REPRESENTATIONS OF COMPACT GROUPS

We say that a topological group G is a compact group if its underlying space is compact and Hausdorff. The classical groups are all compact topological groups in this sense. It turns out that the theory of continuous finite dimensional complex representations of compact groups is completely analogous to the theory of finite dimensional complex represenations of finite groups, except that there will typically be a countably infinite number of non-isomorphic irreducible such representations. We first define the generalization to compact groups of the regular representation for finite groups. It will a representation on a complex Hilbert space $L^2(G)$, the definition of which requires some input from analysis, which we will assume.

One can show that there exists a Borel measure μ on G that is both left-invariant and right-invariant in the sense that for every Borel subset $A \subset G$ and $g \in G$,[1]

$$\mu(g \cdot A) = \mu(A) = \mu(A \cdot g),$$

and regular in the sense that for every Borel subset $A \subset G$,

$$\mu(A) = \inf\{\mu(U) \mid A \subset U,\, U \subset G \text{ open}\} = \sup\{\mu(K) \mid K \subset A \text{ compact}\}.$$

Moreover, such a measure, which is called a Haar measure, is unique up to scaling. In particular, there exists a unique Haar measure on G that is a probability measure in the sense that $\mu(G) = 1$.

Let $C^0(G, \mathbb{R})$ to be the (right) real vector space given by the set consisting of all continuous functions $\varphi \colon G \to \mathbb{R}$ equipped with pointwise vector sum and pointwise scalar multiplication. Given a Haar measure μ on G, we define a linear map

$$C^0(G, \mathbb{R}) \xrightarrow{I} \mathbb{R}$$

as follows. We choose a real number $0 < d < 1$ and define

$$A_{n,r}(\varphi) = \{x \in G \mid nd^r \leq \varphi(x) < (n+1)d^r\} \subset G,$$

for all integers n and positive integers r. Since $\varphi \colon G \to \mathbb{R}$ is continuous and G compact, the subset $\varphi(G) \subset \mathbb{R}$ is compact and therefore bounded. It follows that for every positive integer r, the subset $A_{n,r}(\varphi) \subset G$ is non-empty for only finitely many integers n. It is a Borel subset, and hence, we may form the sum

$$\textstyle\sum_{n \in \mathbb{Z}} nd^r \mu(A_{n,r}(\varphi)) \in \mathbb{R}.$$

One may show that the limit

$$I(\varphi) = \textstyle\lim_{r \to \infty} \sum_{n \in \mathbb{Z}} nd^r \mu(A_{n,r}(\varphi)) \in \mathbb{R}$$

exists and is independent of the choice of $0 < d < 1$. Finally, one may show that the function $I \colon C^0(G, \mathbb{R}) \to \mathbb{R}$ defined in this way is indeed linear.

Similarly, let $C^0(G, \mathbb{C})$ be the (right) complex vector space given by the set of all continuous complex functions $\varphi \colon G \to \mathbb{C}$ equipped with pointwise vector sum

[1] More generally, if G is locally compact, then there exists a left-invariant, but not necessarily right-invariant, measure on G.

and scalar multiplication. Let $f\colon \mathbb{R} \to \mathbb{C}$ be the canonical inclusion. Then we have the map of right real vector spaces

$$C^0(G,\mathbb{R}) \longrightarrow f_*C^0(G,\mathbb{C})$$

that to $\varphi\colon G \to \mathbb{R}$ assigns $f \circ \varphi\colon G \to \mathbb{C}$, and its adjunct map

$$f^*C^0(G,\mathbb{R}) = C^0(G,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow C^0(G,\mathbb{C})$$

is an isomorphism of complex vector spaces. Hence, we obtain a $\mathbb{C}$-linear map

$$C^0(G,\mathbb{C}) \xrightarrow{I_{\mathbb{C}}} \mathbb{C}$$

defined to be the adjunct of the composite $\mathbb{R}$-linear map

$$C^0(G,\mathbb{R}) \xrightarrow{I} \mathbb{R} \xrightarrow{f} f_*\mathbb{C}.$$

We will only consider $\mathbb{C}$-valued continuous functions on G, so we will abbreviate and write $C^0(G)$ instead of $C^0(G,\mathbb{C})$ and $I(\varphi)$ or $\int_G f(x)d\mu(x)$ instead of $I_{\mathbb{C}}(\varphi)$.

Given $\varphi, \psi \in C^0(G)$, we define $\varphi\psi \in C^0(G)$ to be the pointwise product product of φ and ψ, and we define φ^* to be the pointwise complex conjugate of φ. Since the map $I\colon C^0(G) \to \mathbb{C}$ is $\mathbb{C}$-linear, it follows immediately that the map

$$C^0(G) \times C^0(G) \xrightarrow{\langle -,- \rangle} \mathbb{C}$$

defined by $\langle \varphi, \psi \rangle = I(\varphi^*\psi)$ is a hermitian form. Moreover, this map is a hermitian inner product. Indeed, if $\varphi \in C^0(G)$ and $\langle \varphi, \varphi \rangle = I(|\varphi|^2) = 0$, then $\varphi = 0$.

If $(V, \langle -,- \rangle)$ is complex vector space with hermitian inner product, then the inner product gives rise to a metric $d\colon V \times V \to \mathbb{R}_{\geq 0}$ defined by

$$d(v,w) = \sqrt{\langle v-w, v-w \rangle},$$

and we say that $(V, \langle -,- \rangle)$ is a Hilbert space if the metric space (V,d) is complete.[2] If both $(U, \langle -,- \rangle_U)$ and $(V, \langle -,- \rangle_V)$ are complex vector spaces with hermitian inner products, then we say that a linear map $f\colon V \to U$ is Cauchy-continuous if for every sequence $v\colon \mathbb{Z}_{\geq 0} \to V$ that is Cauchy with respect to d_V, the sequence $f \circ v\colon \mathbb{Z}_{\geq 0} \to U$ is Cauchy with respect to d_U.[3] Let $\mathrm{Herm}_{\mathbb{C}}$ be the category, whose objects are the complex vector spaces with hermitian inner products, and whose morphisms are the Cauchy-continuous linear maps between these, and let $\mathrm{Hilb}_{\mathbb{C}}$ be the full subcategory of Hilbert spaces. In this situation, there is an adjunction

$$\begin{tikzcd} \mathrm{Herm}_{\mathbb{C}} \arrow[r, shift left, "i^{\wedge}"] & \mathrm{Hilb}_{\mathbb{C}}, \arrow[l, shift left, "i_{\wedge}"] \end{tikzcd}$$

where the right adjoint functor $i_\wedge$ is the canonical inclusion, and where the left adjoint functor $i^\wedge$ takes a complex vector space with hermitian inner product

[2] This means that for every sequence in V that is Cauchy with respect to d converges with respect to d. A sequence $v\colon \mathbb{Z}_{\geq 0} \to V$ is Cauchy with respect to d, if for all $\epsilon > 0$, there exists $N \in \mathbb{Z}_{\geq 0}$ such that $d(v_i, v_j) < \epsilon$, for all $i, j \geq N$, and it converges with respect to d, if there exists $v \in V$ such that for all $\epsilon > 0$, there exists $N \in \mathbb{Z}_{\geq 0}$ such that $d(v, v_i) < \epsilon$, for all $i \geq N$.

[3] Every Cauchy-continuous map between two metric spaces is continuous, and every continuous map between two complete metric spaces is Cauchy-continuous.

$(U, \langle -,- \rangle_U)$ to a Hilbert space $(V, \langle -,- \rangle_V)$ such that the underlying metric space (V, d_V) is the completion of the metric space (U, d_U). The unit map

$$(U, \langle -,- \rangle_U) \xrightarrow{\eta} (\widehat{U}, \langle -,- \rangle_{\widehat{U}}) = (i_\wedge \circ i^\wedge)(U, \langle -,- \rangle)$$

is injective and its image $\eta(U) \subset \widehat{U}$ is a dense subset of the metric space $(\widehat{U}, d_{\widehat{U}})$. In the following, we will omit the hermitian inner products from the notation.

The complex vector space with hermitian inner product $C^0(G)$ is not a Hilbert space, unless G is finite, and we now define the Hilbert space

$$L^2(G) = \widehat{C^0(G)}$$

to be its completion. As just explained the unit map

$$C^0(G) \xrightarrow{\eta} L^2(G)$$

is injective and its image is dense in $L^2(G)$. Hence, every element of $L^2(G)$ can be written, non-canonically, as a limit of a Cauchy sequence of continuous $\mathbb{C}$-valued functions on G, but a general element of $L^2(G)$ is not a $\mathbb{C}$-valued function on G, unless G is finite. In particular, the value "$f(x)$" of $f \in L^2(G)$ at $x \in G$ is not meaningful.[4] We will see below that the Hilbert space $L^2(G)$ is separable in the sense that it admits a countably dimensional dense subspace.

Lemma 1. *The map $I\colon C^0(G) \to \mathbb{C}$ is Cauchy-continuous.*

Proof. We must show that if the sequence $\varphi\colon \mathbb{Z}_{\geq 0} \to C^0(G)$ is Cauchy, then so is the sequence $I \circ \varphi\colon \mathbb{Z}_{\geq 0} \to \mathbb{C}$. It suffices to show that for all $\varphi, \psi \in C^0(G)$,

$$|I(\varphi) - I(\psi)| = |I(\varphi - \psi)| \leq I(|\varphi - \psi|),$$

which follows immediately from the definition of $I\colon C^0(G) \to \mathbb{C}$. ☐

Since $\mathbb{C}$ is complete, we conclude that $I\colon C^0(G) \to \mathbb{C}$ extends uniquely to a continuous, or equivalently, Cauchy-continuous linear map

$$L^2(G) \xrightarrow{I} \mathbb{C}.$$

Example 2. If G is a finite group, which we consider as a compact topological group with the discrete topology, then the Haar probability measure on G is given by the normalized counting measure that to $A \subset G$ assigns $\mu(A) = |A|/|G|$. It follows that the corresponding integral $I\colon C^0(G) \to \mathbb{C}$ is given by

$$I(f) = |G|^{-1} \textstyle\sum_{x \in G} f(x),$$

so we find that $L^2(G) = C^0(G) = \mathbb{C}[G]$.

We wish to extend the definition of the two-sided regular representation from finite groups to compact groups. So let G be a compact topological group. Given $(g_1, g_2) \in G \times G$ and $\varphi \in C^0(G)$, the formula

$$\mathrm{Reg}(g_1, g_2)(\varphi)(x) = \varphi(g_2^{-1} x g_1)$$

[4] The linear map $\mathrm{ev}_x\colon C^0(G) \to \mathbb{C}$ defined by $\mathrm{ev}_x(\varphi) = \varphi(x)$ is not Cauchy-continuous, and hence, does not extend to a map $\mathrm{ev}_x\colon L^2(G) \to \mathbb{C}$. However, it is possible to identify $L^2(G)$ with the quotient of the complex vector space consisting of the functions $f\colon G \to \mathbb{C}$ that are Haar measurable and square-integrable by the subspace of functions that are zero almost everywhere.

defines an element $\mathrm{Reg}(g_1, g_2)(\varphi) \in C^0(G)$. Moreover, since a Haar measure on G is both left-invariant and right-invariant, the map

$$C^0(G) \xrightarrow{\mathrm{Reg}(g_1,g_2)} C^0(G)$$

is a linear isometry with respect to $\langle -, - \rangle$. Indeed, we have

$$\|\mathrm{Reg}(g_1, g_2)(\varphi)\|^2 = \textstyle\int_G |\varphi(g_2^{-1} x g_1)|^2 d\mu(x) = \int_G |\varphi(x)|^2 d\mu(x) = \|\varphi\|^2.$$

In particular, it is Cauchy-continuous, and therefore, it induces a map

$$L^2(G) \xrightarrow{\mathrm{Reg}(g_1,g_2)} L^2(G)$$

which is a linear isometry with inverse $\mathrm{Reg}(g_1^{-1}, g_2^{-1})$. This defines a map

$$G \times G \xrightarrow{\mathrm{Reg}} U(L^2(G))$$

to the group of linear isometric isomorphisms of $L^2(G)$.[5] We wish to say that this is a map of topological groups, so we much define a topology on $U(L^2(G))$ and show that the map is continuous. It turns out that the appropriate topology on $U(L^2(G))$ is the so-called strong operator topology.[6]

Proposition 3. *The two-sided regular representation*

$$G \times G \xrightarrow{\mathrm{Reg}} U(L^2(G))$$

is continuous with respect to the strong operator topology.

Proof. The strong operator topology has the property that the map Reg in question is continuous if and only if for every $\varphi \in L^2(G)$, the composite map

$$G \times G \xrightarrow{\mathrm{Reg}} U(L^2(G)) \xrightarrow{\mathrm{ev}_\varphi} L^2(G)$$

is continuous. Let us write Reg_φ for this map. Since $G \times G$ is a topological group, it suffices to prove that this map is continuous at $(g_1, g_2) = (e, e)$.

We first let $\varphi \in C^0(G)$ and prove that Reg_φ is continuous at (e, e). We have

$$\|\mathrm{Reg}_\varphi(g_1, g_2) - \mathrm{Reg}_\varphi(e, e)\|^2 = \textstyle\int_G |\varphi(g_2^{-1} x g_1) - \varphi(x)|^2 d\mu(x)$$

and wish to prove that this quantity goes to 0 as $(g_1, g_2) \to (e, e)$. Since both φ and multiplication and inversion in G are continuous, we have every $x \in G$,

$$\lim_{(g_1,g_2)\to(e,e)} |\varphi(g_2^{-1} x g_1) - \varphi(x)|^2 = 0.$$

Moreover, for all $x \in G$, the integrand is dominated by

$$|\varphi(g_2^{-1} x g_1) - \varphi(x)|^2 \leq 4 \cdot \sup\{|\varphi(h)| \mid h \in G\},$$

so the dominated convergence theorem for the integral shows that

$$\lim_{(g_1,g_2)\to(e,e)} \textstyle\int_G |\varphi(g_2^{-1} x g_1) - \varphi(x)|^2 d\mu(x) = 0$$

as desired.

[5] Traditionally, linear isometric isomorphisms of a Hilbert space $\mathfrak{h}$ are called unitary operators, and therefore, we write $U(\mathfrak{h})$ for the group consisting of these operators.

[6] The uniform operator topology, which is given by the operator norm, is stronger than the strong operator topology. It turns out that it is too strong for our purposes, since, even for $G = U(1)$, the map Reg is not continuous with respect to this topology.

We next prove that for any $\varphi \in L^2(G)$, the map Reg_φ is continuous at (e,e). Given $\epsilon > 0$, we choose $\varphi_\epsilon \in C^0(G)$ such that $\|\varphi - \varphi_\epsilon\| < \epsilon$, which is possible, because $C^0(G)$ is dense in $L^2(G)$. Now

$$\begin{aligned}\|\mathrm{Reg}_\varphi(g_1,g_2) - \varphi\| &\le \|\mathrm{Reg}_\varphi(g_1,g_2) - \mathrm{Reg}_{\varphi_\epsilon}(g_1,g_2)\| \\ &\qquad + \|\mathrm{Reg}_{\varphi_\epsilon}(g_1,g_2) - \varphi_\epsilon\| + \|\varphi_\epsilon - \varphi\| \\ &= 2\|\varphi - \varphi_\epsilon\| + \|\mathrm{Reg}_{\varphi_\epsilon}(g_1,g_2) - \varphi_\epsilon\| \\ &< 2\epsilon + \|\mathrm{Reg}_{\varphi_\epsilon}(g_1,g_2) - \varphi_\epsilon\|,\end{aligned}$$

and by the first case, there exists an open neighborhood $(e,e) \in U \subset G \times G$ such that $\|\mathrm{Reg}_{\varphi_\epsilon}(g_1,g_2) - \varphi_\epsilon\| < \epsilon$, for all $(g_1,g_2) \in U$, we conclude that

$$\|\mathrm{Reg}_\varphi(g_1,g_2) - \varphi\| < 3\epsilon,$$

for all $(g_1,g_2) \in U$. This proves that Reg_φ is continuous at (e,e). $\square$

If (V,π) is a finite dimensional complex representation of G, then we define the associated space of matrix coefficients $M(\pi)$ to be the image of the map

$$V \otimes V^* \xrightarrow{\mu_\pi} C^0(G) \subset L^2(G)$$

defined by $\mu_\pi(v \otimes h)(g) = h(\pi(g)(v))$. One verifies immediately that it intertwines between $\pi \boxtimes \pi^*$ and Reg, so that we obtain a map

$$\pi \boxtimes \pi^* \xrightarrow{\mu_\pi} \mathrm{Reg}_{M(\pi)}$$

of continuous representations of $G \times G$. It is an isomorphism, if π is an irreducible representation of G, because then $\pi \boxtimes \pi^*$ is an irreducible representation of $G \times G$.

Lemma 4. *Let G be a compact topological group, let π_1 and π_2 be irreducible finite dimensional complex representations of G, and let $M(\pi_1), M(\pi_2) \subset L^2(G)$ the their subspaces of matrix coefficients.*

1. *If $\pi_1 \simeq \pi_2$, then $M(\pi_1) = M(\pi_2)$.*
2. *If $\pi_1 \not\simeq \pi_2$, then $M(\pi_1) \perp M(\pi_2)$.*

Proof. To prove (1), we let V_1 and V_2 be the representation spaces of π_1 and π_2, respectively, and let $h\colon V_1 \to V_2$ be a linear isomorphism that is intertwining between π_1 and π_2. In this situation, the diagram

$$\begin{tikzcd} V_1 \otimes V_2^* \arrow[r, "\mathrm{id} \otimes h^*"] \arrow[d, "h \otimes \mathrm{id}"] & V_1 \otimes V_1^* \arrow[d, "\mu_{\pi_1}"] \\ V_2 \otimes V_2^* \arrow[r, "\mu_{\pi_2}"] & L^2(G) \end{tikzcd}$$

commutes, and therefore,

$$M(\pi_1) = \mathrm{im}(\mu_{\pi_1}) = \mathrm{im}(\mu_{\pi_1} \circ (\mathrm{id} \otimes h^*)) = \mathrm{im}(\mu_{\pi_2} \circ (h \otimes \mathrm{id})) = \mathrm{im}(\mu_{\pi_2}) = M(\pi_2).$$

To prove (2), we consider the composition

$$M(\pi_1) \xrightarrow{i} L^2(G) \xrightarrow{p} M(\pi_2)$$

of the canonical inclusion of $M(\pi_1)$ and the orthogonal projection onto $M(\pi_2)$. The map i is intertwining between $\mathrm{Reg}_{M(\pi_1)}$ and Reg, since $M(\pi_1)$ is a Reg-invariant subspace, and the map p is intertwining between Reg and $\mathrm{Reg}_{M(\pi_2)}$, since Reg is a unitary representation. Therefore, the composite map $p \circ i$ is intertwining between $\mathrm{Reg}_{M(\pi_1)}$ and $\mathrm{Reg}_{M(\pi_2)}$, which are non-isomorphic irreducible finite dimensional complex representations of $G \times G$, so by Schur's lemma, $p \circ i = 0$ as stated. □

The theorem of Peter and Weyl states if G is a compact topological group, then two-sided regular representation of $G \times G$ decomposes as the completed direct sum of the spaces of matrix coefficients, one for each isomorphism class of irreducible finite dimensional continuous complex representations of G.

Theorem 5 (Peter–Weyl)**.** *Let G be a compact topological group, and let $\widehat{G}$ be the set of isomorphism classes of finite dimensional complex representations of G. For every $\sigma \in \widehat{G}$, let (V_σ, π_σ) be a representative of the class σ. The map*

$$\widehat{\bigoplus}_{\sigma \in \widehat{G}} \pi_\sigma \boxtimes \pi_\sigma^* \xrightarrow{\mu} \mathrm{Reg},$$

whose σth component is given by $\mu_{\pi_\sigma}(v \otimes h)(g) = h(\pi_\sigma(g)(v))$, is an isomorphism of continuous representations of $G \times G$.

Proof. We will only prove the theorem for compact groups G that admit a faithful continuous representation $\rho \colon G \to \mathrm{GL}_n(\mathbb{C})$; for a proof in the general case, we refer to [1, Theorem 5.4.1]. By Lemma 4, the canonical map

$$\bigoplus_{\sigma \in \widehat{G}} M(\pi_\sigma) \longrightarrow C^0(G)$$

is injective, and we proceed to prove that its image is dense with respect to the L^2-norm. To this end, we let $a_{ij} = \mu_\rho(e_j \otimes e_i^*) \in C^0(G)$ be the matrix coefficients of $\rho \colon G \to \mathrm{GL}_n(\mathbb{C})$ and consider the sub-$\mathbb{C}$-algebra $\mathbb{C}[G] \subset C^0(G)$ given by the image of the unique $\mathbb{C}$-algebra homomorphism

$$\mathbb{C}[X_{ij}, Y_{ij} \mid 1 \leq i, j \leq n] \longrightarrow C^0(G)$$

that to X_{ij} and $Y_{i,j}$ assign a_{ij} and a_{ij}^*. We claim that $\mathbb{C}[G] \subset C^0(G)$ is dense with respect to the L^2-norm. Indeed, by the Stone–Weierstrass theorem, $\mathbb{C}[G] \subset C^0(G)$ is dense with respect to the supremum norm $\|{-}\|_\infty$, and since G has finite volume $\mu(G)$, the calculation

$$\|\varphi\|_2^2 = \int_G |\varphi(x)|^2 d\mu(x) \leq \int_G \|\varphi\|_\infty^2 d\mu(x) = \|\varphi\|_\infty^2 \mu(G)$$

shows that $\mathbb{C}[G] \subset C^0(G)$ is also dense with respect to the L^2-norm.

Now, for all $m \geq 0$, we consider the finite dimensional subspace

$$\mathrm{Fil}_m \mathbb{C}[G] \subset \mathbb{C}[G]$$

given by the image by the $\mathbb{C}$-algebra homomorphism

$$\mathbb{C}[X_{ij}, Y_{i,j} \mid 1 \leq i, j \leq n] \longrightarrow C^0(G)$$

of the subspace of polynomials of degree $\leq m$. It is Reg-invariant, since the matrix coefficients a_{ij} transform linearly under left and right translation on G, and

$$\bigcup_{m \geq 0} \mathrm{Fil}_m \mathbb{C}[G] = \mathbb{C}[G].$$

We consider the representation $R_m \colon G \to \mathrm{GL}(\mathrm{Fil}_m \mathbb{C}[G])$ given by the restriction of the right regular representation of G on $L^2(G)$ to this subspace. Since it is finite dimensional, it decomposes as a direct sum of irreducible finite dimensional representations of G, so by Lemma 4, the inclusion $M(R_m) \to C^0(G)$ factors as

$$M(R_m) \longrightarrow \bigoplus_{\sigma \in \widehat{G}} M(\pi_\sigma) \longrightarrow C^0(G).$$

We define $\epsilon \colon C^0(G) \to \mathbb{C}$ to be the linear map given by $\epsilon(\varphi) = \varphi(e)$ and consider the map $\nu_m \colon \operatorname{Fil}_m \mathbb{C}[G] \to M(R_m)$ given by $\nu_m(\varphi) = \mu_{R_m}(\varphi \otimes \epsilon)$. The calculation

$$\nu_m(\varphi)(g) = \mu_{R_m}(\varphi \otimes \epsilon)(g) = \epsilon(R_m(g)(\varphi)) = R_m(g)(\varphi)(e) = \varphi(e \cdot g) = \varphi(g)$$

shows that the composite map

$$\operatorname{Fil}_m \mathbb{C}[G] \xrightarrow{\nu_m} M(R_m) \longrightarrow \bigoplus_{\sigma \in \widehat{G}} M(\pi_\sigma) \longrightarrow C^0(G)$$

is equal to the canonical inclusion, and hence, the canonical inclusion of $\mathbb{C}[G]$ into $C^0(G)$ factors as a composition

$$\mathbb{C}[G] = \bigcup_{m \geq 0} \operatorname{Fil}_m \mathbb{C}[G] \longrightarrow \bigoplus_{\sigma \in \widehat{G}} M(\pi_\sigma) \longrightarrow C^0(G).$$

Since the image of the composite map is dense with respect to the L^2-norm, so is the image of the right-hand map. This completes the proof. □

Remark 6. Let G be a linear compact topological group, let $\rho \colon G \to \mathrm{GL}_n(\mathbb{C})$ be a faithful continuous representation, and let $\mathbb{C}[G] \subset C^0(G)$ be the subalgebra of polynomial functions on G defined in the proof of Theorem 5. We claim that

$$\mathbb{C}[G] = \bigoplus_{\sigma \in \widehat{G}} M(\pi_\sigma) \subset C^0(G).$$

For otherwise, there exists $\tau \in \widehat{G}$ such that $M(\pi_\tau) \not\subset \mathbb{C}[G]$, and since $\mathbb{C}[G]$ is a direct sum of irreducible finite dimensional representations, it follows from Lemma 4 that $M(\pi_\tau) \perp \mathbb{C}[G]$. But this contradicts the fact that $\mathbb{C}[G] \subset \bigoplus_{\sigma \in \widehat{G}} M(\pi_\sigma)$ is dense.

Remark 7. In general, a unitary representation of a topological group G is defined to be a pair $(\mathfrak{h}, \pi)$ of a Hilbert space $\mathfrak{h}$ and a continuous group homomorphism

$$G \xrightarrow{\pi} U(\mathfrak{h})$$

from G to the group $U(\mathfrak{h})$ of linear isometric isomorphisms of $\mathfrak{h}$ equipped with the strong operator topology. As a consequence of the Peter–Weyl theorem, one can show that for every such representation admits a finite dimensional π-invariant subspace $V \subset \mathfrak{h}$; for a proof, see [1, p. 301]. In particular, every irreducible unitary representation of a compact topological group G is finite dimensional. By contract, locally compact topological groups such as $G = \mathrm{GL}_n(\mathbb{C})$ that are not compact have irreducible unitary representations that are infinite dimensional.

Example 8. We let $G = U(1)$ and let $\tau \colon G \to \mathrm{GL}(V)$ be the standard representation on $V = \mathbb{C}$. For every $n \geq 0$, we have the representation

$$\tau_n = \operatorname{Sym}^n_{\mathbb{C}}(\tau)$$

of G on $\operatorname{Sym}^n_{\mathbb{C}}(V)$. It is an irreducible representation, because the complex vector space $\operatorname{Sym}^n_{\mathbb{C}}(V)$ is 1-dimensional. Let (e_1) be the standard basis of V so that (e_1^n) is a basis of $\operatorname{Sym}^n_{\mathbb{C}}(V)$. Then for $z \in G$, we have

$$\tau_n(z)(e_1^n) = (e_1 z)^n = e_1^n z^n.$$

The dual representation $\tau_{-n} = \tau_n^*$ is also 1-dimensional, and hence, irreducible, and

$$\tau_{-n}(z)((e_1^*)^n) = ((e_1 z)^*)^n = (e_1^*)^n z^{-n}.$$

So for all $m, n \in \mathbb{Z}$, we have $\tau_m \simeq \tau_n$ if and only if $m = n$. Up to isomorphism, these are all irreducible finite dimensional continuous complex representations of G. Hence, by the Peter–Weyl theorem, the map of unitary $G \times G$-representations

$$\widehat{\bigoplus}_{n\in\mathbb{Z}} \tau_n \boxtimes \tau_n^* \xrightarrow{\mu} \mathrm{Reg}$$

is an isomorphism.

Example 9. Let $G = SU(2)$ and let $\pi\colon G \to \mathrm{GL}(V)$ be the standard representation on $V = \mathbb{C}^2$. For every $n \geq 0$, we have the representation

$$\pi_n = \mathrm{Sym}^n_{\mathbb{C}}(\pi)$$

of G on the $(n+1)$-dimensional complex vector space $\mathrm{Sym}^n_{\mathbb{C}}(V)$. Let (e_1, e_2) be the standard basis of V so that $(e_1^{n-i}e_2^i \mid 0 \leq i \leq n)$ is a basis of $\mathrm{Sym}^n_{\mathbb{C}}(V)$. We let $f\colon U(1) \to SU(2)$ be the group homomorphism defined by $f(z) = \mathrm{diag}(z, z^{-1})$ and consider the representation $f^*(\pi_n)$ of $U(1)$. For $z \in U(1)$, the calculation

$$\pi_n(f(z))(e_1^{n-i}e_2^i) = (e_1 z)^{n-i}(e_2 z^{-1})^i = e_1^{n-i}e_2^i z^{n-2i}$$

shows that the $\mathbb{C}$-linear isomorphism

$$\bigoplus_{0\leq i\leq n} \mathrm{Sym}^{2n-i}_{\mathbb{C}}(\mathbb{C}) \xrightarrow{h} \mathrm{Sym}^n_{\mathbb{C}}(V),$$

whose ith component is given by $h_i(v_i^{n-2i}) = e_1^{n-i}e_2^i v_i^{n-2i}$, is intertwining with respect to $\bigoplus_{0\leq i\leq n} \tau_{n-2i}$ and $f^*(\pi_n)$. Therefore, every $f^*(\pi_n)$-invariant subspace of $\mathrm{Sym}^n_{\mathbb{C}}(V)$ is of the form $W = h(\bigoplus_{i\in S} \mathrm{Sym}^{n-2i}_{\mathbb{C}}(\mathbb{C}))$ with $S \subset \{0, 1, \dots, n\}$. In particular, if $x = \sum_{0\leq i\leq n} e_1^{n-i}e_2^i x_i \in W$ and $x_i \neq 0$, then $e_1^{n-i}e_2^i \in W$.

If $W \subset \mathrm{Sym}^n_{\mathbb{C}}(V)$ is a non-zero π_n-invariant subspace, then W is in particular an $f^*(\pi_n)$-invariant subspace. Hence, there exists $0 \leq i \leq n$ such that $e_1^{n-i}e_2^i \in W$. We now consider

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G$$

and first calculate

$$g \cdot e_1^{n-i}e_2^i = e_1^{n-i}(e_1 + e_2)^i = e_1^n + \sum_{0<j\leq i} \binom{i}{j} e_1^{n-i-j}e_2^j,$$

which shows that $e_1^n \in W$, and next calculate

$$g^* \cdot e_1^n = (e_1 + e_2)^n = \sum_{0\leq j\leq n} \binom{n}{j} e_1^{n-j}e_2^j,$$

which shows that $e_1^{n-j}e_2^j \in W$ for all $0 \leq j \leq n$. Therefore, $W = \mathrm{Sym}^n_{\mathbb{C}}(V)$, and hence, π_n is irreducible. We will show later that, up to isomorphism, these are all irreducible finite dimensional continuous complex representations of G. Hence, by the Peter–Weyl theorem, the map of unitary $G \times G$-representations

$$\widehat{\bigoplus}_{n\in\mathbb{Z}_{\geq 0}} \pi_n \boxtimes \pi_n^* \xrightarrow{\mu} \mathrm{Reg}$$

is an isomorphism.

Example 10. Let $G = SO(\mathfrak{su}(2)) \simeq SO(3)$. We recall from last time that restriction along the adjoint representation

$$SU(2) \xrightarrow{\mathrm{Ad}} SO(\mathfrak{su}(2))$$

defines an equivalence of categories from $\mathrm{Rep}_{\mathbb{C}}(SO(\mathfrak{su}(2))$ onto the full subcategory of $\mathrm{Rep}_{\mathbb{C}}(SU(2))$ that is spanned by the representations (V,π) of $SU(2)$ for which $\pi(-I)=\mathrm{id}_V$. Now, for the representation π_n defined in Example 9, we have

$$\pi_n(-I)(e_1^{n-i}e_2^i)=(-e_1)^{n-i}(-e_2)^i=(-1)^n e_1^{n-i}e_2^i.$$

So there exists $\bar{\pi}_n\in\mathrm{Rep}_{\mathbb{C}}(SO(\mathfrak{su}(2))$ such that $\pi_n\simeq\mathrm{Ad}^*(\bar{\pi}_n)\in\mathrm{Rep}_{\mathbb{C}}(SU(2))$ if and only if $n=2m$ is even. Therefore, by the Peter–Weyl theorem, we conclude that the map of unitary $G\times G$-representations

$$\widehat{\bigoplus}_{m\in\mathbb{Z}_{\geq 0}}\bar{\pi}_{2m}\boxtimes\bar{\pi}_{2m}^*\xrightarrow{\mu}\mathrm{Reg}$$

is an isomorphism.

Appendix: Tensors

Let k be a field and V a vector space.[7] The tensor algebra of V is defined to be the graded associative k-algebra given by the graded k-vector space

$$T_k(V)=\bigoplus_{n\geq 0}T_k^n(V),$$

where $T_k(V)=V^{\otimes_k n}$, equipped with the multiplication given by

$$(x_1\otimes\cdots\otimes x_m)\cdot(y_1\otimes\cdots\otimes y_n)=x_1\otimes\cdots\otimes x_m\otimes y_1\otimes\cdots\otimes y_n.$$

The symmetric algebra of V is defined to be the graded commutative k-algebra given by the quotient

$$\mathrm{Sym}_k(V)=\bigoplus_{n\geq 0}\mathrm{Sym}_k^n(V)=T_k(V)/I$$

of the tensor algebra of V by the graded two-sided ideal $I\subset T_k(V)$ generated by the family $(x\otimes y-y\otimes x\mid x,y\in V)$, and the exterior algebra of V is defined to be the graded anticommutative k-algebra given by the quotient

$$\Lambda_k(V)=\bigoplus_{n\geq 0}\Lambda_k^n(V)=T_k(V)/J$$

of the tensor algebra of V by the graded two-sided ideal $J\subset T_k(V)$ generated by the family $(x\otimes x\mid x\in V)$. If $f\colon V\to U$ is a k-linear map, then the map

$$T_k^n(V)\xrightarrow{T_k^n(f)}T_k^n(U)$$

that to $x_1\otimes\cdots\otimes x_n$ assigns $f(x_1)\otimes\cdots\otimes f(x_n)$ is k-linear and induce maps

$$\mathrm{Sym}_k^n(V)\xrightarrow{\mathrm{Sym}_k^n(f)}\mathrm{Sym}_k^n(U)\qquad\qquad\Lambda_k^n(V)\xrightarrow{\Lambda_k^n(f)}\Lambda_k^n(U)$$

that also are k-linear. This makes $T_k^n(-)$, $\mathrm{Sym}_k^n(-)$, and $\Lambda_k^n(-)$ functors from the category of k-vector spaces and k-linear maps to itself.

In particular, if $\pi\colon G\to\mathrm{GL}(V)$ is a representation of a group G on a k-vector space V, then the composite map

$$G\xrightarrow{\pi}\mathrm{GL}(V)\xrightarrow{\mathrm{Sym}_k^n}\mathrm{GL}(\mathrm{Sym}_k^n(V))$$

[7] We only use that k is a commutative ring and that V is a k-module. It is important, however, that k be commutative, so $k=\mathbb{H}$ is not an option.

is a representation of G on the k-vector space $\mathrm{Sym}_k^n(V)$, which we, by abuse of notation, denote by $\mathrm{Sym}_k^n(\pi)$. Similarly, we define k-linear representations $T_k^n(\pi)$ and $\Lambda_k^n(\pi)$ on $T_k^n(V)$ and $\Lambda_k^n(V)$.

We denote the classes of $v_1 \otimes \cdots \otimes v_n \in T_k^n(V)$ in $\mathrm{Sym}_k^n(V)$ and $\Lambda_k^n(V)$ by $v_1 \ldots v_n$ and $v_1 \wedge \cdots \wedge v_n$, respectively. If $\sigma \in \Sigma_n$ is a permutation, then we

$$v_{\sigma(1)} \ldots v_{\sigma(n)} = v_1 \ldots v_n \in \mathrm{Sym}_k^n(V)$$

and

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = \mathrm{sgn}(\sigma) v_1 \wedge \cdots \wedge v_n \in \Lambda_k^n(V).$$

These statements both follow immediately from the definitions. However, it is a non-trivial theorem that if $(e_i)_{i \in I}$ is a basis of V then the family

$$(e_{i_1} \otimes \cdots \otimes e_{i_n} \mid i_1, \ldots, i_n \in I)$$

is a basis of $T_k^n(V)$, and that if we choose a total order "$\leq$" on I, then

$$(e_{i_1} \ldots e_{i_n} \mid i_1, \ldots, i_n \in I, i_1 \leq \cdots \leq i_n)$$

is a basis of $\mathrm{Sym}_k^n(V)$, and

$$(e_{i_1} \wedge \cdots \wedge e_{i_n} \mid i_1, \ldots, i_n \in I, i_1 < \cdots < i_n)$$

is a basis of $\Lambda_k^n(V)$. For instance, if $\dim_k(V) = d$ and $(e_1, \ldots, e_d)$ is a basis V, then the fact that $\dim_k(\Lambda_k^d(V)) = 1$ with basis $e_1 \wedge \cdots \wedge e_d$ is equivalent to the existence of the determinant.

References

[1] E. Kowalski, *An Introduction to the Representation Theory of Groups*, Graduate Studies in Mathematics, vol. 135, Amer. Math. Soc., Providence, RI, 2014.