Today: Chapter 3 (3.1 ~ 3.4)

Recall dual vector space: If

$$V = (V, +, \cdot)$$

is a right k-vector space, then its dual is the left k-vector space

$$V^* = (\mathrm{Hom}_k(V, k), +, \cdot),$$

where for $f, g \in \mathrm{Hom}_k(V, k)$, and $a \in k$,

$$(f+g)(x) = f(x) + g(x)$$

$$(a \cdot f)(x) = a \cdot f(x)$$

Check that $a \cdot f \in \mathrm{Hom}_k(V, k)$:

$$(a \cdot f)(x \cdot b) := a \cdot f(x \cdot b)$$

$$= a \cdot (f(x) \cdot b) = (a \cdot f(x)) \cdot b$$

$$=: (a \cdot f)(x) \cdot b \qquad \text{OK}$$

Similarly, if W is a left k-vector space, then its dual is the right k-vector space

$$W^* = (\mathrm{Hom}_k(W, k), +, \cdot)$$

where

$$(f+g)(x) := f(x) + g(y)$$

$$(f \cdot a)(x) := f(x) \cdot a \, .$$

So if V is a right (resp. left) k-vector space then so is its double-dual

$$V^{**} = (V^*)^* \, .$$

Moreover, there is a canonical k-linear map

$$V \xrightarrow{\delta} V^{**}$$

defined by

$$\delta(x)(f) = f(x) \, .$$

If $\dim_k(V) < \infty$, then δ is an isomorphism.

A basis $(e_1, \dots, e_n)$ of V gives rise to a basis $(e_1^*, \dots, e_n^*)$ of V^* defined by

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \, . \end{cases}$$

We say that $(e_1^*, \dots, e_n^*)$ is the dual basis of $(e_1, \dots, e_n)$. So

$$\dim_k(V^*) = \dim_k(V),$$

provided that $\dim_k(V) < \infty$.

<u>Warning</u>: There is no canonical way to compare V and V^*. If V is a right k-vector space, then V^* is a left k-vector space, or equivalently a right k^{op}-vector space, so to convert V^* to a right k-vector space, we must first choose a ring homomorphism.

$$k \xrightarrow{\sigma} k^{op}.$$

For $k = \mathbb{C}$, we can choose σ to be the identity, but we can also choose σ to be complex conjugation. Next, we must choose a map

$$V \xrightarrow{b} \sigma_*(V^*)$$

of right k-vector spaces. The map b determines and is determined by the map

$$V \times V \xrightarrow{\langle -,- \rangle} k$$

$$\langle x, y \rangle := b(x)(y)$$

The map b is a well-defined k-linear map if and only if the map $\langle -, - \rangle$ satisfies:

$$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

$$\langle x, y \cdot a \rangle = \langle x, y \rangle \cdot a$$

$$\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

$$\langle x \cdot a, y \rangle = \sigma(a) \cdot \langle x, y \rangle.$$ //

Note that if $(e_1, \ldots, e_n)$ is a basis of V, then

$$V \xrightarrow{\delta} V^{**}$$

maps $\delta(e_i) = e_i^{**}$. Indeed,

$$\delta(e_i)(e_j^*) = e_j^*(e_i) = \delta_{ij}.$$

So δ is an isom., if $\dim_k(V) < \infty$, but not otherwise!

(Book writes V' instead of V^*.)

A k-linear map

$$W \xrightarrow{\varphi} V$$

induces a k-linear map

$$V^* \xrightarrow{\varphi^*} W^*$$

in the opposite direction,

$$\varphi^*(f)(y) = f(\varphi(y)).$$

Note that

$$(\varphi \circ \psi)^* = \psi^* \circ \varphi^*.$$

Suppose that $(v_1, \dots, v_m)$ and $(w_1, \dots, w_n)$ are bases of V and W, respectively, and let

$$A = (a_{ij}) \in M_{m,n}(k)$$

be the matrix that represents $\varphi : W \to V$ w.r.t. these bases. Then the matrix that repr. $\varphi^* : V^* \to W^*$ w.r.t. the dual bases $(v_1^*, \dots, v_m^*)$ and $(w_1^*, \dots, w_n^*)$ is the the transpose

$$A^t = (a_{ji}) \in M_{n,m}(k).$$

<u>Def</u> Given a k-linear repr.

$$G \xrightarrow{\pi} GL(V),$$

its dual representation

$$G \xrightarrow{\pi^*} GL(V^*)$$

is defined by

$$\pi^*(g) := \pi(g^{-1})^* . \qquad //$$

Check that π^* is a group homomorphism:

$$\pi^*(g.h) = \pi((gh)^{-1})^*$$
$$= \pi(h^{-1}g^{-1})^*$$
$$= (\pi(h^{-1}) \circ \pi(g^{-1}))^*$$
$$= \pi(g^{-1})^* \circ \pi(h^{-1})^*$$
$$= \pi^*(g) \circ \pi^*(h) . \qquad \text{OK}$$

If $A = A(g)$ represents

$$V \xrightarrow{\pi(g)} V$$

with respect to a basis $(e_1, \dots, e_n)$ of V, then the matrix that represents

$$V^* \xrightarrow{\pi^*(g)} V^*$$

with respect to the dual basis $(t_1^*, \ldots, t_n^*)$ is

$$A(g^{-1})^t = (A(g)^{-1})^t = (A^{-1})^t.$$

In particular, if $k = \mathbb{R}$ and π is an orthogonal representation, then

$$\pi^* \simeq \pi.$$

Indeed, there exists an inner product $\langle -, - \rangle$ on V s.t.

$$\langle \pi(g)(x), \pi(g)(y) \rangle = \langle x, y \rangle$$

for all $g \in G$ and $x, y \in V$, and the matrix $Q = Q(g)$ that represents $\pi(g)$ w.r.t. to a basis $(t_1, \ldots, t_n)$ that is orthonormal w.r.t. $\langle -, - \rangle$ is orthogonal, i.e.

$$Q = (Q^{-1})^t.$$

So $\pi(g)$ and $\pi^*(g)$ give equivalent matrix representations.

Better argument: The map

$$V \xrightarrow{b} V^*$$

$$b(x)(y) = \langle x, y \rangle$$

is a k-linear isomorphism that is intertwining between π and π^*.

<u>Thm 1</u> Let $\pi : G \to GL(V)$ be a f. d. k-lin. repr. Then:

(1) π is irreducible if and only if π^* is irreducible.

(2) π is completely reducible if and only if π^* is so.

<u>Pf</u> The sequence

$$0 \to U \to V \to W \to 0$$

is exact (resp. split exact) if and only if the sequence

$$0 \to W^* \to V^* \to U^* \to 0$$

is exact (resp. split exact). //

<u>Rmk</u> In particle physics,

$$\begin{array}{ccc} \text{elem. part.} & = & \text{irrep } \pi \\ \updownarrow & & \updownarrow \\ \text{anti-part.} & = & \text{irrep. } \pi^* \end{array}$$ //

Let V_1, V_2 be two k-vector spaces, and let $V_1 \oplus V_2$ be their direct sum. The map

$$GL(V_1) \times GL(V_2) \xrightarrow{\oplus} GL(V_1 \oplus V_2)$$

$$(f_1, f_2) \longmapsto f_1 \oplus f_2$$

is a group homomorphism. So given representations

$$G_1 \xrightarrow{\pi_1} GL(V_1)$$

$$G_2 \xrightarrow{\pi_2} GL(V_2),$$

we get the new repr.

$$\begin{tikzcd} G_1 \times G_2 \arrow[r, "\pi_1 \times \pi_2"] \arrow[dr, "\pi_1 \boxplus \pi_2"'] & GL(V_1) \times GL(V_2) \arrow[d, "\oplus"] \\ & GL(V_1 \oplus V_2). \end{tikzcd}$$

Call this the exterior sum.

The diagonal map

$$\begin{aligned} G &\xrightarrow{\Delta} G \times G \\ g &\longmapsto (g, g) \end{aligned}$$

is also a group homom., so if

$$G \xrightarrow{\pi_1} GL(V_1)$$

$$G \xrightarrow{\pi_2} GL(V_2)$$

are two repr. of the same group, then we get the new representation

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[dr, "\pi_1 \oplus \pi_2"'] & G \times G \arrow[d, "\pi_1 \boxplus \pi_2"] \\ & GL(V_1 \oplus V_2) \end{tikzcd}$$

(Book writes $\pi_1 + \pi_2$ for $\pi_1 \oplus \pi_2$.)

In other words,

$$\begin{aligned} &(\pi_1 \oplus \pi_2)(g)(x_1 \oplus x_2) \\ &\quad = \pi_1(g)(x_1) \oplus \pi_2(g)(x_2). \end{aligned}$$

Ex Let $\pi : G \to GL(V)$ be a repr., and let $U_1, U_2 \subset V$ be π-inv. subspaces. Then the canonical inclusion maps

$$U_1 \xhookrightarrow{i_1} V \xhookleftarrow{i_2} U_2$$

are intertwining between π_{U_i} and π, and hence, the induced map

$$U_1 \oplus U_2 \xrightarrow{i_1 + i_2} V$$

$$x_1 \oplus x_2 \longmapsto x_1 + x_2$$

is intertwining between $\pi_{U_1} \oplus \pi_{U_2}$ and π. Moreover,

$$i_1 + i_2 \text{ surj.} \iff U_1 + U_2 = V$$

$$i_1 + i_2 \text{ inj.} \iff U_1 \cap U_2 = \{0\}$$

So if $i_1 + i_2$ is bijective, then

$$\pi \simeq \pi_{U_1} \oplus \pi_{U_2} .$$

//

Can restate Thm. 2 and Thm. 3 on page 18 of the book as follows:

<u>Thm 2</u> A f.d. repr. π is compl. reducible if and only if

$$\pi \simeq \pi_1 \oplus \cdots \oplus \pi_m$$

with $\pi_1, \ldots, \pi_m$ irreducible. ✓

<u>Thm. 3</u> Let $\pi : G \to GL(V)$ be a repr. and suppose

$$\pi \simeq \pi_1 \oplus \cdots \oplus \pi_m$$

with $\pi_1, \ldots, \pi_m$ irreducible. If $U \subset V$ is π-invariant, then

$$\pi_U \simeq \text{sum of some } \pi_i$$

$$\pi_{V/U} \simeq \text{sum of remaining } \pi_i\text{.} \checkmark$$

<u>Lemma</u> If $\pi : G \to GL(V)$ is a repr., and if $V_1, \ldots, V_m \subset V$ are π-invariant subspaces s.t. $\pi_{V_1}, \ldots, \pi_{V_m}$ are irreducible and

$$\pi_{V_i} \simeq \pi_{V_j} \iff i = j.$$

Then the sum

$$V_1 + \cdots + V_m \subset V$$

is direct.

__Pf__ By induction on $m \geq 0$. If $m=0$, then the statement is true. So we assume lemma proved for $m = r-1$ and prove it for $m=r$. By induction,

$$V_1 + \cdots + V_{r-1} \subset V$$

is direct. Now, if

$$(V_1 + \cdots + V_{r-1}) \cap V_r \neq \{0\},$$

then

$$V_r \subset V_1 + \cdots + V_{r-1}.$$

But $\pi_{V_r} \not\cong \pi_{V_i}$ for $1 \leq i < r$, so this is impossible, by Thm. 3. //

__Thm. 4__ Let $\pi_1, \ldots, \pi_m$ and $\rho_1, \ldots, \rho_n$ be irred. repr. of G. If

$$\pi_1 \oplus \cdots \oplus \pi_m \cong \rho_1 \oplus \cdots \oplus \rho_n,$$

then $m=n$ and, up to reordering,

$$\pi_i \cong \rho_i$$

for all i.

<u>Pf</u> By induction on $m \geq 0$, the case $m = 0$ being trivial. So we assume that the statement has been proved for $m < r$ and prove it for $m = r$. We choose any $0 < s < r$ and consider

$$\pi_1 \oplus \cdots \oplus \pi_s, \quad \pi_{s+1} \oplus \cdots \oplus \pi_r$$

$$\subset \pi_1 \oplus \cdots \oplus \pi_r \simeq \rho_1 \oplus \cdots \oplus \rho_n.$$

By Thm. 3, we have

$$\pi_1 \oplus \cdots \oplus \pi_s = \text{sum of some } \rho_i.$$

$$\pi_{s+1} \oplus \cdots \oplus \pi_r = \text{sum of the remaining } \rho_i$$

Since both $s < r$ and $r - s < r$, the inductive hypothesis shows that, up to reordering,

$$\pi_i \simeq \rho_i$$

for $1 \leq i \leq s$ and for $s+1 \leq i \leq r$. So we are done by induction. //

Note that Thm. 4 is similar to the unique factorization of a natural number as a product of prime numbers, up to ordering.

Let V_1, V_2 be two k-vecter spaces, and let $V_1 \otimes V_2$ be their tensor product. Recall that there is a k-bilinear map

$$\begin{array}{ccc} V_1 \times V_2 & \xrightarrow{p} & V_1 \otimes V_2 \\ (x_1, x_2) & \longmapsto & x_1 \otimes x_2 \end{array}$$

and that for every k-bilinear map

$$V_1 \times V_2 \xrightarrow{b} W,$$

there exists a unique k-linear map

$$V_1 \otimes V_2 \xrightarrow{\hat{b}} W$$

such that $b = \hat{b} \circ p$.

If $f_i : V_i \to W_i$ are linear maps, then we see that there is a unique linear map $f_1 \otimes f_2$ such that

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "p"] \arrow[d, "f_1 \times f_2"] & V_1 \otimes V_2 \arrow[d, "f_1 \otimes f_2"] \\ W_1 \times W_2 \arrow[r, "q"] & W_1 \otimes W_2 \end{tikzcd}$$

commutes. Indeed, $q \circ (f_1 \times f_2)$ is bilinear. Now, the map

$$GL(V_1) \times GL(V_2) \xrightarrow{\otimes} GL(V_1 \otimes V_2)$$

$$(f_1, f_2) \longmapsto f_1 \otimes f_2$$

is a group homomorphism. So given representations

$$G_1 \xrightarrow{\pi_1} GL(V_1)$$

$$G_2 \xrightarrow{\pi_2} GL(V_2),$$

we get the new repr.

$$\begin{tikzcd} G_1 \times G_2 \arrow[r, "\pi_1 \times \pi_2"] \arrow[dr, "\pi_1 \boxtimes \pi_2"'] & GL(V_1) \times GL(V_2) \arrow[d, "\otimes"] \\ & GL(V_1 \otimes V_2), \end{tikzcd}$$

and if $G_1 = G_2 = G$, then we further have the repr.

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[dr, "\pi_1 \otimes \pi_2"'] & G \times G \arrow[d, "\pi_1 \boxtimes \pi_2"] \\ & GL(V_1 \otimes V_2) \end{tikzcd}$$

We call $\pi_1 \boxtimes \pi_2$ (resp. $\pi_1 \otimes \pi_2$)

the exterior tensor product (resp. the tensor product) of π_1 and π_2.

__Warning__ The book uses the following non-standard notation:

$$\pi_1 \boxtimes \pi_2 \quad = \quad \pi_1 \otimes \pi_2 \quad \text{— book}$$

$$\pi_1 \otimes \pi_2 \quad = \quad \pi_1 \pi_2$$

Examples:

1) Direct sum and tensor product satisfies a distributive law, namely, the canonical map

$$(U \otimes V_1) \oplus (U \otimes V_2) \longrightarrow U \otimes (V_1 \oplus V_2)$$

is an isomorphism. So if $\tau : G \to GL(V)$ is the trivial representation on $V = k^n$ and if $\pi : G \to GL(U)$ is any repr., then

$$\pi \oplus \cdots \oplus \pi \qquad (n \text{ times})$$

$$\simeq (\pi \otimes k) \oplus \cdots \oplus (\pi \otimes k)$$

$$\simeq \pi \otimes (k \oplus \cdots \oplus k) \simeq \pi \otimes \tau$$

2) Let U and V be right k-vector spaces. There is a natural isomorphism

$$V \otimes U^* \xrightarrow{\alpha} \mathrm{Hom}_k(U, V)$$

defined by

$$\alpha(y \otimes f)(x) = y \cdot f(x)$$

Here "natural" means that given linear maps $\varphi : U_1 \to U_2$ and $\psi : V_1 \to V_2$, the diagr.

$$\begin{tikzcd}
V_1 \otimes U_1^* \arrow[r, "\alpha"] \arrow[d, "\psi \otimes id"] & \mathrm{Hom}_k(U_1, V_1) \arrow[d, "\mathrm{Hom}(U_1, \psi)"] \\
V_2 \otimes U_1^* \arrow[r, "\alpha"] & \mathrm{Hom}_k(U_1, V_2) \\
V_2 \otimes U_2^* \arrow[r, "\alpha"] \arrow[u, "id \otimes \varphi^*"] & \mathrm{Hom}_k(U_2, V_2) \arrow[u, "\mathrm{Hom}(\varphi, V_2)"']
\end{tikzcd}$$

commutes. In particular, if

$$G \xrightarrow{\pi} GL(V)$$

is a representation, then for all $g \in G$, the diagram

$$\begin{tikzcd}
V \otimes V^* \arrow[r, "\alpha"] \arrow[d, "\pi(g) \otimes \mathrm{Id}"] & \mathrm{Hom}_k(V, V) \arrow[d, "\mathrm{Hom}_k(V, \pi(g))"] \\
V \otimes V^* \arrow[r, "\alpha"] \arrow[d, "\mathrm{id} \otimes \pi(g^{-1})^*"] & \mathrm{Hom}_k(V, V) \arrow[d, "\mathrm{Hom}(\pi(g^{-1}), V)"] \\
V \otimes V^* \arrow[r, "\alpha"] & \mathrm{Hom}_k(V, V)
\end{tikzcd}$$

commutes. The outer diagram is

$$\begin{tikzcd}
V \otimes V^* \arrow[r, "\alpha"] \arrow[d, "(\pi \otimes \pi^*)(g)"] & L(V) \arrow[d, "\mathrm{Ad}(\pi(g))"] \\
V \otimes V^* \arrow[r, "\alpha"] & L(V).
\end{tikzcd}$$

So we see that α is an isomorphism of repr.

$$\pi \otimes \pi^* \simeq \mathrm{Ad} \circ \pi,$$

where the right-hand side denotes the composite map

$$G \xrightarrow{\pi} GL(V) \xrightarrow{\mathrm{Ad}} GL(L(V)).$$

⌜Recall from lecture 1 that

$$\mathrm{Ad}(g)(f) = g \circ f \circ g^{-1}.$$ ⌟

In particular, let $G = GL(V)$ and let $\pi = \mathrm{id}$. So π is irreducible, but

$$\pi \otimes \pi^* \simeq \mathrm{Ad}$$

is not!

If π_1 and π_2 are irreducible representations, then it is an important problem to determine how $\pi_1 \otimes \pi_2$ decomposes as a sum of irred. repr.,

$$\pi_1 \otimes \pi_2 \simeq \rho_1 \oplus \cdots \oplus \rho_m$$

Physicists call this process "Scattering."