Today: Chapter 1

Invariant subspaces.

<u>Def</u> Let $\pi : G \to GL(V)$ be a k-lin. repr. A subspace $U \subset V$ is said to be π-invariant if for all $g \in G$ and $x \in U$,

$$\pi(g)(u) \in U.$$

<u>Ex</u> $G = (\mathbb{R}, +)$, $V = \{\text{pol. fct. } f : \mathbb{R} \to \mathbb{R}\}$, $L : G \to GL(V)$, left reg. repr.

$$(L(t)(f))(x) = f(x-t).$$

Then $U = \{\text{pol. fct. of deg.} \leq n\} \subset V$ is L-invariant.

Suppose $\dim_k(V) < \infty$. Choose basis $(e_1, \dots, e_m)$ for $U \subset V$ and extend to basis $(e_1, \dots, e_m, e_{m+1}, \dots, e_{m+n})$ for V. That U is π-invariant means that the matrix that represents $\pi(g) : V \to V$ w.r.t. to this basis has the form

$$\begin{array}{cc} \left(\begin{array}{c|c} A(g) & C(g) \\ \hline 0 & B(g) \end{array}\right) & \begin{array}{l} \}\, m \\ \}\, n \end{array} \\ \begin{array}{cc} \underbrace{\qquad\quad}_{m} & \underbrace{\qquad\quad}_{n} \end{array} & \end{array}$$

In general, if $\pi : G \to GL(V)$ is a k-linear repr. and $U \subset V$ a π-invariant subspace, then we define

$$G \xrightarrow{\pi_U} GL(U)$$

$$g \longmapsto \pi|_U$$

We say that π_U is a subrepr. of π. We also define

$$G \xrightarrow{\pi_{V/U}} GL(V/U)$$

by

$$\pi_{V/U}(g)(x+U) = \pi(g)(x) + U$$

and say that $\pi_{V/U}$ is a quotient repr. of π.

⌈ Check that $\pi_{V/U}$ is a well-def. repr. Recall that the quotient vector space V/U is defined by

$$V/U = \{x + U \subset V \mid x \in V\}$$

with vector sum and scalar mult.

$$(x+U) + (x'+U) := (x+x') + U$$

$$a \cdot (x+U) := (a \cdot x) + U.$$

Moreover, a linear map $f : V \to V$

(such as $f = \pi(g)$) such that $f(U) \subset U$ induces a linear map

$$V/U \xrightarrow{\bar{f}} V/U$$

$$x + U \longmapsto f(x) + U$$

and if $f_1, f_2 : V \to V$ are two lin. maps s.t. $f_i(U) \subset U$, then

$$\overline{f_1 \circ f_2} = \bar{f}_1 \circ \bar{f}_2 .$$

If $\dim_k(V) < \infty$ and if we choose bases $(e_1, \dots, e_m)$ of U and $(e_1, \dots, e_m, e_{m+1}, \dots, e_{m+n})$ of V, then

$$(e_{m+1} + U, \dots, e_{m+n} + U)$$

is a basis of V/U, and the matrices that represent $\pi_U(g)$ and $\pi_{V/U}(g)$ w.r.t. these bases are $A(g)$ and $B(g)$, respectively, where

$$\begin{array}{c} \left(\begin{array}{c|c} A(g) & C(g) \\ \hline 0 & B(g) \end{array}\right) \begin{array}{l} \}\, m \\ \}\, n \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{m} & \underbrace{\qquad}_{n} \end{array} \end{array}$$

represents $\pi(g)$ w.r.t. $(e_1, \dots, e_{m+n})$

<u>Def</u> A repr. $\pi : G \to GL(V)$ is irreducible (or simple) if $V \neq \{0\}$ and if the only π-invariant subspaces of V are $\{0\}$ and V. ✓

Note: The axiom $V \neq \{0\}$ is missing in the book. ✓

The trivial representation

$$G \xrightarrow{\pi} GL(k)$$
$$g \longmapsto \mathrm{id}_k$$

is irreducible.

<u>Ex</u> 1) Every 1-dimensional repr. is irreducible.

2) The identity repr.

$$G = GL(V) \xrightarrow{\pi = \mathrm{id}} GL(V)$$

is irreducible for all $V \neq \{0\}$.

3) The 2-dim'l repr.

$$G = (\mathbb{R}, +) \xrightarrow{\pi} GL_2(\mathbb{R})$$

$$t \longmapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is irreducible.

4) Let $V = \{\text{pol. fct. } f: \mathbb{R} \to \mathbb{R}\}$. The representation

$$G = (\mathbb{R}, +) \xrightarrow{L} GL(V)$$

$$(L(t)(f))(x) = f(x-t)$$

is not irreducible. Indeed, for all $n \geq 0$, the subspace

$$U = \{\text{pol. fct. } f: \mathbb{R} \to \mathbb{R} \text{ of deg.} \leq n\}$$

is L-invariant, and $\{0\} \subsetneqq U \subsetneqq V$.

5) Let $G = S_n$ be the symmetric group on n letters, and let

$$G \xrightarrow{\pi} GL_n(k)$$

$$\sigma \longmapsto P(\sigma)$$

be the standard representation on $V = k^n$. The subspaces

$$V_0 = \{x \in V \mid \sum x_i = 0\} \subset V$$

$$V_1 = k \cdot (1, \ldots, 1) \subset V$$

are π-invariant and if n is not divisible by char(k), then

$$V_1 \not\subset V_0$$

So in this case

$$V = V_0 \oplus V_1 .$$

We claim that both the subrepr.

$$G \xrightarrow{\pi_i} GL(V_i)$$

$$\sigma \longmapsto \pi(\sigma)|V_i$$

are irreducible. This is clear for V_1, since $\dim_k(V_1) = 1$. To prove that V_0 is irreducible, we let $\{0\} \neq U \subset V_0$ be π_0-invariant and show that $U = V_0$. So let

$$0 \neq x = \sum e_i x_i \in U .$$

Since $x \notin V_1$, not all x_i are equal, say, $x_1 \neq x_2$. But then

$$\pi_0((12))(x) - x$$

$$= (e_1 - e_2)(x_2 - x_1) \in U,$$

so $e_1 - e_2 \in U$. Since U is π_0-invariant, it follows that for all $1 \leq i < j \leq n$, $e_i - e_j \in U$. But

$$\mathrm{Span}(e_i - e_j \mid 1 \leq i < j \leq n) = V_0,$$

so $U = V_0$. Hence, π_0 is irred.

Def A repr. $\pi : G \to GL(V)$ is completely reducible if for every π-invariant subspace $U \subset V$, there exists a π-invariant subspace $W \subset V$ s.t.

$$V = U \oplus W.$$

We say that W is a π-inv. complement of U. In this case the composite map

$$W \hookrightarrow V \longrightarrow V/U$$

$$w \longmapsto w, \quad v \longmapsto v+U$$

is an isomorphism of repr.

$$(W, \pi_W) \xrightarrow{\sim} (V/U, \pi_{V/U}).$$

Note that if a π-inv. compl. $W \subset V$ of $U \subset V$ exists, then it is (typically) not unique.

If $\dim_k(V) < \infty$, then a π-inv. subspace $U \subset V$ admits a π-inv. complement if and only if we can find bases

$(e_1, \dots, e_m)$ of U

$(e_1, \dots, e_m, e_{m+1}, \dots, e_{m+n})$ of V

s.t. for all $g \in G$, the matrix that repr. $\pi(g)$ w.r.t. to this basis has the form

$$\begin{array}{c} \left(\begin{array}{c|c} A(g) & 0 \\ \hline 0 & B(g) \end{array}\right) \begin{array}{l} \}\, m \\ \}\, n \end{array} \\ \begin{array}{cc} \underbrace{\qquad\quad}_{m} & \underbrace{\qquad\quad}_{n} \end{array} \end{array}$$

Ex Consider two representations

$$G = (\mathbb{R}, +) \xrightarrow{\pi} GL(\mathbb{R}^2)$$

$$t \longmapsto e^{tA}$$

1) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A^2 = 0$, so

$$e^{tA} = E + tA = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

The subspace $U = \operatorname{Span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is π-invariant, but has no π-inv. complement. So π is not completely reducible.

2) $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A^2 = A$, so

$$e^{tA} = E + (e^t - 1)A = \begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}$$

The subspace $U = \text{Span}\begin{pmatrix}1\\0\end{pmatrix}$ is π-inv. with π-inv. complement

$$W = \text{Span}\begin{pmatrix}1\\-1\end{pmatrix}.$$

Moreover, $U, W \subset V = \mathbb{R}^2$ are the only 1-dim'l π-inv. subspaces, so π is completely reducible. (Compare Theorem 3 below.)

Three theorems:

<u>Thm 1</u> Let $\pi : G \to GL(V)$ be a repr., and let $U \subset V$ be a π-invariant subspace. If π is completely reducible, then so is π_U.

Pf Let $U_1 \subset U$ be a π_U-invariant subspace. We must show that U_1 admits a π_U-invariant compl. $W_1 \subset U$. Now, $U_1 \subset V$ is π-inv., so, by assumption, there exists a π-inv. subspace $W \subset V$ s.t.

$$V = U_1 \oplus W.$$

But then

$$U = (U_1 \oplus W) \cap U = U_1 \oplus (W \cap U),$$

so $W_1 = W \cap U \subset U$ is a π_U-inv. compl. //

<u>Thm 2</u> Let $\pi : G \to GL(V)$ be a completely reducible representation with $\dim_k(V) < \infty$. There exist π-invariant subspaces

$$V_1, \dots, V_m \subset V$$

such that

(1) $V = V_1 \oplus \cdots \oplus V_m$

(2) For all $1 \leq i \leq m$, the subrepr.

$$G \xrightarrow{\pi_{V_i}} GL(V_i)$$

is irreducible.

<u>Pf</u> We argue by induction on

$$n = \dim_k(V).$$

If $n=0$, then the statement is trivial, so we assume that the theorem has been proved for $n < r$ and prove it for $n = r$. Since $r < \infty$, there exists a π-invariant subspace $V_1 \subset V$ s.t. $\pi_{V_1} : G \to GL(V_1)$ is irreducible. By assumption, there exists a π-inv. subspace $W \subset V$ s.t. $V = V_1 \oplus W$. But

$$\dim_k(W) < \dim_k(V) = r,$$

so by the inductive hypothesis,

$$W = V_2 \oplus \cdots \oplus V_m$$

with $V_i \subset W$ π_W-inv. and $\pi|V_i : G \to GL(V_i)$ irred. Hence,

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$$

with $V_i \subset V$ π-inv. and $\pi|V_i$ irreducible. //

<u>Thm 3</u> Let $\pi : G \to GL(V)$ be a repr. and suppose that there exists π-inv. subspaces $V_1, \ldots, V_m \subset V$ s.t.

$$V = V_1 + \cdots + V_m$$

and s.t. each $\pi|V_i : G \to GL(V_i)$ is irreducible. Then for every π-inv. subspace $U \subset V$, there exists $\{i_1, \ldots, i_p\} \subset \{1, \ldots, m\}$ such that

$$V = U \oplus V_{i_1} \oplus \cdots \oplus V_{i_p}.$$

In particular, π is completely reducible.

Pf Among the sets of indices

$$\{i_1, \ldots, i_p\} \subset \{1, 2, \ldots, m\}$$

s.t. the sum

$$U + V_{i_1} + \cdots + V_{i_p} \subset V$$

is direct, we choose a maximal one. (It could be $\emptyset$.) We claim that

$$U \oplus V_{i_1} \oplus \cdots \oplus V_{i_p} = V.$$

Since $V = V_1 + \cdots + V_m$, it will suffice to show that

$$V_i \subset U \oplus V_{i_1} \oplus \cdots \oplus V_{i_p}$$

for all $1 \le i \le m$. We can assume that $i \notin \{i_1, \ldots, i_p\}$, so by the maximality of $\{i_1, \ldots, i_p\}$, the sum

$$U + V_{i_1} + \cdots + V_{i_p} + V_i$$

is not direct. Since $U \oplus V_{i_1} \oplus \cdots \oplus V_{i_p}$ is direct, this implies that

$$(U \oplus V_{i_1} \oplus \cdots \oplus V_{i_p}) \cap V_i \neq \{0\}.$$

But V_i, by assumption, is irred.,

so we conclude that

$$(U \oplus V_{i_1} \oplus \cdots \oplus V_{i_p}) \cap V_i = V_i,$$

as desired. //

Rmk Thm 2 + 3 show that, for finite dimensional repr.,

"completely reducible"

= "semisimple."

Cor Let $\pi : G \to GL(V)$ be a representation, and let $U \subset V$ be a π-invariant subspace. If $\dim_k(V) < \infty$ and if π is completely reducible, then $\pi_{V/U}$ is completely reducible.

Pf By Thm. 2 the hypothesis of Thm. 3 holds. So

$$V/U \simeq V_{i_1} \oplus \cdots \oplus V_{i_p}$$

with $\pi|_{V_i} : G \to GL(V_i)$ irred. So we can apply Thm. 3 again to conclude that $\pi_{V/U}$ is completely reducible. //

<u>Ex</u> Three examples, all with

$$G = GL(V), \quad \dim_k(V) = n < \infty.$$

1) Representation by left mult. on vector space

$$L(V) = \mathrm{End}_k(V)$$

of all linear maps $f : V \to V$:

$$G \xrightarrow{\lambda} GL(L(V))$$

$$\lambda(g)(f) = g \circ f$$

<u>Lemma</u> λ is completely red.

<u>Pf</u> Choose basis $(v_1, \dots, v_n)$ of V and define

$$L_j = \{ f \in L(V) \mid f(v_i) = 0 \text{ for } i \neq j \}.$$

The subspace $L_j < L(V)$ is λ-inv. Indeed, if $g \in G$ and $f \in L_j$, then

$$\lambda(g)(f)(v_i) = g(f(v_i))$$

which is zero if $f(v_i)$ is zero, because g is linear. Also, the map $h_j : L_j \to V$ given by

$$h_j(f) = f(v_j)$$

is a k-linear isomorphism. It is also intertwining:

$$h_j(\lambda(g)(f)) = h_j(g \circ f)$$
$$= (g \circ f)(v_j) = g(f(v_j))$$
$$= \mathrm{id}(g)(f(v_j)).$$

So $(L_j, \lambda_{L_j}) \simeq (V, \mathrm{id})$ is irred. Finally,

$$L(V) = L_1 \oplus \cdots \oplus L_n,$$

since every $f \in L(V)$ can be written uniquely as

$$f = f_1 + \cdots + f_n$$

with $f_j \in L_j$ defined by

$$f_j(v_i) = \begin{cases} f(v_j) & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

By Thm. 3, we conclude that λ is completely reducible.

2) The adjoint representation

$$G = GL(V) \xrightarrow{Ad} GL(L(V))$$

$$Ad(g)(f) = g \circ f \circ g^{-1}.$$

<u>Prop</u> Suppose that char(k) does not divide $n = \dim_k(V)$. Then

$$GL(V) \xrightarrow{Ad} GL(L(V))$$

is completely reducible.

<u>Pf</u> (incomplete) The subspaces

$$t = \mathrm{Span}(\mathrm{id}) \subset L(V)$$

$$\mathfrak{sl}_n = \{ f \in L(V) \mid tr(f) = 0 \} \subset L(V)$$

are Ad-invariant. In the case of $\mathfrak{sl}_n$, we use that

$$tr(gfg^{-1}) = tr(f).$$

Now $tr(\mathrm{id}) = n \neq 0$, since char(k) does not divide n, so

$$L(V) = t \oplus \mathfrak{sl}_n.$$

It is clear that t is irred., but not easy to see that also $\mathfrak{sl}_n$ is so. //

3) Bilinear forms.

$$B(V) = \{ f : V \times V \to k \mid f \text{ bilinear} \}$$
$$\cong \mathrm{Hom}_k(V \otimes_k V, k).$$

It is a k-vector space of dimension n^2. Define

$$GL(V) \xrightarrow{\pi} GL(B(V))$$

$$\pi(g)(f)(x,y) = f(g^{-1}(x), g^{-1}(y))$$

Prop If $\mathrm{char}(k) \neq 2$, then π is completely reducible.

Pf (incomplete) Let

$$B^{\pm}(V) \subset B(V)$$

be the subspaces of symmetric and skew-symmetric bilinear forms. Recall that

$$f \in B^+(V) \iff$$

$$f(x,y) = f(y,x), \quad \forall x,y \in V$$

$$f \in B^-(V) \iff$$

$$f(x,y) = -f(y,x), \quad \forall x,y \in V.$$

Since $\mathrm{char}(k) \neq 2$, we have

$$B(V) = B^+(V) \oplus B^-(V).$$

Clearly, $B^\pm(V) \subset B(V)$ are π-invariant. It is more difficult to see that they are irreducible.