LIE GROUPS

Definition 1. A Lie group is a group object $G = ((G, \mathcal{O}_G), \mu, \iota)$ in the category of smooth manifolds and morphisms of smooth manifolds.[1] A morphism of Lie groups is a homomorphism of group objects in the category of smooth manifolds and morphisms of smooth manifolds.

One defines complex Lie groups similarly to be a group objects in the category of complex manifolds and morphism of complex manifolds.

There is a "forgetful" functor from the category of Lie groups and morphisms of Lie groups to that of topological groups and continuous group homomorphisms that to $((G, \mathcal{O}_G), \mu, \iota)$ assigns (G, μ, ι). One can prove that this functor is fully faithful,[2] so in particular, the sheaf $\mathcal{O}_G$ is uniquely determined, up to unique isomorphism, by the remaining data. Hence, we may view "being a Lie group" as a property of a topological group.

Example 2. By using the implicit function theorem, we see that the classical groups all are (real) Lie groups. The groups $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})$ are examples of complex Lie groups.

If $((G, \mathcal{O}_G), \mu, \iota)$ is a Lie group, then we may consider the tangent space

$$\mathfrak{g} = T(G, \mathcal{O}_G)_e$$

of the smooth manifold $(G, \mathcal{O}_G)$ at the identity element $e \in G$. It is a real vector space of dimension $n = \dim(e)$. We proceed to show that the group structure morphisms μ and ι give rise to a structure of Lie algebra $[-,-]$ on this real vector space. Let us first define Lie algebras.

Definition 3. Let k be a field. A Lie algebra over k is a pair $\mathfrak{g} = (\mathfrak{g}, [-,-])$ of a right k-vector space $\mathfrak{g}$ and a k-linear map $[-,-]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that:

(LA1) For all $x \in \mathfrak{g}$, $[x, x] = 0$.

(LA2) For all $x, y, z \in \mathfrak{g}$, $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

A morphism of Lie algebras $f\colon (\mathfrak{h}, [-,-]) \to (\mathfrak{g}, [-,-])$ is a k-linear map $f\colon \mathfrak{h} \to \mathfrak{g}$ such that for all $x, y \in \mathfrak{h}$, $f([x, y]) = [f(x), f(y)]$.

We call $[-,-]$ the "Lie bracket" and we refer to (LA1) and (LA2) by saying that the Lie bracket is alternating and satisfies the Jacobi identity, respectively. It follows that the Lie bracket is antisymmetric in that for all $x, y \in \mathfrak{g}$, $[x, y] = -[y, x]$. We warn the reader that the Lie bracket is neither associative nor does it have an identity element, except in trivial cases. A Lie algebra $\mathfrak{a}$ is defined to be abelian if $[x, x] = 0$ for all $x \in \mathfrak{a}$.

[1] If (G, μ, ι) is a topological group and if $\{e\} \subset G$ is closed, then the space G is automatically Hausdorff. For the diagonal $\Delta(G) \subset G \times G$ is the preimage by the continuous map $\mu \circ (\mathrm{id} \times \iota)$ of the closed subset $\{e\} \subset G$ and hence closed.

[2] See [2, Theorem 9.2.16].

Example 4. (1) An associative k-algebra A determines a Lie algebra with the same underlying k-vector space as A and with Lie bracket $[a,b] = a\cdot b - b\cdot a$. In particular, if V is a right k-vector space, then $\mathrm{End}_k(V)$ is an associative k-algebra under composition of k-linear maps. The associated Lie algebra is denoted $\mathfrak{gl}(V)$.

(2) If $(X, \mathcal{O}_X)$ is a smooth manifold, then the real vector space $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ has a structure of real Lie algebra with Lie bracket $[-,-]$ defined by[3]

$$[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1.$$

Hence, there is a unique structure of Lie algebra on $\mathrm{Vect}(X, \mathcal{O}_X)$ for which the directional derivative is an isomorphism of Lie algebras.

In particular, a morphism of Lie groups $\pi\colon G \to \mathrm{GL}(V)$ gives rise to a morphism of Lie algebras $d\pi_e\colon \mathfrak{g} \to \mathfrak{gl}(V)$. So a representation of a Lie group determines a representation of its Lie algebra. This assignment is a functor, and we will prove that its restriction to the full subcategory of connected Lie groups is faithful.

Let $G = ((G, \mathcal{O}_G), \mu, \iota)$ be a Lie group. Given $g \in G$, we write

$$(G, \mathcal{O}_G) \xrightarrow{L_g} (G, \mathcal{O}_G)$$

for the morphism of smooth manifolds defined by $L_g(x) = \mu(g,x) = gx$ and call it "left multiplication by $g \in G$." The map L_g is not a group homomorphisms, but it is an automorphism of smooth manifolds, so we get a map

$$G \xrightarrow{L} \mathrm{Aut}(G, \mathcal{O}_G)$$

from G to the group of automorphism of the smooth manifold $(G, \mathcal{O}_G)$, and this map is a group homomorphism. We wish to consider the induced actions on the "space" of tangent vector fields. We first prove a general result.

Proposition 5. *Let $f\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a morphism of smooth manifolds, and let $D_u \in \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ and $D_v \in \mathrm{Der}_k(\mathcal{O}_Y, \mathcal{O}_Y)$ be the directional derivatives along two tangent vector fields $u \in \mathrm{Vect}(X, \mathcal{O}_X)$ and $v \in \mathrm{Vect}(Y, \mathcal{O}_Y)$, respectively. The following statements are equivalent.*

(a) *The diagram of smooth manifolds and morphisms of smooth manifolds*

$$\begin{tikzcd} T(Y, \mathcal{O}_Y) \arrow[r, "df"] & T(X, \mathcal{O}_X) \\ (Y, \mathcal{O}_Y) \arrow[u, "v"] \arrow[r, "f"] & (X, \mathcal{O}_X) \arrow[u, "u"'] \end{tikzcd}$$

commutes.

(b) *The diagram of sheaves of $\mathcal{O}_X$-modules and k-linear maps*

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "f^\sharp"] \arrow[d, "D_u"] & f_*\mathcal{O}_Y \arrow[d, "f_*D_v"] \\ \mathcal{O}_X \arrow[r, "f^\sharp"] & f_*\mathcal{O}_Y \end{tikzcd}$$

commutes.

[3] If $\delta_1, \delta_2\colon \mathcal{O}_X \to \mathcal{O}_X$ are k-linear derivations, then $\delta_1 \circ \delta_2\colon \mathcal{O}_X \to \mathcal{O}_X$ is typically not a k-linear derivation. So (2) is not a special case of (1).

Proof. We first assume (a) and prove (b). We must show that for all $U \subset X$ open with $V = f^{-1}(U) \subset Y$ and for all $\varphi \in \Gamma(V, \mathcal{O}_Y)$, the identity

$$D_v(\varphi \circ f|_V) = f|_V \circ D_u(\varphi)$$

holds. But this follows from the chain rule. Indeed, we consider the diagram

$$\begin{tikzcd} T(V, \mathcal{O}_Y|_V) \arrow[r, "df|_V"] & T(U, \mathcal{O}_X|_U) \arrow[r, "d\varphi"] & T(\mathbb{R}, \mathcal{O}_{\mathbb{R}}^{\mathrm{sm}}) \\ (V, \mathcal{O}_Y|_V) \arrow[r, "f|_V"] \arrow[u, "v|_V"] & (U, \mathcal{O}_X|_U) \arrow[r, "\varphi"] \arrow[u, "u|_U"] & (\mathbb{R}, \mathcal{O}_{\mathbb{R}}^{\mathrm{sm}}) \arrow[u, "w"] \end{tikzcd}$$

where w is the constant vector field defined by $w(t) = (t, e_1)$. We have

$$\begin{aligned} & D_v(\varphi \circ f|_V) \cdot (w \circ \varphi \circ f|_V) = d(\varphi \circ f|_V) \circ v|_V = d\varphi \circ d(f|_V) \circ v|_V \\ & = d\varphi \circ u|_U \circ f|_V = (D_u(\varphi) \circ f|_V) \cdot (w \circ \varphi \circ f|_V) \end{aligned}$$

where the first and last identity hold by the definition of D_u and D_v, the second identity holds by the chain rule, and the third identity holds by (a).

We next assume (b) and prove (a). Since p_X is a submersion, the implicit function theorem shows that the base-change of p_X along f exists,

$$\begin{tikzcd} T(X, \mathcal{O}_X)' \arrow[r, "f'"] \arrow[d, "p'_X"] & T(X, \mathcal{O}_X) \arrow[d, "p_X"] \\ (Y, \mathcal{O}_Y) \arrow[r, "f"] & (X, \mathcal{O}_X). \end{tikzcd}$$

We repeat the definition of the directional derivative to define a map

$$\mathrm{Vect}(X, \mathcal{O}_X)' \xrightarrow{D'} \mathrm{Der}_k(\mathcal{O}_X, f_*\mathcal{O}_Y)$$

from the set of morphism of smooth manifolds $s \colon (Y, \mathcal{O}_Y) \to T(X, \mathcal{O}_X)'$ such that $p'_X \circ s = \mathrm{id}_Y$ to the set of k-linear derivations $\delta \colon \mathcal{O}_X \to f_*\mathcal{O}_Y$. Given $U \subset X$ open with $V = f^{-1}(U) \subset Y$ and $\varphi \in \Gamma(U, \mathcal{O}_X)$, we consider the diagram

$$\begin{tikzcd} T(U, \mathcal{O}_X|_U)' \arrow[r, "(f|_V)'"] \arrow[d, "p'_U"] & T(U, \mathcal{O}_X|_U) \arrow[r, "d\varphi"] \arrow[d, "p_U"] & T(\mathbb{R}, \mathcal{O}_{\mathbb{R}}^{\mathrm{sm}}) \arrow[d, "p_{\mathbb{R}}"] \\ (V, \mathcal{O}_Y|_V) \arrow[r, "f|_V"] & (U, \mathcal{O}_X|_U) \arrow[r, "\varphi"] & (\mathbb{R}, \mathcal{O}_{\mathbb{R}}^{\mathrm{sm}}) \end{tikzcd}$$

and define $D'_s(\varphi) \in \Gamma(U, f_*\mathcal{O}_Y)$ to be the unique element such that

$$D'_s(\varphi) \cdot (w \circ \varphi \circ f|_V) = d\varphi \circ (f|_V)' \circ s|_V.$$

Now, the two composites $u \circ f$ and $df \circ v$ of the morphisms in the top diagram in the statement are both elements of $\mathrm{Vect}(X, \mathcal{O}_X)'$, and we have

$$D'_{u \circ f} = f^\sharp \circ D_u = f_* D_v \circ f^\sharp = D'_{df \circ v}.$$

Indeed, the first and last identity follow immediately from the definitions of D and D', and the middle identity is (b). Hence, it will suffice to prove that the map D' is injective.[4] To this end, we proceed as in the proof of Proposition 15 last time.

[4] The map D' need not be surjective, because the sheaf $f_*\mathcal{O}_Y$ can be very complicated.

We first observe that the map in question is equal to the map of global sections induced by a map of sheaves of $\mathcal{O}_Y$-modules

$$\mathcal{V}ect(X, \mathcal{O}_X)' \xrightarrow{D'} \mathcal{D}er_k(\mathcal{O}_X, f_*\mathcal{O}_Y).$$

Hence, we may assume that $(X, \mathcal{O}_X)$ is equal to $(U, \mathcal{O}_U^{\mathrm{sm}})$ with $U \subset \mathbb{R}^m$ an open subset. In this case, the $\Gamma(Y, \mathcal{O}_Y)$-module $\mathrm{Vect}(X, \mathcal{O}_X)'$ is free of rank m, and a basis is given by the family $(s_1, \dots, s_m)$ with $s_i = w_i \circ f$, where $w_i(x) = (x, e_i)$ and where $(e_1, \dots, e_m)$ is the standard basis of $\mathbb{R}^m$. Moreover, we have

$$D'_{s_i} = D'_{w_i \circ f} = f^\sharp \circ D_{w_i} = f^\sharp \circ (\partial/\partial x_i),$$

and since $f^\sharp \colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is a ring homomorphism, we find that

$$D'_{s_j}(x_i) = f^\sharp \circ (\partial x_i/\partial x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

This shows that the family $(f^\sharp \circ (\partial/\partial x_1), \dots, f^\sharp \circ (\partial/\partial x_m))$ is linear independent, which, in turn, shows that D' is injective as desired. $\square$

Now, if $(X, \mathcal{O}_X)$ is a smooth manifold, then we obtain a group homomorphism

$$\mathrm{Aut}(X, \mathcal{O}_X) \xrightarrow{\tau} \mathrm{Aut}_k(\mathrm{Vect}(X, \mathcal{O}_X))$$

defined by $\tau(f)(v) = u$, where $u, v \in \mathrm{Vect}(X, \mathcal{O}_X)$ are related as in the statement of Proposition 5. We note that the map $\tau(f)$ is not a $\Gamma(X, \mathcal{O}_X)$-linear automorphism, but, instead, it is a $\Gamma(X, \mathcal{O}_X)$-linear isomorphism

$$\mathrm{Vect}(X, \mathcal{O}_X) \xrightarrow{\tau(f)} f^{\sharp *}\,\mathrm{Vect}(X, \mathcal{O}_X)$$

from $\mathrm{Vect}(X, \mathcal{O}_X)$ to the left $\Gamma(X, \mathcal{O}_X)$ obtained from $\mathrm{Vect}(X, \mathcal{O}_X)$ by extension of scalars along $f^\sharp \colon \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$. We will not explore this further here and will simply consider $\tau(f)$ as a k-linear automorphism of $\mathrm{Vect}(X, \mathcal{O}_X)$. However, it is clear from Proposition 5 that for all $v_1, v_2 \in \mathrm{Vect}(X, \mathcal{O}_X)$,

$$[\tau(f)(v_1), \tau(f)(v_2)] = \tau(f)([v_1, v_2]),$$

so we may view τ as a group homomorphism

$$\mathrm{Aut}(X, \mathcal{O}_X) \xrightarrow{\tau} \mathrm{Aut}_k(\mathrm{Vect}(X, \mathcal{O}_X), [-,-])$$

to the group of automorphisms of the real Lie algebra of tangent vector fields on the smooth manifold $(X, \mathcal{O}_X)$.

We return to the case of a Lie group G. We define "left translation of tangent vector fields" to be the composite group homomorphism

$$G \xrightarrow{L} \mathrm{Aut}(G, \mathcal{O}_G) \xrightarrow{\tau} \mathrm{Aut}_k(\mathrm{Vect}(G, \mathcal{O}_G), [-,-]),$$

and we define a "left-invariant tangent vector field" to be a tangent vector v that is fixed under left translation by every $g \in G$.

Definition 6. The Lie algebra of a Lie group G is the sub-Lie algebra

$$(\mathrm{Lie}(G), [-,-]) = (\mathrm{Vect}(G, \mathcal{O}_G), [-,-])^G$$

of left-invariant tangent vector fields.[5]

We also write $\mathfrak{g}$ instead of $\mathrm{Lie}(G)$. We now show that the k-vector space $\mathrm{Lie}(G)$ is finite dimensional, and that the assignment of $\mathrm{Lie}(G)$ to G extends to a functor from the category of Lie groups and morphisms of Lie groups to the category of Lie algebras and morphisms of Lie algebras.[6]

Proposition 7. *If G is a Lie group, then the map $\epsilon_G \colon \mathfrak{g} \to T(G, \mathcal{O}_G)_e$ defined by $\epsilon_G(v) = v(e)$ is a k-linear isomorphism. Moreover, if $f \colon H \to G$ is a morphism of Lie groups, the unique k-linear map* $\mathrm{Lie}(f)$ *that makes the diagram*

$$\begin{tikzcd} \mathrm{Lie}(H) \arrow[r, "\mathrm{Lie}(f)"] \arrow[d, "\epsilon_H"] & \mathrm{Lie}(G) \arrow[d, "\epsilon_G"] \\ T(H, \mathcal{O}_H)_e \arrow[r, "df_e"] & T(G, \mathcal{O}_G)_e \end{tikzcd}$$

commute is a morphism of Lie algebras.

Proof. A tangent vector field $u \in \mathrm{Vect}(G, \mathcal{O}_G)$ is left-invariant if for all $g \in G$,

$$u(g) = dL_{g,e}(u(e)),$$

so the first part of the statement is clear. To prove the second part of the statement, we note that if $v \in \mathrm{Vect}(H, \mathcal{O}_H)$, then $u = \mathrm{Lie}(f)(v) \in \mathrm{Vect}(G, \mathcal{O}_G)$ is characterized as the unique left-invariant vector field such that $u \circ f = df \circ v$. Equivalently, by Proposition 5, the directional derivative $D_u \in \mathrm{Der}_k(\mathcal{O}_G, \mathcal{O}_G)$ is characterized in terms of $D_v \in \mathrm{Der}_k(\mathcal{O}_H, \mathcal{O}_H)$ by the properties that (1) the diagram

$$\begin{tikzcd} \mathcal{O}_G \arrow[r, "f^\sharp"] \arrow[d, "D_u"] & f_*\mathcal{O}_H \arrow[d, "f_*D_v"] \\ \mathcal{O}_G \arrow[r, "f^\sharp"] & f_*\mathcal{O}_H \end{tikzcd}$$

commutes, and (2) for all $g \in G$, the diagram

$$\begin{tikzcd} \mathcal{O}_G \arrow[r, "L_g^\sharp"] \arrow[d, "D_u"] & L_{g*}\mathcal{O}_G \arrow[d, "L_{g*}D_u"] \\ \mathcal{O}_G \arrow[r, "L_g^\sharp"] & L_{g*}\mathcal{O}_G \end{tikzcd}$$

[5] We could of course just as well have chosen to use right-invariant tangent vector fields, but note that, in general, being left-invariant and being right-invariant are different properties.

[6] The assignment of $\mathrm{Vect}(G, \mathcal{O}_G)$ to G does not extend to a functor between these categories.

commutes. More generally, if $s \in \mathrm{End}_k(\mathcal{O}_G)$ and $t \in \mathrm{End}_k(\mathcal{O}_H)$ are any k-linear morphisms, then we may ask that (1) the diagram

$$\begin{tikzcd} \mathcal{O}_G \arrow[r, "f^\sharp"] \arrow[d, "s"] & f_*\mathcal{O}_H \arrow[d, "f_*t"] \\ \mathcal{O}_G \arrow[r, "f^\sharp"] & f_*\mathcal{O}_H \end{tikzcd}$$

commutes, and (2) for all $g \in G$, the diagram

$$\begin{tikzcd} \mathcal{O}_G \arrow[r, "L_g^\sharp"] \arrow[d, "s"] & L_{g*}\mathcal{O}_G \arrow[d, "L_{g*}s"] \\ \mathcal{O}_G \arrow[r, "L_g^\sharp"] & L_{g*}\mathcal{O}_G \end{tikzcd}$$

commutes. Let us write $s \sim t$ if this is the case. We now let $v_i \in \mathrm{Lie}(H)$, and let $u_i = \mathrm{Lie}(f)(v_i) \in \mathrm{Lie}(G)$ so that $D_{u_i} \sim D_{v_i}$. Then the composite k-linear morphisms $D_{u_1} \circ D_{u_2}, D_{u_2} \circ D_{u_1} \in \mathrm{End}_k(\mathcal{O}_H)$ and $D_{v_1} \circ D_{v_2}, D_{v_2} \circ D_{v_1} \in \mathrm{End}_k(\mathcal{O}_G)$ also satisfy that $D_{u_1} \circ D_{u_2} \sim D_{v_1} \circ D_{v_2}$ and $D_{u_2} \circ D_{u_1} \sim D_{v_2} \circ D_{v_1}$. But then

$$[D_{u_1}, D_{u_2}] = D_{u_1} \circ D_{u_2} - D_{u_2} \circ D_{u_1} \sim D_{v_1} \circ D_{v_2} - D_{v_2} \circ D_{v_1} = [D_{v_1}, D_{v_2}],$$

which shows that

$$[u_1, u_2] = \mathrm{Lie}(f)([v_1, v_2]),$$

as desired. $\square$

Remark 8. Let G be a Lie group, and let us identify $\mathfrak{g} = T(G, \mathcal{O}_G)_e$. The Lie bracket on $\mathfrak{g}$ may also be defined as follows. The group structure on $(G, \mathcal{O}_G)$ induces a group structure on $T(G, \mathcal{O}_G)$, and the maps

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "i"] & T(G, \mathcal{O}_G) \arrow[r, "p"] & G \arrow[l, bend left, "0"] \end{tikzcd}$$

where $i = i_e$ is the kernel of $p = p_G$ and where $0 = 0_G$ is the zero section, all are morphisms of Lie groups.[7] Moreover, they exhibit the Lie group $T(G, \mathcal{O}_G)$ as the semidirect product of the Lie group G and the k-vector space $\mathfrak{g}$ considered as a Lie group under addition. This determines a morphism of Lie groups

$$G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}_k(\mathfrak{g})$$

called the adjoint representation. The induced map of tangent spaces at the identity element $e \in G$ is a k-linear map

$$\mathfrak{g} \xrightarrow{\mathrm{ad}} \mathrm{End}_k(\mathfrak{g})$$

the adjunct of which is a k-linear map

$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{[-,-]} \mathfrak{g}.$$

[7] The fact that i is a group homomorphism was the subject of the problem set for week 14.

To see that it satisfies the Jacobi identity, we argue as follows. If $f\colon H \to G$ is a morphism of Lie groups, then the map $\mathrm{Lie}(f) = df_e\colon \mathfrak{h} \to \mathfrak{g}$ satisfies

$$\mathrm{Lie}(f)([x,y]) = [\mathrm{Lie}(f)(x), \mathrm{Lie}(f)(y)]$$

for all $x, y \in \mathfrak{g}$. Moreover, if $G = \mathrm{GL}(V)$, then bracket $[-,-]$ defined here is equal to the one defined in Example 4. In particular, for $\mathrm{ad} = \mathrm{Lie}(\mathrm{Ad})$, we find that

$$\mathrm{ad}([x,y]) = [\mathrm{ad}(x), \mathrm{ad}(y)] = \mathrm{ad}(x) \circ \mathrm{ad}(y) - \mathrm{ad}(y) \circ \mathrm{ad}(x)$$

for all $x, y \in \mathfrak{g}$, which is equivalent to the Jacobi identity.

We next compare the representation theory of a Lie group G to that of its Lie algebra $\mathfrak{g}$. We will restrict our attention to representations (V, π), where V is a finite dimensional complex vector spaces, and where

$$G \xrightarrow{\pi} \mathrm{GL}(V)$$

is a morphism of Lie groups. If we apply the Lie algebra functor to this morphism, then we obtain a morphism of Lie algebras

$$\mathfrak{g} \xrightarrow{\mathrm{Lie}(\pi)} \mathfrak{gl}(V).$$

Hence, a representation π of a Lie group G on a finite dimensional complex vector space V gives rise to the representation $\mathrm{Lie}(\pi)$ of the Lie algebra $\mathfrak{g}$ on the same vector space V. In particular, if $\mathrm{Lie}(\pi)$ is irreducible, then π is necessarily also irreducible. We will now use the existence and uniqueness theorem for solutions to ordinary differential equations to show that if G is connected, then the representation π is completely determined by the representation $\mathrm{Lie}(\pi)$.

A global flow on a smooth manifold $(X, \mathcal{O}_X)$ is defined to be a left action

$$(\mathbb{R}, \mathcal{O}_{\mathbb{R}}^{\mathrm{sm}}) \times (X, \mathcal{O}_X) \xrightarrow{\phi} (X, \mathcal{O}_X)$$

in the category of smooth manifolds and morphisms of smooth manifolds, of the group object $\mathbb{R} = ((\mathbb{R}, \mathcal{O}_{\mathbb{R}}^{\mathrm{sm}}), +, -)$ on the object $(X, \mathcal{O}_X)$. There is a unique tangent vector field $v \in \mathrm{Vect}(X, \mathcal{O}_X)$ that makes the diagram

$$\begin{tikzcd}
T(\mathbb{R} \times X, \mathcal{O}_{\mathbb{R}\times X}) \arrow[r, "d\phi"] & T(X, \mathcal{O}_X) \\
(\mathbb{R} \times X, \mathcal{O}_{\mathbb{R}\times X}) \arrow[r, "\phi"] \arrow[u, "w\times 0"] & (X, \mathcal{O}_X) \arrow[u, "v"']
\end{tikzcd}$$

commute. Indeed, let $i\colon X \to \mathbb{R} \times X$ be the inclusion defined by $i(x) = (0, x)$. Since $\phi \circ i = \mathrm{id}_X$, we are forced to define v to be the composite morphism

$$v = v \circ \phi \circ i = d\phi \circ (w \times 0) \circ i,$$

and with this definition, we have

$$v \circ \phi = d\phi \circ (w \times 0) \circ i \circ \phi = d\phi \circ (w \circ 0),$$

where the second non-trivial identity holds, because ϕ is an action. We say that v is the infinitesimal generator of the flow ϕ.

Conversely, given $v \in \mathrm{Vect}(X, \mathcal{O}_X)$, the existence and uniqueness theorem for solutions to ordinary differential equations shows that there exists a morphism of smooth manifolds $\phi\colon (U, \mathcal{O}_{\mathbb{R}\times X}|_U) \to (X, \mathcal{O}_X)$ with $\{0\} \times X \subset U \subset \mathbb{R} \times X$ open which makes the diagram

$$\begin{tikzcd} T(U, \mathcal{O}_{\mathbb{R}\times X}|_U) \arrow[r, "d\phi"] & T(X, \mathcal{O}_X) \\ (U, \mathcal{O}_{\mathbb{R}\times X}|_U) \arrow[u, "(w\times 0)|_U"'] \arrow[r, "\phi"] & (X, \mathcal{O}_X) \arrow[u, "v"'] \end{tikzcd}$$

commute and satisfies $\phi(0, x) = x$ and $\phi(s, \phi(t, x)) = \phi(s+t, x)$ whenever this makes sense. We say that ϕ is a local flow with infinitesimal generator v. In particular, if there exists a global flow ϕ with infinitesimal generator v, then ϕ is uniquely determined by v. If this is the case, then we say that v is complete.

If G is a Lie group, and if $v \in \mathfrak{g}$ is a left-invariant vector field, then, by using the group structure, one shows that every local flow with infinitesimal generator v extends uniquely to a global flow $\phi = \phi_v$ with infinitesimal generator v. We define the exponential map of the Lie group G to be the map

$$\mathfrak{g} \xrightarrow{\exp} G$$

given by $\exp(v) = \phi_v(1, e)$. We note that exp is not a group homomorphism, unless the Lie algebra $\mathfrak{g}$ is abelian.

Theorem 9. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The exponential map is a morphism of smooth manifolds*

$$(\mathfrak{g}, \mathcal{O}_{\mathfrak{g}}^{\mathrm{sm}}) \xrightarrow{\exp} (G, \mathcal{O}_G).$$

Moreover, it is étale at $0 \in \mathfrak{g}$.[8]

Proof. The structure of group object on the smooth manifold $(G, \mathcal{O}_G)$ gives rise to a structure of group object on $T(G, \mathcal{O}_G)$. Moreover, there is a left-invariant tangent vector field u on the Lie group $T(G, \mathcal{O}_G)$ such that for every left-invariant tangent vector field v on $(G, \mathcal{O}_G)$, the diagram

$$\begin{tikzcd} T(G, \mathcal{O}_G) \arrow[r, "dv"] & T(T(G, \mathcal{O}_G)) \\ (G, \mathcal{O}_G) \arrow[u, "v"'] \arrow[r, "v"] & T(G, \mathcal{O}_G) \arrow[u, "u"'] \end{tikzcd}$$

commutes. Now, there is a global flow φ_u on $T(G, \mathcal{O}_G)$ with infinitesimal generator u, and it follows from the uniqueness of solutions to ordinary differential equations that for every $v \in \mathrm{Vect}(G, \mathcal{O}_G)$ with global flow φ_v on $(G, \mathcal{O}_G)$, the diagram

$$\begin{tikzcd} (\mathbb{R}, \mathcal{O}_{\mathbb{R}}^{\mathrm{sm}}) \times (G, \mathcal{O}_G) \arrow[r, "\varphi_v"] \arrow[d, "\mathrm{id} \times v"] & (G, \mathcal{O}_G) \arrow[d, "v"] \\ (\mathbb{R}, \mathcal{O}_{\mathbb{R}}^{\mathrm{sm}}) \times T(G, \mathcal{O}_G) \arrow[r, "\varphi_u"] & T(G, \mathcal{O}_G) \end{tikzcd}$$

[8] The exponential map may have critical points. One can show that $x \in \mathfrak{g}$ is a critical point for exp if and only if some $0 \neq \lambda \in 2\pi i\mathbb{Z} \subset \mathbb{C}$ is an eigenvalue of $\mathrm{ad}(x) \in \mathrm{End}_k(\mathfrak{g})$.

commutes. Therefore, the exponential map is equal to the composite map

$$(\mathfrak{g}, \mathcal{O}_{\mathfrak{g}}^{\mathrm{sm}}) \xrightarrow{i_1 \times i_e} (\mathbb{R}, \mathcal{O}_{\mathbb{R}}^{\mathrm{sm}}) \times T(G, \mathcal{O}_G) \xrightarrow{\varphi_u} T(G, \mathcal{O}_G) \xrightarrow{p_G} (G, \mathcal{O}_G)$$

and since each of these three maps is a morphism of smooth manifolds, so is the exponential map. Finally, it follows immediately from the definition that

$$\mathfrak{g} = T(\mathfrak{g}, \mathcal{O}_{\mathfrak{g}})_0 \xrightarrow{d\exp_0} T(G, \mathcal{O}_G)_e$$

is equal to the isomorphism ϵ_G in Proposition 7, and therefore, the inverse function theorem shows that exp is étale at $0 \in \mathfrak{g}$ as stated. □

Corollary 10. *If G is a connected Lie group, then every $g \in G$ can be written as a product $g = \exp(x_1) \cdots \exp(x_n)$ with $n \geq 0$ and $x_1, \dots, x_n \in \mathfrak{g}$.*

Proof. By Theorem 9, there exists open subsets $0 \in U \subset \mathfrak{g}$ and $e \in V \subset G$ such that $\exp|_U \colon (U, \mathcal{O}_U^{\mathrm{sm}}) \to (V, \mathcal{O}_G|_V)$ is a diffeomorphism. Hence, it suffices to show that the subgroup $H \subset G$ generated by V is equal to G.[9] Since $V \subset G$ is open, so is $H \subset G$. But then $gH \subset G$ is open, for all $g \in G$, which implies that

$$H = G \smallsetminus \left(\textstyle\bigcup_{g \in G \smallsetminus H} gH\right) \subset G$$

is closed. Since G is connected, we conclude that $H = G$ as desired. □

Corollary 11. *Let G and H be Lie groups. If H is connected, then the map*

$$\mathrm{Hom}(H, G) \xrightarrow{\mathrm{Lie}} \mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$$

is injective.

Proof. Let $f \colon H \to G$ be a morphism of Lie groups. The diagram

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\exp_H"] \arrow[d, "\mathrm{Lie}(f)"] & H \arrow[d, "f"] \\ \mathfrak{g} \arrow[r, "\exp_G"] & G \end{tikzcd}$$

commutes, by naturality of the exponential map. By Corollary 10, every element of H is a product of elements of $\exp_H(\mathfrak{h}) \subset H$. Since f is a group homomorphism, this implies that it is uniquely determined by the map $\mathrm{Lie}(f)$. □

We use the last corollary to show that if π_1 and π_2 are two finite dimensional real or complex representations of a connected Lie group G, then $\pi_1 \simeq \pi_2$ if and only if $\mathrm{Lie}(\pi_1) \simeq \mathrm{Lie}(\pi_2)$. In effect, we prove the following more precise result.

Corollary 12. *Let $\pi_1 \colon G \to \mathrm{GL}(V_1)$ and $\pi_2 \colon G \to \mathrm{GL}(V_2)$ be representations of a connected Lie group on finite dimensional real or complex vector spaces. A linear isomorphism $f \colon V_1 \to V_2$ intertwines between π_1 and π_2 if and only if it intertwines between $\mathrm{Lie}(\pi_1)$ and $\mathrm{Lie}(\pi_2)$.*

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[9] Here we also use that $\exp(x)^{-1} = \exp(-x)$, since $[x, -x] = -[x, x] = 0$.

Proof. That f intertwines between π_1 and π_2 means that the diagram of Lie groups

$$\begin{tikzcd} & \mathrm{GL}(V_1) \arrow[d, "c_f"] \\ G \arrow[ur, bend left, "\pi_1"] \arrow[dr, bend right, "\pi_2"'] & \\ & \mathrm{GL}(V_2) \end{tikzcd}$$

where $c_f(h) = f \circ h \circ f^{-1}$, commutes. But then the diagram of Lie algebras

$$\begin{tikzcd} & \mathfrak{gl}(V_1) \arrow[d, "\mathrm{Lie}(c_f)"] \\ \mathfrak{g} \arrow[ur, bend left, "\mathrm{Lie}(\pi_1)"] \arrow[dr, bend right, "\mathrm{Lie}(\pi_2)"'] & \\ & \mathfrak{gl}(V_2) \end{tikzcd}$$

commutes, and since $\mathrm{Lie}(c_f)(h) = f \circ h \circ f^{-1}$, this shows that f intertwines between $\mathrm{Lie}(\pi_1)$ and $\mathrm{Lie}(\pi_2)$. This part of the statement only uses that $\mathrm{Lie}(-)$ is a functor and not that G is connected. Conversely, if f intertwines between $\mathrm{Lie}(\pi_1)$ and $\mathrm{Lie}(\pi_2)$, then the bottom diagram commutes, and since G is connected, this implies, by Corollary 11 that the top diagram commutes. $\square$

So this is really marvelous. To a large extent, we have replaced the differential geometric problem of finding representations of a Lie group with the linear algebraic problem of finding representations of its Lie algebra. We illustrate for $G = SU(2)$, which is a compact connected Lie group. We have already proved that for every integer $n \geq 0$, the representation π_n given by the nth symmetric power

$$\pi_n = \mathrm{Sym}^n_{\mathbb{C}}(\pi_1)$$

of the standard representation π of $SU(2)$ on $V = \mathbb{C}^2$ is an irreducible representation of dimension $n+1$. The associated representation of the Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ is a morphism of real Lie algebras

$$\mathfrak{su}(2) \xrightarrow{\mathrm{Lie}(\pi_n)} f_*\mathfrak{gl}(\mathrm{Sym}^n_{\mathbb{C}}(V))$$

from the real Lie algebra $\mathfrak{su}(2)$ to the real Lie algebra obtained by restriction of scalars along $f\colon \mathbb{R} \to \mathbb{C}$ from the complex Lie algebra $\mathfrak{gl}(\mathrm{Sym}^n_{\mathbb{C}}(V))$. The adjunct of $\mathrm{Lie}(\pi_n)$ is a morphism of complex Lie algebras

$$\mathfrak{su}(2)_{\mathbb{C}} = f^*\mathfrak{su}(2) \xrightarrow{\widetilde{\mathrm{Lie}(\pi_n)}} \mathfrak{gl}(\mathrm{Sym}^n_{\mathbb{C}}(V)).$$

We have earlier identified $\mathfrak{su}(2)$ with the real vector space of traceless skew-hermitian complex 2×2-matrices. It has a basis given by the family (A_1, A_2, A_3), where

$$A_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad A_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The Lie bracket on $\mathfrak{su}(2)$ is given by $[A, B] = AB - BA$. Similarly, the complex Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of the complex Lie group $\mathrm{SL}_2(\mathbb{C})$ is given by the complex vector space of all traceless complex 2×2-matrices with the Lie bracket given by the same formula. So the inclusion of the set of traceless skew-hermitian complex 2×2-matrices in the set of all traceless complex 2×2-matrices defines a morphism of real Lie algebras $\mathfrak{su}(2) \to f_*\mathfrak{sl}(2,\mathbb{C})$, the adjunct of which is a morphism

$$\mathfrak{su}(2)_{\mathbb{C}} = f^*\mathfrak{su}(2) \longrightarrow \mathfrak{sl}(2,\mathbb{C}).$$

of complex Lie algebras. We claim that the latter map is an isomorphism. Indeed, one readily verifies that the family (A_1, A_2, A_3) is a basis of both of complex vector spaces. Moreover, under this identification, the representation

$$\mathfrak{sl}(2,\mathbb{C}) \xrightarrow{\widetilde{\mathrm{Lie}(\pi_n)}} \mathfrak{gl}(\mathrm{Sym}^n_{\mathbb{C}}(V))$$

is equivalent to the nth symmetric power of the standard representation of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ on V.

Now, the complex vector space $\mathfrak{sl}(2,\mathbb{C})$ has the much more convenient basis given by the family (X, H, Y), where[10]

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Indeed, in this basis, the Lie bracket is given by the simple formulas

$$[X, Y] = H, \quad [H, X] = 2X, \quad \text{and} \quad [H, Y] = -2Y.$$

The complex representations of $\mathfrak{sl}(2,\mathbb{C})$ can be completely understood, and this, in turn, is the starting point for understanding the representation theory of all complex reductive Lie algebras and Lie groups. Serre's book [3] is a very readable introduction to this beautiful theory.

Let $\pi\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V)$ be a representation on a complex vector space V, which, at the moment, we do not assume to be finite dimensional. We write $V^\lambda \subset V$ for the eigenspace corresponding to the eigenvalue $\lambda \in \mathbb{C}$ of $\pi(H)\colon V \to V$, and we say that $x \in V^\lambda$ has weight λ. The canonical map

$$\bigoplus_{\lambda\in\mathbb{C}} V^\lambda \longrightarrow V$$

is always injective. If the dimension of V is finite, then it is also surjective, but, in general, this is not the case. If x has weight λ, then the calculation

$$\begin{aligned}
(\pi(H)\circ\pi(X))(x) &= \pi([H,X])(x) + (\pi(X)\circ\pi(H))(x) \\
&= \pi(2X)(x) + \pi(X)(\lambda x) \\
&= (\lambda+2)\pi(X)(x) \\
(\pi(H)\circ\pi(Y))(x) &= \pi([H,Y])(x) + (\pi(Y)\circ\pi(H))(x) \\
&= -\pi(2Y)(x) + \pi(Y)(\lambda x) \\
&= (\lambda-2)\pi(Y)(x)
\end{aligned}$$

shows that $\pi(X)(x)$ has weight $\lambda+2$ and that $\pi(Y)(x)$ has weight $\lambda-2$. We say that an element $e \in V$ is primitive of weight λ if $e \neq 0$ and if $\pi(H)(e) = \lambda e$ and $\pi(X)(e) = 0$.

[10] The alternative notation e, h, and f for these matrices is also common.

Theorem 13. *Let π be an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ on a complex vector space V of finite dimension $n+1$. The following hold.*

(1) *There exists a primitive element $e \in V$ of weight $\lambda = n$.*
(2) *The family $(e_0, \dots, e_n)$, where $e_k = \pi(Y)^k(e)/k!$, is a basis of V.*
(3) *In this basis, the representation π is given by*

$$\pi(H)(e_k) = (\lambda - 2k)e_k$$

$$\pi(X)(e_k) = \begin{cases} 0 & \text{if } k = 0 \\ (\lambda - k + 1)e_{k-1} & \text{if } 0 < k \leq n \end{cases}$$

$$\pi(Y)(e_k) = \begin{cases} (k+1)e_{k+1} & \text{if } 0 \leq k < n \\ 0 & \text{if } k = n. \end{cases}$$

Conversely, the formulas (3) define an irreducible representation of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ on a complex vector space with basis $(e_0, \dots, e_n)$.

Proof. Since $\mathbb{C}$ is algebraically closed, there exists an eigenvector $x \in V$ of the linear endomorphism $\pi(H)\colon V \to V$. The vectors $\pi(X)^k(x)$ with $k \geq 0$ are either eigenvectors of $\pi(H)$ or zero. Since V is finite dimensional, there exists a maximal $k \geq 0$ such that $e = \pi(X)^k(x) \neq 0$ and $\pi(X)(e) = 0$. Hence, this element e is a primitive element of some weight $\lambda \in \mathbb{C}$.

Now, for all $k \geq 0$, we consider the elements $e_k \in V$ defined by

$$e_k = \pi(Y)^k(e)/k!,$$

and we also set $e_{-1} = 0$. We claim that for all $k \geq 0$, the following hold:

(a) $\pi(H)(e_k) = (\lambda - 2k)e_k$
(b) $\pi(Y)(e_k) = (k+1)e_{k+1}$
(c) $\pi(X)(e_k) = (\lambda - k + 1)e_{k-1}$.

Indeed, (b) holds, by definition, and (a) holds by the observation that $\pi(Y)$ lowers weight by 2. We prove (c) by induction on $k \geq -1$, the case $k = -1$ being trivial. Assuming that (c) holds for $k < m$, the calculation

$$\begin{aligned} m\pi(X)(e_m) &= (\pi(X) \circ \pi(Y))(e_{m-1}) \\ &= \pi([X,Y])(e_{m-1}) + (\pi(Y) \circ \pi(X))(e_{m-1}) \\ &= \pi(H)(e_{m-1}) + (\lambda - m + 2)\pi(Y)(e_{m-2}) \\ &= (\lambda - 2m + 2 + (\lambda - m + 2)(m-1))e_{m-1} \\ &= m(\lambda - m + 1)e_{m-1}, \end{aligned}$$

shows that (c) holds for $k = m$. This proves the claim.

Next, if the elements e_k with $k \geq 0$ all are non-zero, then $(e_k)_{k \geq 0}$ is a family of eigenvectors for $\pi(H)$ with pairwise distinct eigenvalues. But then this family is linearly independent, which is not possible, because V is finite dimensional. We also observe from (b) that $e_k = 0$ implies that $e_{k+1} = 0$. So there exists $m \geq 0$ such that $e_k \neq 0$ for $0 \leq k \leq m$ and $e_k = 0$ for $k > m$. Moreover, by (c), we have

$$0 = \pi(X)(e_{m+1}) = (\lambda - m)e_m,$$

so we conclude that $\lambda = m$.

Finally, it follows immediately from (a)–(c) that the subspace $W \subset V$ spanned by $(e_0, \dots, e_m)$ is π-invariant. It is also non-zero, since $0 \neq e = e_0 \in W$, and since (V, π) was assumed to be irreducible, we conclude that $W = V$ and $m = n$. $\square$

Corollary 14. *Let $n \geq 0$ be an integer.*

(1) *The complex Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ has a unique isomorphism class of irreducible complex representations of dimension $n + 1$.*

(2) *The real Lie algebra $\mathfrak{su}(2)$ has a unique isomorphism class of irreducible complex representations of dimension $n + 1$.*

(3) *The real Lie group $SU(2)$ has a unique isomorphism class of irreducible complex representations of dimension $n + 1$.*

Proof. First, (1) follows immediately from Theorem 13. Second, (2) follows from (1) and from the extension-of-scalars/restriction-of-scalars adjunction, since we have an isomorphism of complex Lie algebras $\mathfrak{su}(2)_{\mathbb{C}} \to \mathfrak{sl}(2, \mathbb{C})$. Finally, we conclude from (2) and from Corollary 12 that the connected Lie group $SU(2)$ has at most one isomorphism class of irreducible complex representations of dimension $n+1$. But we have already proved that $\pi_n \colon SU(2) \to \mathrm{GL}(\mathrm{Sym}^n_{\mathbb{C}}(V))$ is such a representation, so (3) follows. $\square$

Example 15. The adjoint representation

$$SU(2) \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathfrak{su}(2))$$

is a 3-dimensional real representation. One can show that the adjoint representation is irreducible, and that its complexification

$$SU(2) \xrightarrow{\mathrm{Ad}_{\mathbb{C}}} \mathrm{GL}(\mathfrak{su}(2)_{\mathbb{C}})$$

also is irreducible. Therefore, by Corollary 14, it is isomorphic to the symmetric square π_2 of the standard representation $\pi = \pi_1$.

In elementary particle physics, a gauge theory begins with a compact Lie group G of "internal symmetries," and the complexified adjoint representation

$$G \xrightarrow{\mathrm{Ad}_{\mathbb{C}}} \mathrm{GL}(\mathfrak{g}_{\mathbb{C}})$$

provides the "gauge bosons" of the theory; they are the elements of a basis of the complex vector space $\mathfrak{g}_{\mathbb{C}}$. For example, physicists write (W^+, W^0, W^-) for the basis (X, H, Y) of $\mathfrak{su}_{\mathbb{C}} \simeq \mathfrak{sl}(2, \mathbb{C})$. Its elements are the W-bosons, which mediate the weak force. Let me explain what this means. The "elementary fermions" in the gauge theory are basis elements of certain irreducible finite dimensional complex representations of G. The selection of the irreducible representations that should be considered the "elementary fermions" of the theory, however, is entirely empirical. If $\pi \colon G \to \mathrm{GL}(V)$ is an irreducible finite dimensional complex representation, then

$$\mathfrak{g}_{\mathbb{C}} \xrightarrow{\mathrm{Lie}(\pi)_{\mathbb{C}}} \mathfrak{gl}(V)$$

is a representation of the complexified Lie algebra on V, and moreover, this map is intertwining with respect to the G-representations $\mathrm{Ad}_{\mathbb{C}}$ on the domain and $\mathrm{End}(\pi)$ on the target. It is by means of this Lie algebra representation that the gauge bosons acts on the elementary fermions. See the article [1] by Baez–Huerta for more on this.

References

[1] J. Baez and J. Huerta, *The algebra of grand unified theories*, Bull. Amer. Math. Soc. **47** (2010), 483–552.

[2] J. Hilgert and K.-H. Neeb, *Structure and geometry of Lie groups*, Springer Monographs in Mathematics, Springer, New York, 2012.

[3] J.-P. Serre, *Complex semisimple Lie algebras. Translated from the French by G. A. Jones*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2001.