Today: Rest of Chap. 3

Extension / restriction of scalars:

Let $f: A \longrightarrow B$ be a ring homom.
If $N = (N, +, \cdot)$ is a right $B$-module, then we define

$$f_*(N) = (N, +, *)$$

to be the right $A$-module, where for $y \in N$ and $a \in A$,

$$y * a = y \cdot f(a).$$

If $h: N_1 \longrightarrow N_2$ is a $B$-linear map between right $B$-modules, then the same map

$$f_*(N_1) \xrightarrow{f_*(h) = h} f_*(N_2)$$

is an $A$-linear map between right $A$-modules.

If $M = (M, +, \cdot)$ is a right $A$-module, then we define

$$f^*(M) = (M \otimes_A B, +, \cdot)$$

to be the right $B$-module, where for $x \in M$ and $b_1, b_2 \in B$,

$$(x \otimes b_1) \cdot b_2 := x \otimes (b_1 \cdot b_2).$$

If $g : M_1 \to M_2$ is an $A$-lin. map between right $A$-modules, then we define

$$f^*(M_1) \xrightarrow{f^*(g)} f^*(M_2)$$

to be the unique $B$-linear map s.t. for $x \in M_1$, $b \in B$,

$$f^*(g)(x \otimes b) = g(x) \otimes b$$

It is well-defined, since $g$ is $A$-linear. Indeed, if $x \in M_1$, $a \in A$, and $b \in B$, then

$$x \cdot a \otimes b = x \otimes a \cdot b,$$

but we also have

$$f^*(g)(x \cdot a \otimes b) = g(x \cdot a) \otimes b$$

$$= g(x) \cdot a \otimes b = g(x) \otimes a \cdot b$$

$$=: f^*(g)(x \otimes a \cdot b).$$

We call $f^*$ extension of scalars along $f$, and we call $f_*$ restriction of scalars along $f$.

<u>Ex</u> If $\sigma : \mathbb{C} \to \mathbb{C}$ is cx. conj.

$$\sigma(a+ib) = a - ib,$$

and if $V = (V, +, \cdot)$ is a $\mathbb{C}$-vector space, then

$$\sigma_*(V) = (V, +, *)$$

is the conjugate vector space $\overline{V}$. //

We define <u>natural</u> maps

$$\begin{aligned} M &\xrightarrow{\eta} f_* f^*(M) \\ x &\longmapsto x \otimes 1 \end{aligned}$$

$$\begin{aligned} f^* f_*(N) &\xrightarrow{\varepsilon} N \\ y \otimes b &\longmapsto y \cdot b \end{aligned}$$

called the unit map and the counit map. For every right $A$-module $M$ and right $B$-module $N$, the diagrams

$$\begin{tikzcd} f^*(M) \arrow[r, "f^*\eta"] \arrow[dr, equal] & f^* f_* f^*(M) \arrow[d, "\varepsilon f_*"] \\ & f^*(M) \end{tikzcd}$$

and

$$\begin{tikzcd} f_*(N) \arrow[r, "\eta f_*"] \arrow[dr, equal] & f_* f^* f_*(N) \arrow[d] \\ & f_*(N) \end{tikzcd}$$

commute.

<u>Proposition</u> In the situation above, the maps

$$\mathrm{Hom}_B(f^*(M), N) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Hom}_A(M, f_*(N))$$

defined by

$$\alpha(g) = f_*(g) \circ \eta$$

$$\beta(h) = \varepsilon \circ f^*(h)$$

are each other inverses.

<u>Pf</u> By definition, $\alpha(g)$ is the composite map

$$M \xrightarrow{\eta} f_* f^*(M) \xrightarrow{f_*(g)} f_*(N),$$

so $(\beta \circ \alpha)(g) = \beta(\alpha(g))$ is the composition of the upper horizontal and right-hand vertical maps in the diagram

$$\begin{tikzcd}
f^*(M) \arrow[r, "f^*(\eta)"] \arrow[dr, equal] & f^*f_*f^*(M) \arrow[r, "f^*f_*(g)"] \arrow[d, "\varepsilon"] & f^*f_*(N) \arrow[d, "\varepsilon"] \\
 & f^*(M) \arrow[r, "g"] & N
\end{tikzcd}$$
.

But the square commutes, by naturality of $\varepsilon$, and we saw above that the triangle commutes. So $(\beta \circ \alpha)(g) = g$.

Similarly, $\beta(h)$ is the comp.

$$f^*(M) \xrightarrow{f^*(h)} f^*f_*(N) \xrightarrow{\varepsilon} N,$$

and $(\alpha \circ \beta)(h)$ is the composition of the left-hand vertical map and the bottom horizontal maps in the diagram

$$\begin{tikzcd}
M \arrow[r, "h"] \arrow[d, "\eta"] & f_*(N) \arrow[d, "\eta"] \arrow[dr, equal] & \\
f_*f^*(M) \arrow[r, "f_*f^*(h)"] & f_*f^*f_*(N) \arrow[r, "f_*\varepsilon"] & f_*(N)
\end{tikzcd}$$
.

The square commutes, by naturality of $\eta$, and we saw above that the triangle commutes. So $(\alpha \circ \beta)(h) = h$. //

Let $f: k \to k'$ be an extension of fields. If $V$ is a $k$-vector space, then

$$GL(V) \xrightarrow{f^*} GL(f^*(V))$$
$$g \longmapsto f^*(g)$$

is a group homomorphism. So a representation

$$G \xrightarrow{\pi} GL(V)$$

on a $k$-vector space $V$ gives rise to the representation

$$G \xrightarrow{\pi} GL(V) \xrightarrow{f^*} GL(f^*(V))$$

on the $k'$-vector space $f^*(V)$. We write $f^*\pi$ for this repr. and call it the representation obtained from $\pi$ by extension of scalars along $f$.

Similarly, if $V'$ is a $k'$-vector space, then

$$GL(V') \xrightarrow{f_*} GL(f_*(V'))$$
$$g \longmapsto f_*(g)$$

is a group homomorphism. So a representation

$$G \xrightarrow{\pi'} GL(V')$$

on a $k'$-vector space $V'$ gives rise to the representation

$$G \xrightarrow{\pi'} GL(V') \xrightarrow{f_*} GL(f_*(V'))$$

on the $k$-vector space $f_*(V)$. We write $f_*\pi'$ for this repr. and call it the representation obtained from $\pi'$ by restriction of scalars along $f$.

<u>Rmk</u> We have

$$\dim_{k'}(f^*(V)) = \dim_k(V)$$

$$\dim_k(f_*(V')) = d \cdot \dim_{k'}(V'),$$

where $d = [k' : k]$. //

<u>Thm</u> Let $f : k \to k'$ be a field ext., and let $\pi_1 : G \longrightarrow GL(V_1)$ and $\pi_2 : G \to GL(V_2)$ be f.d. $k$-lin. representations of a grp. $G$. The following are equivalent:

(1) $\pi_1 \simeq \pi_2$

(2) $f^*\pi_1 \simeq f^*\pi_2$.

Pf (1) $\Rightarrow$ (2) : If $h: V_1 \to V_2$ is a $k$-linear isom. that is intertwining between $\pi_1$ and $\pi_2$, then

$$f^*(V_1) \xrightarrow{f^*(h)} f^*(V_2)$$

is a $k'$-linear isom. that is intertwining between $f^*\pi_1$ and $f^*\pi_2$.

(2) $\Rightarrow$ (1) : The proof here (and on p. 39 in the book) uses that $k$ is infinite. A different proof based on the Krull-Schmidt theorem works for all $k$. We first note that

$$\dim_k(V_1) = \dim_{k'}(f^*(V_1)) \overset{(2)}{=} \dim_{k'}(f^*(V_2)) = \dim_k(V_2)$$

so we may consider $\pi_1, \pi_2$ as matrix representations

$$G \xrightarrow{\pi_1,\, \pi_2} GL_n(k).$$

That $\pi_1 \simeq \pi_2$ means that there exists $C \in L_n(k)$ s.t.

(a) for all $g \in G$, $C \cdot \pi_1(g) = \pi_2(g) \cdot C$

(b) $\det(C) \neq 0$.

Here (a) is a system of linear equations over $k$ in the $n^2$ variables. By Gauss elimination, a general solution has the form

$$C = t_1 C_1 + t_2 C_2 + \cdots + t_m C_m$$

where $(C_1, \ldots, C_m)$ is a linearly independent family in $L_n(k)$, and $(t_1, \ldots, t_m) \in k^m$. And (b) is the requirement that there exist $(t_1, \ldots, t_m) \in k^m$ such that the value of the polynomial

$$p(x_1, \ldots, x_m) = \det(x_1 C_1 + \cdots + x_m C_m)$$

$$\in k[x_1, \ldots, x_m]$$

at $(x_1, \ldots, x_m) = (t_1, \ldots, t_m)$ is nonzero.

Similarly, that $f^*\pi_1 \cong f^*\pi_2$ means that there exists $C' \in L_n(k')$ such that

(a') for all $g \in G$, $C' \cdot \pi_1(g) = \pi_2(g) \cdot C'$

(b') $\det(C') \neq 0$.

But (a') is the same system of linear equations over $k$, so Gauss elimination shows that a general solution over $k'$ has the form

$$c' = t'_1 c_1 + t'_2 c_2 + \cdots + t'_m c_m$$

with $(c_1, \ldots, c_m)$ as before and with $(t'_1, \ldots, t'_m) \in (k')^m$. And (b') is the requirement that there exists $(t'_1, \ldots, t'_m) \in (k')^m$ s.t. the value of the polynomial

$$p \in k[x_1, \ldots, x_m] \subset k'[x_1, \ldots, x_m]$$

at $(x_1, \ldots, x_m) = (t'_1, \ldots, t'_m)$ is nonzero. Since $k$ and hence $k'$ is infinite, the $k'$-linear map

$$k'[x_1, \ldots, x_m] \xrightarrow{ev} \mathrm{Map}((k')^m, k')$$

is injective. So our assumption (2) implies that the polynomial

$$p(x_1, \ldots, x_m) \in k[x_1, \ldots, x_m]$$

is nonzero. But

$$k[x_1, \ldots, x_m] \xrightarrow{ev} \mathrm{Map}(k^m, k)$$

is injective, so there exists $(t_1, \ldots, t_m) \in k^m$ s.t. $p(t_1, \ldots, t_m) \neq 0$. Hence, (1) holds.

//

Suppose that $f: k \to k'$ is a Galois extension with Galois grp.

$$\Gamma = \mathrm{Aut}_k(k').$$

If $V$ is a $k$-vector space, then we get a $k$-linear repr.

$$\Gamma \xrightarrow{\rho} GL(f_* f^*(V))$$

defined by

$$\rho(\gamma)(x \otimes b) = x \otimes \gamma(b),$$

where $x \in V$ and $b \in k'$. If

$$G \xrightarrow{\pi} GL(V)$$

is a $k$-lin. repr. of some group $G$, then the induced $k$-linear repr.

$$G \xrightarrow{f_* f^* \pi} GL(f_* f^*(V))$$

is given by

$$f_* f^* \pi(g)(x \otimes b) = \pi(x) \otimes b.$$

Note that for all $g \in G$, $f_* f^* \pi(g)$ is intertwining w.r.t. $\rho$ and that for all $\gamma \in G$, $\rho(\gamma)$ is intertwining w.r.t. $f_* f^* \pi$. Indeed,

$$\rho(\gamma)\left(f_* f^* \pi(g)(x \otimes b)\right)$$
$$= \rho(\gamma)\left(\pi(g)(x) \otimes b\right)$$
$$= \pi(g)(x) \otimes \gamma(b)$$
$$= f_* f^* \pi(g)\left(x \otimes \gamma(b)\right)$$
$$= f_* f^* \pi(g)\left(\rho(\gamma)(x \otimes b)\right).$$

Equivalently, the map

$$G \times \Gamma \xrightarrow{\tau} GL(f_* f^*(V))$$

$$\tau(g, \gamma)(x \otimes b) = \pi(g)(x) \otimes \gamma(b)$$

is a $k$-linear representation of the product group $G \times \Gamma$. It follows that the subspace

$$W = (f_* f^*(V))^\Gamma \subset f_* f^*(V)$$
$$\| $$
$$\{ y \in f_* f^*(V) \mid \rho(\gamma)(y) = y \ \forall \gamma \in \Gamma \}$$

is $\pi$-invariant. Moreover, the map

$$V \xrightarrow{\eta} f_* f^*(V)$$

is intertwining w.r.t. $\pi$ and $f_* f^* \pi$ and induces a map

$$V \xrightarrow{\tilde{\eta}} (f_* f^*(V))^\Gamma$$

that is intertwining w.r.t. $\pi$ and $(f_* f^* \pi)_W$.

<u>Thm</u> If $[k' : k] < \infty$, then

$$V \xrightarrow{\tilde{\eta}} (f_* f^*(V))^\Gamma$$

is an isomorphism.

<u>Pf</u> (Incomplete) This is a case of "faithfully flat descent" for modules. //

This was rather abstract! We now specialize to the case

$$k = \mathbb{R} \xrightarrow{f} k' = \mathbb{C}$$

which is Galois with group

$$\Gamma = \{1, \sigma\},$$

where $\sigma : \mathbb{C} \to \mathbb{C}$ is complex conjugation. We also write

$$V_{\mathbb{C}} := f^*(V)$$

and call it the complexification of V. Often, by abuse of notation, $f_*(V')$ is also written $V'$. But this is very confusing, since $V'$ is a $\mathbb{C}$-vector space, whereas $f_*(V')$ is an $\mathbb{R}$-vector space.

Recall that $\sigma \in \Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is complex conjugation. We also the $\mathbb{R}$-linear map

$$f_*(V_{\mathbb{C}}) \xrightarrow{\rho(\sigma)} f_*(V_{\mathbb{C}})$$

for complex conjugation, and given $y \in f_*(V_{\mathbb{C}})$, we write

$$\bar{y} := \rho(\sigma)(y).$$

So if $y = \sum x_i \otimes z_i$ with $x_i \in V$ and $z_i \in \mathbb{C}$, then

$$\bar{y} = \sum x_i \otimes \bar{z}_i.$$

<u>Lemma</u> Let $V$ be an $\mathbb{R}$-vector space, and let $W \subset V_{\mathbb{C}}$ be a sub-$\mathbb{C}$-vector space. The following are equivalent:

(1) $W = \bar{W}$

(2) There exists $U \subset V$ s.t.

$$W = U_{\mathbb{C}} \subset V_{\mathbb{C}}.$$

Pf It is clear that (2) implies (1), so we assume (1) and prove (2). We have the $\mathbb{R}$-linear map $\eta : V \to f_*(V_{\mathbb{C}})$ and define

$$U = \eta^{-1}(f_*(W)) \subset V$$

The $\mathbb{R}$-linear map

$$U \xrightarrow{\eta|U} f_*(W)$$

determines and is determined by a $\mathbb{C}$-linear map

$$U_{\mathbb{C}} = f^*(U) \longrightarrow W.$$

This map is injective, since

$$\begin{array}{ccc} f^*(U) & \longrightarrow & W \\ \cap & & \cap \\ f^*(V) & = & V_{\mathbb{C}} \end{array},$$

and to prove that it is also surjective, we let $y \in W$. Then

$$\bar{y} \in \bar{W} \overset{(1)}{=} W,$$

and hence, so do

$$u = \frac{1}{2}(y + \bar{y}), \quad v = \frac{1}{2i}(y - \bar{y}).$$

But $\bar{u} = u$, $v = \bar{v}$ in $f_*(V_{\mathbb{C}})$, so by theorem on page 13,

$$u, v \in \operatorname{im}\left(V \xrightarrow{\eta} f_*(V_{\mathbb{C}})\right).$$

Since also $u, v \in W$, we have

$$u, v \in \operatorname{im}\left(U \xrightarrow{\eta|_U} f_*(W)\right).$$

But then

$$y = u + iv \in U_{\mathbb{C}} \subset W$$

as desired. //

Ex Recall the 2-dimensional real representation

$$G = (\mathbb{R}, +) \xrightarrow{\pi} GL_2(\mathbb{R})$$

$$\pi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

It is irreducible, but its complexification $\pi_{\mathbb{C}} = f^*\pi$ is not! Indeed, in the complex basis $(e_1 + ie_2, e_1 - ie_2)$,

$$\pi_{\mathbb{C}}(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.$$

//

**Thm** Let $\pi : G \to GL(V)$ be a real representation. Then:

(1) If $\pi_{\mathbb{C}}$ is irreducible, then $\pi$ is irreducible.

(2) If $\pi$ is irreducible, then either $\pi_{\mathbb{C}}$ is irreducible or a sum of two irred.

(3) $\pi$ is completely reducible if and only if $\pi_{\mathbb{C}}$ is so.

Pf (1) If $U \subset V$ is $\pi$-inv., then $U_{\mathbb{C}} \subset V_{\mathbb{C}}$ is $\pi_{\mathbb{C}}$-inv. So $U_{\mathbb{C}} = 0$ or $U_{\mathbb{C}} = V_{\mathbb{C}}$, and hence, $U = 0$ or $U = V$.

(3) follows from (1) + (2).

(2) Let $W \subset V_{\mathbb{C}}$ be $\pi_{\mathbb{C}}$-invariant with $\pi_{\mathbb{C},W}$ irreducible. Then

$$W \cap \bar{W}, \quad W + \bar{W} \subset V_{\mathbb{C}}$$

are $\pi_{\mathbb{C}}$-invariant subspace, and

$$\overline{W \cap \bar{W}} = \bar{W} \cap \bar{\bar{W}} = W \cap \bar{W}$$

$$\overline{W + \bar{W}} = \bar{W} + \bar{\bar{W}} = W + \bar{W}.$$

By lemma, $W \cap \bar{W},\ W + \bar{W} \subset V_{\mathbb{C}}$ are the complexification of $\pi$-inv. subspaces of $V$. Since $V$ is irred., the only possibilities are:

(i) $W \cap \bar{W} = W + \bar{W} = \{0\} \Rightarrow W = \{0\}$

(ii) $W \cap \bar{W} = \{0\} \subset W + \bar{W} = V_{\mathbb{C}}$:

$$V_{\mathbb{C}} = W \oplus \bar{W}$$

(iii) $W \cap \bar{W} = W + \bar{W} = V_{\mathbb{C}} \Rightarrow W = V_{\mathbb{C}}$.

So (2) holds. //

Ex Let $\pi : \Sigma_3 \to GL(\mathbb{R}^3)$ be the standard repr. of the symmetric group. The subsp.

$$V_1 = \{x \in \mathbb{R}^3 \mid x_1 = x_2 = x_3\}$$

$$V_2 = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3\}.$$

are $\pi$-inv. and $\pi_i = \pi_{V_i}$ are irreducible. The repr.

$$\Sigma_3 \xrightarrow{\pi_2} GL(V_2)$$

is faithful. We claim that $\pi_{2,\mathbb{C}}$ is irreducible. If not, then it is a sum of two irreducible repr., and since

$$\dim_{\mathbb{C}}(V_{2,\mathbb{C}}) = \dim_{\mathbb{R}}(V_2) = 2,$$

each of the must be 1-dim'l. But $\pi_{2,\mathbb{C}}$ is again faithful, so this would give an injective group homom.

$$\Sigma_3 \longrightarrow GL_1(\mathbb{C}) \times GL_1(\mathbb{C}),$$

which is impossible, since $\Sigma_3$ is non-commutative! //