Res}_K^G \circ \mathrm{Ind}_H^G$. This is what the base-change theorem allows us to do.

We first determine the set of maps

$$G/H \xrightarrow{f} G/K$$

that a G-equivariant. Given such a map, we have $f(H) = aK$, for some $a \in G$, and then

$$f(gH) = gaK$$

for all $g \in G$, since f is G-equivariant. In particular, we must have $haK = aK$, for all $h \in H$, or equivalently,

$$a^{-1}Ha \subset K.$$

Conversely, given $a \in G$ such that $a^{-1}Ha \subset K$, then $f_a : G/H \to G/K$ defined by $f_a(gH) = gaK$ is a G-equivariant map. Moreover, we have $f_a = f_b$ if and only if $aK = bK$, or equivalently, if and only if

$$a^{-1}b \in K.$$

If $a^{-1}Ha = K$, then $f_a = r_a$ is the G-equivariant map

$$G/H \xrightarrow{r_a} G/a^{-1}Ha$$

given by right multiplication by a,

$$r_a(gH) = gaa^{-1}Ha = gHa.$$

In general, if $a^{-1}Ha \subset K$, then f_a factors in two ways

$$\begin{tikzcd}
G/H \arrow[r, "p_H^{aKa^{-1}}"] \arrow[d, "r_a"'] \arrow[dr, "f_a"] & G/aKa^{-1} \arrow[d, "r_a"] \\
G/a^{-1}Ha \arrow[r, "p_{a^{-1}Ha}^{K}"'] & G/K
\end{tikzcd}$$

as the composition of r_a and the canonical projections.

We now assume that G is finite and consider the cartesian square of left G-sets

$$\begin{tikzcd} X = G/H \times G/K \arrow[r, "p_1"] \arrow[d, "p_2"] & G/H \arrow[d, "p^G_H"] \\ G/K \arrow[r, "p^G_K"] & G/G, \end{tikzcd}$$

where $H, K \subset G$ are subgroups. The base-change theorem gives a canonical natural isomorphism

$$(p^G_K)^* \circ (p^G_H)_* \xrightarrow{\sim} p_{2*} \circ p_1^*,$$

so we wish to understand X as a left G-set. The map $G/K \to X$ that maps $aK \mapsto (H, aK)$ is not equivariant (unless $H = G$), but it induces a surjection

$$\begin{aligned} G/K &\longrightarrow G\backslash X = \pi_0([G\backslash X]) \\ aK &\longmapsto G\cdot(H, aK), \end{aligned}$$

and (H, aK) and (H, bK) are in the same G-orbit in X if and only if $a^{-1}b \in H$. So the map

$$\begin{aligned} H\backslash G/K &\longrightarrow G\backslash X \\ HaK &\longmapsto G\cdot(H, aK) \end{aligned}$$

is a bijection from the set of double cosets onto $G\backslash X$. Moreover,

$$G_{(H,aK)} = H \cap aKa^{-1},$$

since $(H, aK) = (gH, gaK)$ if and only if $g \in H$ and $g \in aKa^{-1}$. We now choose a map

$$\begin{aligned} \{1, 2, \dots, m\} &\longrightarrow G \\ s &\longmapsto a_s \end{aligned}$$

such that the composition

$$\begin{aligned} \{1, 2, \dots, m\} \longrightarrow\ & G \longrightarrow H\backslash G/K \\ s \longmapsto\ & a_s \longmapsto Ha_sK \end{aligned}$$

is a bijection. We say that the family $(a_1, \dots, a_m)$ is a family of double coset representatives. With this choice, we obtain a G-equivariant bijection

$$\coprod_{s=1}^{m} G/H \cap a_s K a_s^{-1} \xrightarrow{u} X$$

$$g(H \cap a_s K a_s^{-1}) \longmapsto (gH, g a_s K).$$

Moreover, we have

$$p_1 \circ u = \sum_{s=1}^{m} p^{H}_{H \cap a_s K a_s^{-1}}$$

$$p_2 \circ u = \sum_{s=1}^{m} r_{a_s} \circ p^{a_s K a_s^{-1}}_{H \cap a_s K a_s^{-1}},$$

where "Σ" is notation for the map from the disjoint union that on i'th summands is given be indicated maps.

Finally, we note that the diagram

$$\begin{CD} B(aKa^{-1}) @>{i_{aKa^{-1}}}>> [G\backslash(G/aKa^{-1})] \\ @V{c_a}VV @VV{r_a}V \\ BK @>{i_K}>> [G\backslash(G/K)] \end{CD},$$

where $c_a : aKa^{-1} \to K$ maps aka^{-1} to k, commutes, up to the natural

Isomorphism

$$i_K \circ c_a \longrightarrow r_a \circ i_{aKa^{-1}}$$

given by the isomorphism

$$\begin{array}{ccc} (i_K \circ c_a)(1) & \longrightarrow & (r_a \circ i_{aKa^{-1}})(1) \\ \| & & \| \\ K & \xrightarrow{(a,K)} & aK \end{array}$$

in the category $[G \backslash (G/K)]$.

The base-change theorem now gives:

Thm (Double coset formula) In the situation above, there is a natural isomorphism

$$\mathrm{Res}^G_K \circ \mathrm{Ind}^G_H$$

$$\simeq \bigoplus_{s=1}^{m} c_{a_s *} \circ \mathrm{Ind}^{a_s K a_s^{-1}}_{H \cap a_s K a_s^{-1}} \circ \mathrm{Res}^H_{H \cap a_s K a_s^{-1}}$$

that depends on the various choices made.

Pf By the base-change thm.,

$$\begin{tikzcd} \mathrm{QCoh}([G\backslash X]) \arrow[d, "p_{2*}"] & \mathrm{QCoh}([G\backslash (G/H)]) \arrow[l, "p_1^*"'] \arrow[d, "(p_H^G)_*"] \\ \mathrm{QCoh}([G\backslash (G/K)]) & \mathrm{QCoh}([G\backslash (G/G)]) \arrow[l, "(p_K^G)^*"'] \end{tikzcd}$$

commutes, up to canonical natural isomorphism. Using the (non-canonical) G-equivariant bijection

$$\coprod_{s=1}^{m} G/H \cap a_s K a_s^{-1} \xrightarrow{u} X = G/H \times G/K,$$

this translates to a diagram

$$\begin{tikzcd}
\prod_{s=1}^{m} \mathrm{Rep}_k(H \cap a_s K a_s^{-1}) \arrow[d, "\prod \mathrm{Ind}_{H \cap a_s K a_s^{-1}}^{a_s K a_s^{-1}}"] & & \mathrm{Rep}_k(H) \arrow[ll, "(\mathrm{Res}^{H}_{H \cap a_s K a_s^{-1}})"'] \arrow[ddd, "\mathrm{Ind}_H^G"] \\
\prod_{s=1}^{m} \mathrm{Rep}_k(a_s K a_s^{-1}) \arrow[d, "\prod c_{a_s *}"] & & \\
\prod_{s=1}^{m} \mathrm{Rep}_k(K) \arrow[d, "\oplus"] & & \\
\mathrm{Rep}_k(K) & & \mathrm{Rep}_k(G) \arrow[ll, "\mathrm{Res}^G_K"']
\end{tikzcd}$$

which commutes, up to a natural isomorphism that depends on the (many) choices made. The translation uses the fact that we discussed at the beginning of today's lecture that adjoints of functors, if they exist, are unique, up to unique natural isomorphism. //

Let (V, σ) be a k-linear repr. of H, and let (W, τ) be a k-linear repr. of K. Using Frobenius reciprocity I + II and the double coset formula, we get:

$$\mathrm{Hom}(\mathrm{Ind}_H^G(\sigma), \mathrm{Ind}_K^G(\tau))$$

$$\simeq \mathrm{Hom}((\mathrm{Res}_K^G \circ \mathrm{Ind}_H^G)(\sigma), \tau)$$

$$\simeq \bigoplus_{s=1}^{m} \mathrm{Hom}((c_{a_s *} \circ \mathrm{Ind}_{H \cap a_s K a_s^{-1}}^{a_s K a_s^{-1}} \circ \mathrm{Res}_{H \cap a_s K a_s^{-1}}^{H})(\sigma), \tau)$$

$$\simeq \bigoplus_{s=1}^{m} \mathrm{Hom}(\mathrm{Res}_{H \cap a_s K a_s^{-1}}^{H}(\sigma), (\mathrm{Res}_{H \cap a_s K a_s^{-1}}^{a_s K a_s^{-1}} \circ c_{a_s}^{*})(\tau)).$$

We note that for $a \in G$,

$$\mathrm{Hom}(\mathrm{Res}_{H \cap a K a^{-1}}^{H}(\sigma), (\mathrm{Res}_{H \cap a K a^{-1}}^{a K a^{-1}} \circ c_a^{*})(\tau))$$

is the k-vector space of all k-lin. maps $f : V \to W$ s.t.

$$f(\sigma(h)(v)) = \tau(a^{-1} h a)(f(v))$$

for all $h \in H \cap a K a^{-1}$ and $v \in V$, or equivalently, such that

$$f \circ \sigma(h) = \tau(k) \circ f$$

for all $(h, k) \in H \times K$ with $ha = ak$.

Let us write $d(\sigma,\tau;s)$ for the dimension of this k-vector space for $a = a_s$. To see that it only depends on σ, τ, and s and not on the choice of $a_s \in Ha_sK \in H\backslash G/K$, we rewrite the calculation of

$$\mathrm{Hom}(\mathrm{Ind}_H^G(\sigma), \mathrm{Ind}_K^G(\tau))$$

in a way that does not involve any choices. Let

$$X_s \xrightarrow{i_s} X = G/H \times G/K$$

be the inclusion of the s'th orbit. Then the calc. becomes:

$$\mathrm{Hom}((p_H^G)_*(\sigma), (p_K^G)_*(\tau))$$

$$\simeq \mathrm{Hom}((p_K^G)^*(p_H^G)_*(\sigma), \tau)$$

$$\simeq \mathrm{Hom}(p_{2*}p_1^*(\sigma), \tau)$$

$$\simeq \bigoplus_{s=1}^m \mathrm{Hom}(p_{2*}i_{s*}i_s^*p_1^*(\sigma), \tau)$$

$$\simeq \bigoplus_{s=1}^m \mathrm{Hom}(p_{2!}i_{s!}i_s^*p_1^*(\sigma), \tau)$$

$$\simeq \bigoplus_{s=1}^m \mathrm{Hom}(i_s^*p_1^*(\sigma), i_s^*p_2^*(\tau))$$

so we have

$$d(\sigma, \tau; s)$$

$$= \dim_k \mathrm{Hom}(i_s^* p_1^*(\sigma), i_s^* p_2^*(\tau)).$$

Taking dimensions everywhere, we get the following theorem of Mackey:

Thm (Intertwining number theorem)
In the situation above,

$$\dim_k \mathrm{Hom}(\mathrm{Ind}_H^G(\sigma), \mathrm{Ind}_K^G(\tau))$$
$$= \sum_{s=1}^{m} d(\sigma, \tau; s). \quad //$$