REPRESENTATIONS OF THE SYMMETRIC GROUPS

Let X be a finite set with n elements, and let $G = \mathrm{Aut}(X)$ be its group of automorphisms. We proceed to construct representatives for all isomorphism classes of irreducible finite dimensional complex representations of G. Since the set of isomorphism classes of irreducible finite dimensional complex representations is bijective to the set of conjugacy classes of elements in G, we first introduce some language to understand the latter set.

Usually, a partition of the set X is defined to be a family $(X_i)_{i\in I}$ of subsets of $X_i \subset X$ that are pairwise disjoint with union X. This definition introduces the index set I, which causes all kinds of complications, so we will give a different definition that avoids this problem. Let Fin be the category of finite sets, and let

$$\mathrm{Fin}^{\wedge} = \mathrm{Fun}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Set})$$

be the category with objects all functors $F\colon \mathrm{Fin}^{\mathrm{op}} \to \mathrm{Set}$ and with morphisms the natural transformations between such functors.[1] We consider the functor

$$\mathrm{Fin} \xrightarrow{h} \mathrm{Fin}^{\wedge}$$

that to a finite set $X \in \mathrm{Fin}$ assigns the functor $h(X) \in \mathrm{Fin}^{\wedge}$ defined by

$$h(X)(S) = \mathrm{Hom}(S, X)$$

and that to a map $g\colon X \to Y$ between finite sets assigns the natural transformation $h(g)\colon h(X) \to h(Y)$, whose value at $S \in \mathrm{Fin}$ is the map

$$\mathrm{Hom}(S, X) \xrightarrow{h(g)_S} \mathrm{Hom}(S, Y)$$

that takes $f\colon S \to X$ to $g \circ f\colon S \to Y$. The following result is very easy to prove, but is nevertheless an extremely powerful result with myriad applications.

Lemma 1 (Yoneda lemma)**.** *For every $X \in \mathrm{Fin}$ and $F \in \mathrm{Fin}^{\wedge}$, the map*

$$\mathrm{Hom}(h(X), F) \xrightarrow{\epsilon} F(X)$$

that to $a\colon h(X) \to F$ assigns $\epsilon(a) = a_X(\mathrm{id}_X)$ is a bijection.

Proof. Let $a\colon h(X) \to F$ be a natural transformation, and let $f\colon S \to X$ be an element of $h(X)(S)$. Then the commutativity of the diagram

$$\begin{tikzcd}
h(X)(X) \arrow[r, "a_X"] \arrow[d, "h(X)(f)"] & F(X) \arrow[d, "F(f)"] \\
h(X)(S) \arrow[r, "a_S"] & F(S)
\end{tikzcd}$$

shows that

$$a_S(f) = a_S(h(X)(f)(\mathrm{id}_X)) = F(f)(a_X(\mathrm{id}_X)) = F(f)(\epsilon(a)).$$

[1] Since the category Fin is large, this definition is not meaningful. However, we can replace Fin by the equivalent small category of hereditarily finite sets to avoid this problem.

Conversely, given $y \in F(X)$, the same formula $a_S(f) = F(f)(y)$ defines a natural transformation $a\colon h(X) \to F$. $\square$

Corollary 2. *For all* $X, Y \in \mathrm{Fin}$*, the map*

$$\mathrm{Hom}(X,Y) \xrightarrow{\ h\ } \mathrm{Hom}(h(X), h(Y))$$

is a bijection.[2]

Proof. Indeed, the map

$$\mathrm{Hom}(h(X), h(Y)) \xrightarrow{\ \epsilon\ } h(Y)(X) = \mathrm{Hom}(X,Y)$$

is the inverse. $\square$

It follows that the functor $h\colon \mathrm{Fin} \to \mathrm{Fin}^\wedge$ allows us to consider Fin to be a full subcategory of $\mathrm{Fin}^\wedge$. If $A \subset X$ is a subset, then $h(A) \subset h(X)$ is a subfunctor in the sense that for all $S \in \mathrm{Fin}$, $h(A)(S) \subset h(X)(S)$. In general, a subfunctor $P \subset h(X)$ is called a sieve on X. The following example shows that there are more sieves on X and there are subsets of X.

Example 3. A partition $(X_i)_{i \in I}$ of $X \in \mathrm{Fin}$ in the traditional sense gives rise to the sieve $P \subset h(X)$, where $P(S) \subset h(X)(S) = \mathrm{Hom}(S,X)$ is defined to be the set of maps $f\colon S \to X$ such that $f(S) \subset X_i$ for some $i \in I$. The sieve $P \subset h(X)$ retains all the essential information in the partition $(X_i)_{i \in I}$, but it does not remember the index set I, which is exactly what we wanted to forget.

We will define a partition of $X \in \mathrm{Fin}$ to be a sieve $P \subset h(X)$ with a property that we now specify. We consider the set $\{f(S) \subset X \mid S \in \mathrm{Fin}, f \in P(S)\}$ to be partially ordered under inclusion.

Definition 4. A partition of $X \in \mathrm{Fin}$ is a sieve $P \subset h(X)$ for which the maximal elements of the partially ordered set $\{f(S) \subset X \mid S \in \mathrm{Fin}, f \in P(S)\}$ are pairwise disjoint and have union X. The set of partitions of X is denoted by $\mathrm{Part}(X)$.

Example 5. (i) In Example 3, the subset of $\{f(S) \subset X \mid S \in \mathrm{Fin}, f \in P(S)\}$ consisting of the elements that are maximal with respect to the partial order given by inclusion is precisely $\{X_i \subset X \mid i \in I\}$.

(ii) If $(P_i)_{i \in I}$ is a family of partitions of $X \in \mathrm{Fin}$, then the sieve

$$(\textstyle\bigcap_{i \in I} P_i)(S) = \bigcap_{i \in I} P_i(S) \subset h(X)(S)$$

is a partition of X, which we call the common refinement of the family $(P_i)_{i \in I}$.

(iii) Inclusion of subfunctors defines a partial order $\preceq$ on $\mathrm{Part}(X)$. There is a unique smallest element with respect to this partial order, which is given by the partition $O \subset h(X)$, where $O(S) \subset h(X)(S) = \mathrm{Hom}(S,X)$ consists of the constant maps.

We define a left action

$$G = \mathrm{Aut}(X) \xrightarrow{\ \mu\ } \mathrm{Aut}(\mathrm{Part}(X))$$

by the formula

$$\mu(g)(P)(S) = g(P(S)) \subset h(X)(S).$$

[2] In particular, the group homomorphism $G = \mathrm{Aut}(X) \xrightarrow{h} \mathrm{Aut}(h(X))$ is an isomorphism.

As usual, we will write $g \cdot P$ or gP instead of $\mu(g)(P)$. If $g \in G$, then we write $\mathrm{Part}(X)^g \subset \mathrm{Part}(X)$ for the subset consisting of the partitions P with $g \cdot P = P$.

Proposition 6. *Let $X \in \mathrm{Fin}$, let $G = \mathrm{Aut}(X)$, and let $C(G)$ be the set of conjugacy classes of elements in G. The map $P\colon G \to \mathrm{Part}(X)$ that to $g \in G$ assigns*

$$P(g) = \textstyle\bigcap_{Q \in \mathrm{Part}(X)^g} Q$$

induces a bijection $G \backslash P\colon C(G) \to G \backslash \mathrm{Part}(X)$.

Proof. We claim that $P\colon G \to \mathrm{Part}(X)$ is equivariant with respect to the left action by G on itself through conjugation and the left action by G on $\mathrm{Part}(X)$ defined above. Indeed, if $g \cdot Q = Q$, then $aga^{-1} \cdot a \cdot Q = a \cdot Q$, so $P(aga^{-1}) = a \cdot P(g)$. Hence, the map $P\colon G \to \mathrm{Part}(X)$ induces a map $G \backslash P\colon C(G) \to G \backslash \mathrm{Part}(X)$ as stated. To produce an inverse map, we let $P \in \mathrm{Part}(X)$ and choose a family $(X_i)_{i \in I}$ of pairwise disjoint subsets of X such that

$$\{X_i \mid i \in I\} \subset \{f(S) \subset X \mid S \in \mathrm{Fin}, f \in P(S)\}$$

is the subset of elements that are maximal with respect to inclusion. For each $i \in I$, we further choose $g_i \in G_P \subset G$ such that the subgroup $H_i \subset G_P$ generated by g_i acts transitively on X_i and trivially on X_j for $i \neq j$. In particular, for all $i, j \in I$, the elements g_i and g_j commute. Hence, the element

$$g = \textstyle\prod_{i \in I} g_i \in G_P \subset G$$

is well-defined, and its conjugacy class does not depend on the choices made. One verifies that the two maps are indeed each others inverses. $\square$

Remark 7. It is common to parametrize the set $G \backslash \mathrm{Part}(X)$ by partitions of the non-negative integer n, defined to be sequences $(\lambda_1, \dots, \lambda_k)$ of non-negative integers such that $\lambda_1 \geq \dots \geq \lambda_k$ and $\lambda_1 + \dots + \lambda_k = n$. If $P \in \mathrm{Part}(X)$, then the cardinalities of the maximal elements in $\{f(S) \subset X \mid S \in \mathrm{Fin}, f \in P(S)\}$ listed in non-increasing order is a partition $(\lambda_1, \dots, \lambda_k)$ of n, which only depends on the orbit of P. Conversely, if $(\lambda_1, \dots, \lambda_k)$ is a partition of n, then we choose a sequence $(X_1, \dots, X_k)$ of pairwise disjoint subsets of X with union X and define $P \subset h(X)$ as in Example 3. The orbit through P is then independent of the choice of $(X_1, \dots, X_k)$. Note that this parametrization of $G \backslash \mathrm{Part}(X)$ is non-canonical. Indeed, we could just as well have defined a partition of n to be a sequence $(\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \leq \dots \leq \lambda_k$ and $\lambda_1 + \dots + \lambda_k = n$. The number of partitions of n increases very fast with n. Indeed, Hardy and Ramanujan have proved the asymptotic formula

$$\mathrm{card}(G \backslash \mathrm{Part}(X)) \;\sim\; \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}.$$

The action by G on $\mathrm{Part}(X)$ is not free, and to remedy this, we introduce Young tableaux. Recall from Example 5 (iii) the partial order $\preceq$ on $\mathrm{Part}(X)$ and the smallest element O with respect to this partial order.

Definition 8. Let $X \in \mathrm{Fin}$. A Young tableau of X is a pair (P, P') of partitions of X such that the following hold:

(1) $P \cap P' = O$.
(2) If (Q, Q') is a pair of partitions of X such that $Q \cap Q' = O$ and such that $P \preceq Q$ and $P' \preceq Q'$, then $(Q, Q') = (P, P')$.

The set of Young tableaux on X is denoted by $\mathrm{Tabl}(X)$.

We give the (finite) set $\mathrm{Tabl}(X)$ the diagonal left G-action defined by

$$g \cdot (P, P') = (g \cdot P, g \cdot P'),$$

and we call $G \setminus \mathrm{Tabl}(X)$ the set of Young diagrams associated with X.

Lemma 9. *Let $X \in \mathrm{Fin}$. The left G-action on $\mathrm{Tabl}(X)$ is free.*

Proof. Let $(P, P') \in \mathrm{Tabl}(X)$. For all $x \in X$, there is, up to isomorphism over X, a unique pair of injective maps $(f \colon S \to X, f' \colon S' \to X)$ such that $f \in P(S)$ and $f' \in P'(S')$, such that $f(S) \subset X$ and $f'(S') \subset X$ are maximal, and such that $f(S) \cap f'(S') = \{x\}$. Hence, if $g \in G_{(P,P')}$, then $g(x) = x$ for all $x \in X$, and therefore, $G_{(P,P')} = \{e\} \subset G$. $\square$

Corollary 10. *Let $X \in \mathrm{Fin}$, and let $(P, P') \in \mathrm{Tabl}(X)$. Then $G_P \cap G_{P'} = \{e\}$.*

Proof. Indeed, we have $G_P \cap G_{P'} = G_{(P,P')}$. $\square$

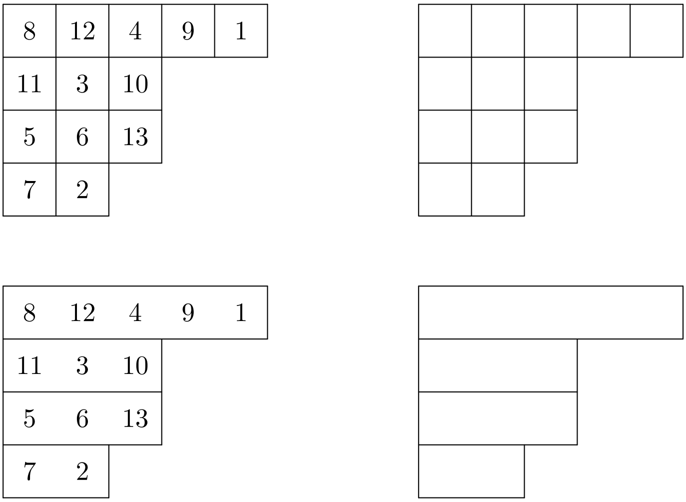
If (P, P') is a Young tableau on X, then we call P and P' the row partition and the column partition, respectively, and we call the isotropy subgroups $G_P \subset G$ and $G_{P'} \subset G$ the row stabilizer and the column stabilizer, respectively.

Proposition 11. *Let $X \in \mathrm{Fin}$. The canonical projections induce bijections*

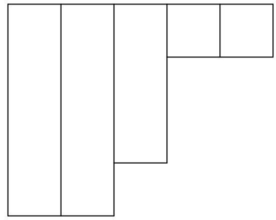
$$G \setminus \mathrm{Part}(X) \xleftarrow{\;p\;} G \setminus \mathrm{Tabl}(X) \xrightarrow{\;p'\;} G \setminus \mathrm{Part}(X).$$

Proof. We leave it to the reader to turn the example sketched by the figure in Example 12 below into a formal proof. $\square$

Example 12. The following figure depits, in the top row, a Young tableau (P, P') and its Young diagram $G \cdot (P, P')$; in the middle row, the row partition P and the orbit $G \cdot P$ through it; and in the bottom row, the column partition P' and the orbit $G \cdot P'$ through it.



8	12	4	9	1
11	3	10		
5	6	13		
7	2			



Translating to partitions of $n = 13$ using "matrix indexing," we find that $G \cdot P$ corresponds to $\lambda = (5,3,3,2)$, and that $G \cdot P'$ corresponds to $\lambda' = (4,4,3,1,1)$.

Corollary 13. *Let $X \in \mathrm{Fin}$, and let (P, P') and (Q, Q') be two tableaux on X. If $P = Q$, then there exists $h \in G_P$ such that $h \cdot P' = Q'$, and if $P' = Q'$, then there exists $h' \in G_{P'}$ such that $h' \cdot P = Q$.*

Proof. If $P = Q$, then $p(G \cdot (P, P')) = p(G \cdot (Q, Q'))$, so by Proposition 11, there exists $g \in G$ such that $g \cdot (P, P') = (Q, Q')$. But $P = Q$, so $g \in G_P$. □

Let Y be a Young diagram and choose a Young tableau $(P, P') \in Y$. We let $H = G_P$ and $K = G_{P'}$ be the row stabilizer and column stabilizer of (P, P'), respectively, and consider the representations

$$\begin{aligned} \pi_Y^+ &= (\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G)(\tau) \\ \pi_Y^- &= (\mathrm{Ind}_K^G \circ \mathrm{Res}_K^G)(\sigma), \end{aligned}$$

where τ is the 1-dimensional trivial representation of G and σ is the 1-dimensional sign representation of G.

Theorem 14. *Let X be a finite set, and let $G = \mathrm{Aut}(X)$.*

(1) *If Y is a Young diagram on X, then, up to isomorphism, there is a unique irreducible finite dimensional complex representation π_Y of G, which occurs in the decompositions of both π_Y^+ and π_Y^-.*
(2) *If Y and Z are distinct Young diagrams on X, then the representations π_Y and π_Z are non-isomorphic.*
(3) *If π is an irreducible finite dimensional complex representation of G, then $\pi \simeq \pi_Y$ for some Young diagram Y on X.*

Proof. To prove (1), it suffices to show that

$$\dim_{\mathbb{C}} \mathrm{Hom}(\pi_Y^+, \pi_Y^-) = 1,$$

and to do so, we will use the results on induced representations that we proved in the last two lectures. We consider the cartesian diagram of left G-sets

$$\begin{tikzcd} G/H \times G/K \arrow[r, "q'"] \arrow[d, "p'"] \arrow[dr, "f"] & G/H \arrow[d, "p"] \\ G/K \arrow[r, "q"] & G/G \end{tikzcd}$$

where we have included the map $f = p \circ q' = q \circ p$. We have canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}(\pi_Y^+, \pi_Y^-) &= \mathrm{Hom}(p_* p^* \tau, q_* q^* \sigma) \simeq \mathrm{Hom}(q^* p_* p^* \tau, q^* \sigma) \\ &\simeq \mathrm{Hom}(q_! \, q^* p_* p^* \tau, \sigma) \simeq \mathrm{Hom}(q_! \, q^* p_! \, p^* \tau, \sigma) \\ &\simeq \mathrm{Hom}(q_! \, p'_! \, q'^* p^* \tau, \sigma) \simeq \mathrm{Hom}(f_! \, f^* \tau, \sigma) \\ &\simeq \mathrm{Hom}(f^* \tau, f^* \sigma).\end{aligned}$$

Moreover, we defined a non-canonical isomorphism of left G-sets

$$\coprod_{1 \le s \le m} G/H \cap a_s K a_s^{-1} \xrightarrow{\;u\;} G/H \times G/K,$$

which depends on a choice of a family $(a_1, \dots, a_m)$ of representatives of the double cosets $H \backslash G/K$, among other things. So we conclude that

$$\mathrm{Hom}(\pi_Y^+, \pi_Y^-) \simeq \prod_{1 \le s \le m} \mathrm{Hom}(\mathrm{Res}^G_{H \cap a_s K a_s^{-1}}(\tau), \mathrm{Res}^G_{H \cap a_s K a_s^{-1}}(\sigma)).$$

Since both $\mathrm{Res}^G_{H \cap aKa^{-1}}(\tau)$ and $\mathrm{Res}^G_{H \cap aKa^{-1}}(\sigma)$ are 1-dimensional representations of $H \cap aKa^{-1}$, we find that

$$\begin{aligned}&\dim_{\mathbb{C}} \mathrm{Hom}(\mathrm{Res}^G_{H \cap aKa^{-1}}(\tau), \mathrm{Res}^G_{H \cap aKa^{-1}}(\sigma)) \\ &\qquad = \begin{cases} 1 & \text{if } \mathrm{sgn}(g) = 1 \text{ for all } g \in H \cap aKa^{-1}, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

For the double coset $HaK = HK$, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}(\mathrm{Res}^G_{H \cap K}(\tau), \mathrm{Res}^G_{H \cap K}(\sigma)) = 1,$$

since $H \cap K = \{e\}$ by Lemma 9. It remains to show that if $a \notin HK$, then there exists $g \in H \cap aKa^{-1}$ such that $\mathrm{sgn}(g) = -1$. To this end, we consider the tableau $a(P, P')$, which has column stabilizer aKa^{-1}. We claim that $P \cap aP' \neq O$. Granting this, we find that there exists a row in P and a column in aP' that have at least two elements in common. But then the transposition g that interchanges these two elements belongs to $H \cap aKa^{-1}$ and has $\mathrm{sgn}(g) = -1$. To prove the claim, we assume that $P \cap aP' = O$. This implies that (P, P'), (P, aP'), and (aP, aP') all are tableaux, so by Corollary 13 there exists $h \in H$ such that $(P, aP') = h(P, P')$ and $k \in K$ such that $(aP, aP') = aka^{-1}(P, aP')$, and therefore,

$$a(P, P') = aka^{-1}h(P, P').$$

By Lemma 9, we conclude that $a = aka^{-1}h$, and hence, that $a = hk \in HK$, which is a contradiction. This proves the claim and hence (1).

To prove (2), it suffices to show that

$$\dim_{\mathbb{C}} \mathrm{Hom}(\pi_Y^+, \pi_Z^-) = 0.$$

So let (P, P') and (Q, Q') be tableaux in Y and Z, respectively, and let $H = G_P$ and $K = G_{Q'}$. Arguing as in the case of (1), it suffices to show that for all $a \in G$, there exists $g \in H \cap aKa^{-1}$ such that $\mathrm{sgn}(g) = -1$. If this is not the case, then we have $P \cap aQ' = O$. But then (P, P'), (P, aQ'), and (aQ, aQ') are all tableaux, so by Corollary 13, there exists $h \in H$ such that $(P, aQ') = h(P, P')$ and $k \in K$ such that $(aQ, aQ') = aka^{-1}(P, aQ')$, and hence,

$$a(Q, Q') = aka^{-1}h(P, P').$$

But this contradicts that $Y \neq Z$, so (2) follows.

Finally, we prove (3). We have constructed the family

$$(\pi_Y)_{Y \in G \setminus \mathrm{Tabl}(X)}$$

of pairwise non-isomorphic irreducible finite dimensional complex representations of G, and Propositions 6 and 11 show that, up to isomorphism, these are all irreducible finite dimensional complex representations of G. □

The representation π_Y is called the Specht representation associated with the Young tableau (P, P'). The isomorphism class of π_Y only depends on the Young diagram $Y = G \cdot (P, P')$.

Remark 15. We defined $\pi_Y^+ = (\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G)(\tau)$ and $\pi_Y^- = (\mathrm{Ind}_K^G \mathrm{Res}_K^G)(\sigma)$, but we could of course just as well have switched τ and σ in this definition. Indeed, some authors (of course) make this other choice.

If $Y = G \cdot (P, P')$ is a Young diagram on X, then $Y' = G \cdot (P', P)$ is again a Young diagram on X. We call Y' the conjugate Young diagram of the Young diagram Y.

Lemma 16. *Let Y be a Young diagram on X, and let Y' be the conjugate Young diagram on X. Then the associated Specht representations are related by*

$$\pi_{Y'} \simeq \pi_Y \otimes \sigma.$$

Proof. Let $(P, P') \in Y$, and let $H = G_P$ and $K = G_{P'}$ be its row stabilizer and column stabilizer. In this situation, we have $(P', P) \in Y'$, and

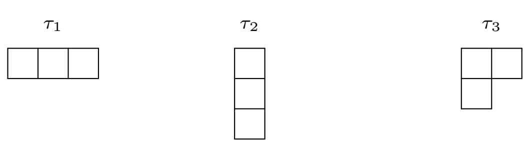
$$\begin{aligned}
\pi_{Y'}^+ &= (\mathrm{Ind}_K^G \circ \mathrm{Res}_K^G)(\tau) \simeq (\mathrm{Ind}_K^G \circ \mathrm{Res}_K^G)(\sigma \otimes \sigma) \\
&\simeq (\mathrm{Ind}_K^G \circ \mathrm{Res}_K^G)(\sigma) \otimes \sigma \simeq \pi_Y^- \otimes \sigma \\
\pi_{Y'}^- &= (\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G)(\sigma) \simeq (\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G)(\tau \otimes \sigma) \\
&\simeq (\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G)(\tau) \otimes \sigma \simeq \pi_Y^+ \otimes \sigma.
\end{aligned}$$

Here we have used that, in general, for $H \subset G$, one has

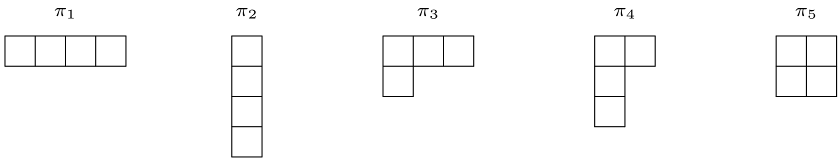
$$\begin{aligned}
\mathrm{Res}_H^G(\pi \otimes \rho) &\simeq \mathrm{Res}_H^G(\pi) \otimes \mathrm{Res}_H^G(\rho) \\
\mathrm{Ind}_H^G(\sigma \otimes \mathrm{Res}_H^G(\rho)) &\simeq \mathrm{Ind}_H^G(\sigma) \otimes \rho
\end{aligned}$$

for all representations π and ρ of G and σ of H. □

Example 17. For $H = \Sigma_3$, we have earlier found three irreducible finite dimensional complex representations of H, namely, the 1-dimensional trivial representation τ_1 and sign representation τ_2, and the 2-dimensional standard representation τ_3. These correspond to the following Specht representations:



Similarly, for $G = \Sigma_4$, we have earlier found five irreducible finite dimensional complex representations of G, namely, the 1-dimensional trivial representation π_1 and sign representation π_2, the 3-dimensional standard representation π_3 and its tensor product $\pi_4 = \pi_2 \otimes \pi_3$ with the sign representation, and the 2-dimensional representation π_5. These correspond to the following Specht representations:



Using Lemma 16, we see immediately from these listings that $\tau_2 \otimes \tau_3 \simeq \tau_3$ and that $\pi_2 \otimes \pi_5 \simeq \pi_5$. If we identify H with the subgroup of G consisting of all $g \in G$ with $g(4) = 4$, then one can also show that, in terms of Young diagrams, Res_H^G takes an irreducible G-representation π to the sum with multiplicity one of all irreducible H-representations τ corresponding to the Young diagrams obtained from the Young diagram for π by removing one box. So we have

$$\begin{aligned}
\mathrm{Res}_H^G(\pi_1) &\simeq \tau_1 \\
\mathrm{Res}_H^G(\pi_2) &\simeq \tau_2 \\
\mathrm{Res}_H^G(\pi_3) &\simeq \tau_1 \oplus \tau_3 \\
\mathrm{Res}_H^G(\pi_4) &\simeq \tau_2 \oplus \tau_3 \\
\mathrm{Res}_H^G(\pi_5) &\simeq \tau_3
\end{aligned}$$

Similarly, one can show that Ind_H^G takes an irreducible H-representation τ to the sum with multiplicity one of all irreducible G-representations π corresponding to the Young diagrams obtained from the Young diagram associated with τ by adding one box. So we find that

$$\begin{aligned}
\mathrm{Ind}_H^G(\tau_1) &\simeq \pi_1 \oplus \pi_3 \\
\mathrm{Ind}_H^G(\tau_2) &\simeq \pi_2 \oplus \pi_3 \\
\mathrm{Ind}_H^G(\tau_3) &\simeq \pi_3 \oplus \pi_4 \oplus \pi_5,
\end{aligned}$$

which is also what we have calculated directly before.

Finally, we mention that for Young diagrams Y and Z, Frobenius has given a formula for the value $\chi_{\pi_Y}(g)$ of the character of the Specht representation π_Y on an element g in the conjugacy class corresponding to Z in terms of combinatorial data that can be read off from the Young diagrams Y and Z directly. The formula is called the Frobenius character formula.