Today: Chapter 2:

Recall that $\pi : G \to GL(V)$ is completely reducible if for every π-inv. subspace $U \subset V$, there exists a π-inv. subspace $W \subset V$ s.t.

$$V = U \oplus W.$$

Assume $k = \mathbb{R}$ or $\mathbb{C}$ and $\dim_k(V) < \infty$.

<u>Def</u> A real (resp. complex) repr.

$$G \xrightarrow{\pi} GL(V)$$

is orthogonal (resp. unitary) if there exists an inner product (resp. a hermitian inner product)

$$V \times V \xrightarrow{\langle -,- \rangle} k$$

s.t. for all $g \in G$ and $x, y \in V$,

$$\langle \pi(g)(x), \pi(g)(y) \rangle = \langle x, y \rangle.$$

equivalently: for all $g \in G$,

$$\pi(g)^* = \pi(g^{-1}).$$

If $(V, \langle -,- \rangle)$ is a real inner product space, then we define the orthogonal group

$$O(V) = O(V, \langle -,- \rangle) \subset GL(V)$$

to be the subgroup of all $\mathbb{R}$-linear maps $f : V \to V$ s.t.

$$\langle f(x), f(y) \rangle = \langle x, y \rangle$$

for all $x, y \in V$.

Similarly, if $(V, \langle -,- \rangle)$ is a hermitian inner product space, then we define the unitary group

$$U(V) = U(V, \langle -,- \rangle) \subset GL(V)$$

to be the subgroup of all $\mathbb{C}$-linear maps $f : V \to V$ s.t.

$$\langle f(x), f(y) \rangle = \langle x, y \rangle$$

for all $x, y \in V$.

So orthogonal representations correspond to group homom.

$$G \xrightarrow{\pi} O(V)$$

and unitary representations correspond to group homom.

$$G \xrightarrow{\pi} U(V).$$

<u>Prop</u> Every orthogonal (resp. unitary) representation

$$G \xrightarrow{\pi} GL(V)$$

is completely reducible.

<u>Pf</u> If $U \subset V$ is π-inv., then so is

$$U^\perp \subset V$$
$$\parallel$$
$$\{x \in U \mid \langle x,y\rangle = 0 \quad \forall y \in U\}$$

Indeed, for all $g \in G$,

$$\pi(g)(U) \subset U \implies$$
$$\pi(g)^*(U^\perp) \subset U^\perp$$
$$\parallel$$
$$\pi(g^{-1})(U^\perp).$$

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We will show that every f.d. real (resp. complex) representation of a compact group G is orthogonal (resp. unitary).

First: G finite

<u>Thm 1</u> Every f. d. real (resp. complex representation

$$G \xrightarrow{\pi} GL(V)$$

of a finite group G is orthogonal (resp. unitary).

<u>Pf</u> Choose any inner product (resp. hermitian inner product) $\langle -,- \rangle_0$ on V and define

$$\langle x, y \rangle = \sum_{h \in G} \langle \pi(h)(x), \pi(h)(y) \rangle_0 .$$

Claim:

1) $\langle -,- \rangle$ is an inner product (resp. herm. inner prod.) on V. (Easy)

2) $\langle -,- \rangle$ is π-invariant:

$$\langle \pi(g)(x), \pi(g)(y) \rangle$$
$$:= \sum_{h \in G} \langle \pi(h)(\pi(g)(x)), \pi(h)(\pi(g)(y)) \rangle_0$$
$$= \sum_{h \in G} \langle \pi(h \circ g)(x), \pi(h \circ g)(y) \rangle_0$$
$$= \sum_{k \in G} \langle \pi(k)(x), \pi(k)(y) \rangle_0 = \langle x, y \rangle .$$

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Def A topological group is a group G with a topology such that

$$G \times G \longrightarrow G, \quad G \longrightarrow G$$
$$(g,h) \longmapsto gh \qquad g \longmapsto g^{-1}$$

are continuous. A compact group is a topological group that is compact (and Hausdorff). //

Ex 1) Finite groups (with discrete topology are compact groups.

2) $O(V)$, $U(V)$ are compact grps.

3) For V a real or complex vector space, $GL(V)$ is a topological group. It is not compact, unless $V = \{0\}$. //

We will always assume that (real or complex) representations of topological groups are continuous, i.e. the map

$$G \xrightarrow{\pi} GL(V)$$

is continuous.

Ex The inclusion $O(V) \xhookrightarrow{\pi} GL(V)$ is a continuous repr. of $G = O(V)$. //

<u>Thm 2</u> Let G be a compact group, and let

$$G \xrightarrow{\pi} GL(V)$$

be a (continuous, finite dim.) real (resp. complex) repr. of G. Then π is orthogonal (resp. unitary).

First proof: Left invariant integration.

<u>Thm</u> Let G be a compact group. There exists a map

$$C^0(G, \mathbb{C}) \longrightarrow \mathbb{C}$$

$$f \longmapsto \int_G f(x)\,dx$$

such that

1) It is linear

2) It is positive, i.e. if $f(x) \geq 0$ for all $x \in G$ and $f \neq 0$, then

$$\int_G f(x)\,dx > 0$$

3) It is left invariant, i.e. for all $g \in G$,

$$\int_G f(gx)\,dx = \int_G f(x)\,dx.$$

Moreover, there is a unique map $f \mapsto \int_G f(x)\,dx$ that satisfies 1) – 3) and

4) $\int_G 1\,dx = 1$. //

Rmk There exists a unique measure $d\mu$ on G (called the Haar measure) s.t.

$$\int_G f(x)\,dx = \int_G f\,d\mu$$

for all $f \in C^0(G, \mathbb{C})$. //

Ex If G is finite, then

$$\int_G f(x)\,dx = \frac{1}{|G|}\sum_{x\in G} f(x).$$

In this case, Haar measure is called counting measure. //

Pf (of Thm. 2) Choose inner prod. $\langle -,-\rangle_0$ on V, define

$$\langle x,y\rangle = \int_G \underbrace{\langle \pi(h^{-1})(x), \pi(h^{-1})(y)\rangle_0}_{\substack{\shortparallel \\ f(h)}}\,dh$$

Show invariance:

$$\begin{aligned}
&\langle \pi(g)(x), \pi(g)(y) \rangle \\
&= \int_G f(g^{-1}h)\,dh \\
&= \int_G f(h)\,dh \\
&= \langle x, y \rangle .
\end{aligned}$$

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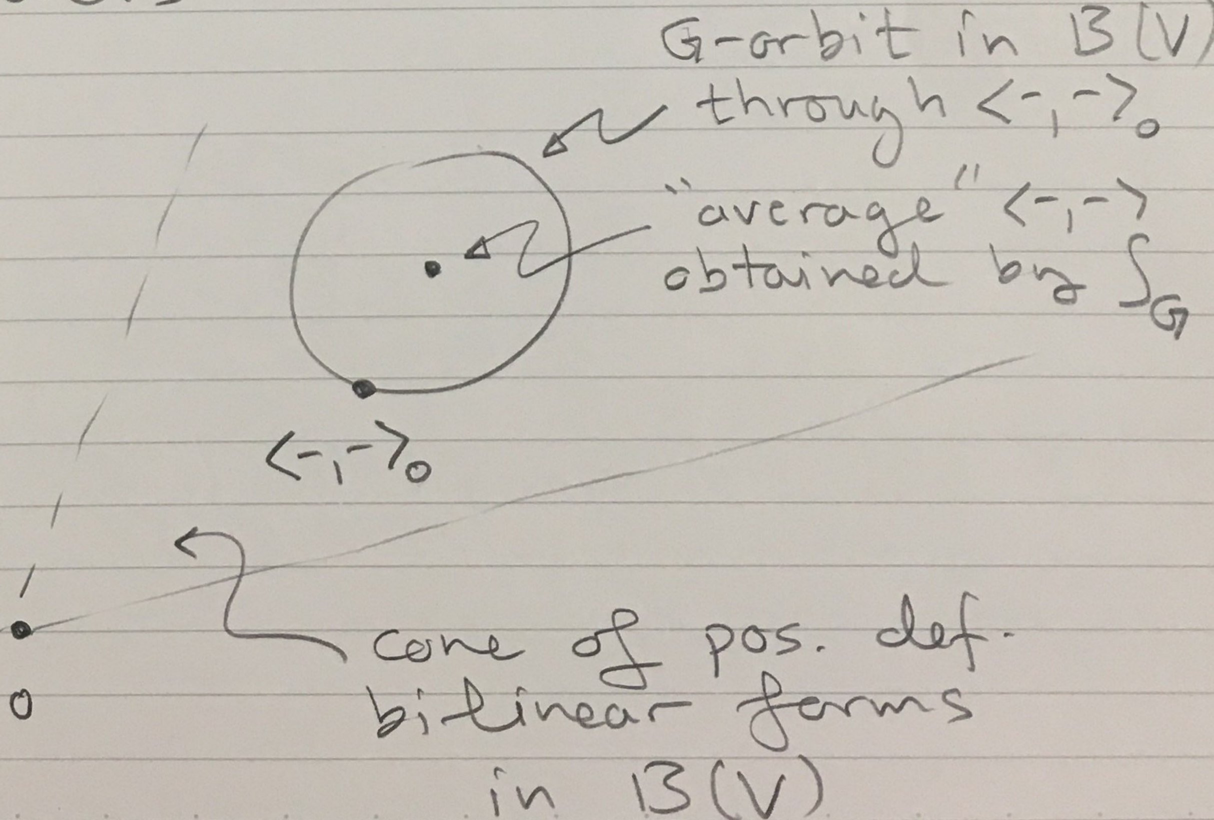
Explain idea of proof: The repr. of G on V induces a repr.

$$G \xrightarrow{\bar{\pi}} GL(B(V))$$

on the vector space $B(V)$ of bilinear forms, which preserves the cone of inner products:



Second proof: Center of mass

Let W be a (f.d.) $\mathbb{R}$-vector space, and let μ be the Lebesgue measure on W.

Def Let $K \subset W$ be compact with pos. volume $\mu(K) > 0$. Call

$$c(K) = \frac{1}{\mu(K)} \int_K x \, dx \in W$$

the center of mass of K. //

Lemma 1 If $f \in GL(W)$, then

$$c(f(K)) = f(c(K)).$$

Pf Let $y = f(x)$, $dy = \det(f)\,dx$.

$$c(f(K)) = \frac{1}{\mu(f(K))} \int_{f(K)} y \, dy$$

$$= \frac{1}{\det(f)\mu(K)} \int_K x \det(f) \, dx$$

$$= \frac{1}{\mu(K)} \int_K x \, dx = c(K). \quad //$$

Recall convex hull of $K \subset W$:

$$\mathrm{conv}(K) = \left\{ \sum_{i=0}^{m} c_i x_i \in V \;\middle|\; \begin{array}{l} m \geq 0, \ x_i \in K \\ c_i \geq 0, \ \sum c_i = 1 \end{array} \right\}$$

Lemma 2 conv(K) is compact.

Pf Let $n = \dim_{\mathbb{R}}(W)$. A theorem of Carathéodory states that for every $x \in \mathrm{conv}(K)$, there exists $x_0, \ldots, x_n \in K$ and $c_0, \ldots, c_n \geq 0$ s.t.

$$x = \sum_{i=0}^{n} c_i x_i .$$

So in fact,

$$\mathrm{conv}(K) = \left\{ \sum_{i=0}^{n} c_i x_i \,\middle|\, x_i \in K, c_i \geq 0, \sum c_i = 1 \right\}$$

Hence, have cont. surjection

$$K^{n+1} \times \Delta^n \twoheadrightarrow \mathrm{conv}(K)$$

$$(x_0, \ldots, x_n, c_0, \ldots, c_n) \longmapsto \sum_{i=0}^{n} c_i x_i$$

where

$$\Delta^n = \left\{ c \in [0,1]^{n+1} \,\middle|\, \sum_{i=0}^{n} c_i = 1 \right\}.$$

So conv(K) is the image of a compact space by a cont. map, hence (quasi-)compact. And $\mathrm{conv}(K) \subset W$ is a subspace of a Hausdorff space, hence Hausdorff. //

<u>Lemma 3</u> $c(K) \in \mathrm{conv}(K)$.

<u>Pf</u> By theory of integration,

$$c(K) = \lim_{k \to \infty} \frac{1}{\mu(K)} \sum_{i=1}^{k} x_i \, \mu(K_i)$$

with

$$K = \coprod_{i=1}^{k} K_i$$

a decomposition of K into k disjoint Lebesgue measurable subsets. By definition,

$$\frac{1}{\mu(K)} \sum_{i=1}^{k} x_i \, \mu(K_i) \in \mathrm{conv}(K),$$

and $\mathrm{conv}(K) \subset W$ is a compact subspace of a Hausdorff space, hence closed. So

$$c(K) \in \mathrm{conv}(K)$$

as desired. //

Pf (of Thm. 2) Let G be a compact group, and let

$$G \xrightarrow{\pi} GL(V)$$

be a real representation. We

must show that π $\mathcal{B}$ orthogonal. Recall from last week the real vector space $W = B^+(V)$ of symmetric bilinear forms on V. The representation of G on V induces a repr.

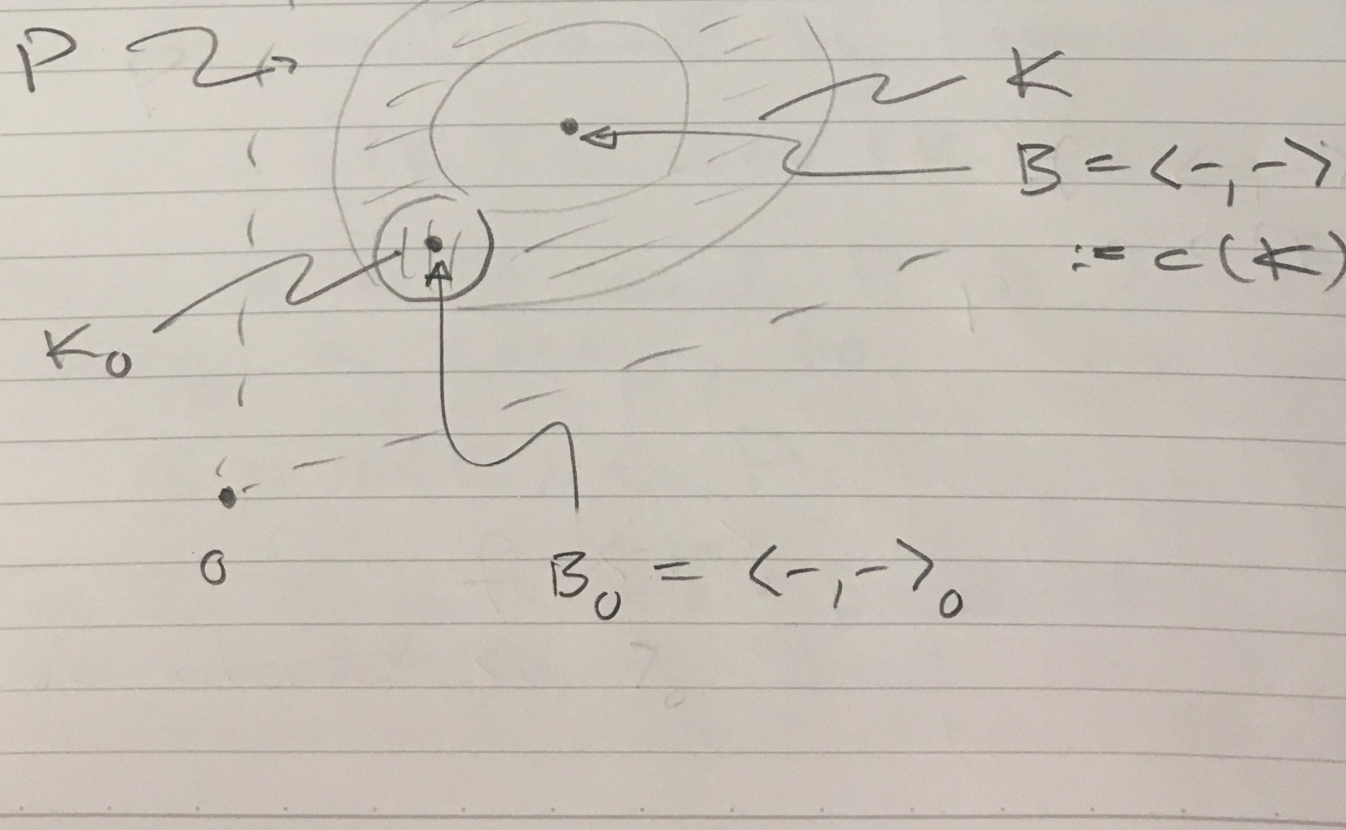
$$G \xrightarrow{\rho} GL(B^+(V))$$

$$\rho(g)(B)(x,y)$$
$$= B(\pi(g^{-1})(x), \pi(g^{-1})(y)).$$

The subset of inner products

$$P \subset B^+(V) = W$$

$\mathcal{B}$ a ρ-invariant open cone.



We choose $B_0 \in P$ and a compact neighborhood

$$B_0 \in K_0 \subset P$$

and define

$$K = \{\rho(g)(B) \mid g \in G, B \in K_0\} \subset P.$$

It is a compact subset, since it is the image of

$$G \times K_0 \longrightarrow P$$

$$(g, B) \longmapsto \rho(g)(B)$$

and $\mu(K) \neq 0$, since $\mu(K_0) \neq 0$. (K_0 is a neighborhood of B_0.) So the center of mass $c(K)$ is defined, and by Lemma 3,

$$B := c(K) \in \operatorname{conv}(K).$$

But $K \subset P$ and P is convex (being a cone), so

$$B = c(K) \in P.$$

So B is an inner product, and Lemma 1 shows that it is ρ-invariant. This shows that

$$G \xrightarrow{\pi} GL(V)$$

is orthogonal as desired.

If π is a complex representation, then we argue in the same way, but with $B^+(V)$ replaced by the real vector space $H^+(V)$ of hermitian bilinear forms and

$$P \subset H^+(V) =: W$$

the open cone of hermitian inner products. //