Today: Chapter 4.

Recall that if (V_1, π_1) and (V_2, π_2) are two k-linear repr. of a group G, then a k-linear map $f: V_1 \to V_2$ is intertwining if for all $g \in G$, the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ \downarrow{\scriptstyle \pi_1(g)} & & \downarrow{\scriptstyle \pi_2(g)} \\ V_1 & \xrightarrow{f} & V_2 \end{array}$$

commutes. We will write

$$\mathrm{Hom}(\pi_1, \pi_2) \subset \mathrm{Hom}_k(V_1, V_2)$$

for the subspace of intertwing maps.

<u>Lemma</u> If $f: V_1 \to V_2$ is intertwinging between repr. (π_1, V_1) and (π_2, V_2) of G, then $\ker(f) \subset V_1$ is π_1-inv. and $\mathrm{im}(f) \subset V_2$ is π_2-invariant.

<u>Pf</u> Exercise. //

<u>Thm 1</u> Let (V_1, π_1) and (V_2, π_2) be irreducible repr. of G. If $f: V_1 \to V_2$ is intertwining between π_1 and π_2, then either f is an isomorphism or f is zero.

Pf π_1 irred. $\longrightarrow$ $\ker(f) = 0$ or V_1
π_2 irred. $\Rightarrow$ $\mathrm{im}(f) = V_2$ or 0. //

Thm 2 Let (V, π) be a repr. of G and suppose that V is the direct sum of π-inv. subspaces $V_1, \dots, V_m$ s.t. $\pi_i = \pi_{V_i}$ is irred. and $\pi_i \not\cong \pi_j$ for $i \neq j$. Then for every π-inv. subspace $U \subset V$, there exists $J \subset \{1, 2, \dots, m\}$ s.t.

$$U = \bigoplus_{j \in J} V_j \subset V.$$

Pf Let $i_j : V_j \longrightarrow V$ be the can. inclusion. We wish to prove that $U \subset V$ is equal to the image of the induced map

$$\bigoplus_{j \in J} V_j \xrightarrow{i = \sum i_j} V$$

$$(x_j)_{j \in J} \longmapsto \sum_{j \in J} x_j .$$

We know from Thm. 3 on p. 33 in the book that $U \subset V$ is equal to the image of some map

$$\bigoplus_{j \in J} V_j \xrightarrow{f} V$$

that is injective and intertw. between $\oplus \pi_j$ and π. Now, for all $i \in \{1, 2, \dots, m\}$ and $j \in J$, Thm. 1

shows that the composite map

$$\begin{tikzcd} \bigoplus_{j \in J} V_j \arrow[r, "f"] & V \arrow[d, "pr_i"] \\ V_j \arrow[u, "in_j"] \arrow[r, "f_{ij}"] & V_i \end{tikzcd}$$

is zero for $i \neq j$ and an isom. for $i = j$. (It cannot also be zero for $i = j$, since f is injective.) Hence, the diagram

$$\begin{tikzcd} \bigoplus_{j \in J} V_j \arrow[r, "f"] \arrow[d, "\bigoplus_{j \in J} f_{jj}"] & V \arrow[d, equal] \\ \bigoplus_{j \in J} V_j \arrow[r, "i"] & V \end{tikzcd}$$

commutes and $\bigoplus_j f_{jj}$ is an isom. In particular,

$$V = \mathrm{im}(f) = \mathrm{im}(i),$$

as we wanted to prove. //

If (V, π) is an irred. repr. of G, then Thm. 1 says that the ring

$$\mathrm{End}(\pi) \subset \mathrm{End}_k(V)$$

is a division algebra over k. In

general, every (f.d.) division alg. D over k can occur as $\mathrm{End}(\pi)$ for some (f.d) irred. k-linear repr. of a group. We make the very simplifying assumption that

$$k = \bar{k}$$

is algebraically closed, so that $D = k$ is the only f.d. division algebra over k. In this case, Thm. 1 gives:

Thm. 3 (Schur's lemma) Let $k = \bar{k}$ be an algebraically closed field, and let (V, π) be a finite dimensional irred. k-linear repr. of G. Then the ring homom.

$$k \xrightarrow{\eta} \mathrm{End}(\pi)$$

$$\lambda \longmapsto \lambda \cdot \mathrm{id}_V$$

is an isomorphism.

Pf Let $f : V \to V$ be a k-linear map that is intertwining w.r.t. π. Since k is alg. closed, the map f has an eigenvalue $\lambda \in k$, and since f is intertwining, the eigenspace

$$0 \neq V_\lambda \subset V$$

is π-inv. Since π is irred., we have $V_\lambda = V$, i.e. $f = \lambda \cdot id_V$. //

<u>Cor</u> Let $\pi_1 \simeq \pi_2$ be isomorphic f.-d. k-linear repr. of a group G. If $k = \bar{k}$ is alg. closed, then

$$\dim_k \mathrm{Hom}(\pi_1, \pi_2) = 1.$$

<u>Pf</u> Let $h : V_1 \to V_2$ be an isom. that intertwines between π_1 and π_2. (Such an h exists, by assumption.) If $f : V_1 \to V_2$ is any k-lin. map that intertwines between π_1 and π_2, then $f \circ h^{-1} : V_2 \to V_2$ is a k-lin. map that is intertwining w.r.t. π_2. So $f \circ h^{-1} = \lambda \cdot id_{V_2}$, by Schur's lemma, and hence, $f = \lambda \cdot h$. //

<u>Rmk</u> Above, the k-vector spaces $\mathrm{End}(\pi)$ and $\mathrm{Hom}(\pi_1, \pi_2)$ are both 1-dim.'l. But while $\mathrm{End}(\pi)$ has the preferred basis (id_V), there is no preferred basis of $\mathrm{Hom}(\pi_1, \pi_2)$. //

<u>Thm. 4</u> Let $k = \bar{k}$ be alg. closed, let (V, π) be a f.-d. k-lin. repr. of G, and let (U, τ) be a trivial f.-d. k-lin. repr. of G. If

$$W \subset V \otimes U$$

is a $\pi\otimes\tau$-inv. subspace s.t. $(\pi\otimes\tau)_W$ is irred., then there exists $u \in U$ such that the map

$$V \xrightarrow{i_u} V\otimes U$$

$$v \longmapsto v\otimes u$$

is an isomorphism on W, which is intertwining between π and $(\pi\otimes\tau)_W$.

Pf The map i_u is clearly intertwining between π and $\pi\otimes\tau$, so we must show that $u\in U$ can be chosen s.t.

$$W = i_u(V) \subset V\otimes U.$$

By Example 1 on p. 35 in the book, every irred. subrepr. of $\pi\otimes\tau$ is isom. to π. So $(\pi\otimes\tau)_W \simeq \pi$ and we choose an isom.

$$W \xrightarrow{h} V$$

that intertwines between $(\pi\otimes\tau)_W$ and π.

Now, for every $f\in U^*$, we define

$$V\otimes U \xrightarrow{c_f} V$$

$$v\otimes u \longmapsto v\cdot f(u)$$

It is linear and intertwining between $\pi \otimes \tau$ and π. Since $k = \bar{k}$, Schur's lemma shows that

$$c_f|_W = \lambda(f) \cdot h$$

for a unique $\lambda(f) \in k$. The map

$$\begin{aligned} U^* &\xrightarrow{\lambda} k \\ f &\longmapsto \lambda(f) \end{aligned}$$

is k-linear, so there exists a unique $u \in U$ s.t.

$$\begin{aligned} U &\xrightarrow{\eta} U^{**} \\ u &\longmapsto \lambda \end{aligned}$$

Indeed, the map η is an isom., since U is f.-d.! We claim that for this $u \in U$, we have

$$W = i_u(V) \subset V \otimes U.$$

To prove this, we choose a basis $(v_1, \ldots, v_n)$ of V and write $w \in W$ as

$$w = \sum_{i=1}^{n} v_i \otimes u_i$$

with $u_i \in U$. (This way of writing w is unique, but we will not use this fact.) Now, for every

$f \in U^*$, we have

$$\begin{array}{ccc} c_f(w) & = & \lambda(f) \cdot h(w) \\ \| & & \| \\ \sum_{i=1}^{n} v_i \, f(u_i) & & f(u) \cdot h(w) \end{array}$$

So if $f(u) = 0$, then for all $1 \leq i \leq n$, $f(u_i) = 0$, and hence,

$$u_i \in \operatorname{span}(u) \subset U.$$

This shows that

$$W \subset i_u(V) \subset V \otimes U,$$

and since $i_u(V) \simeq V$ is irred., we conclude that $W = i_u(V)$ as claimed. //

Abelian groups:

<u>Thm. 5</u> Let $k = \bar{k}$, let A be abelian, and let (V, π) be an irreducible k-linear repr. of A. Then

$$\dim_k(V) = 1.$$

<u>Pf</u> Since A is abelian,

$$\pi(g) \circ \pi(h) = \pi(g \cdot h) = \pi(h \cdot g) = \pi(h) \circ \pi(g)$$

for all $g, h \in A$. Hence, for all $g \in A$,

$$V \xrightarrow{\pi(g)} V$$

is intertwining w.r.t. π. So by Schur's lemma, we have

$$\pi(g) = \lambda(g) \cdot \mathrm{id}_V$$

for some $\lambda(g) \in k$. But this shows that every subspace $W \subset V$ is π-invariant. Since V is irred., this implies $\dim_k(V) = 1$. "

Recall that for every group G, there is a group homom.

$$G \xrightarrow{p} G^{ab}$$

to an abelian group G^{ab} s.t. for every group homom.

$$G \xrightarrow{f} A$$

to an abelian group A there exists a unique group homom.

$$G^{ab} \xrightarrow{f^{ab}} A$$

s.t. $f = f^{ab} \circ p$ This universal property characterizes $p : G \to G^{ab}$

uniquely, up to unique isom. under G. The map p is surjective, and its kernel is the commutator subgroup $[G, G] \subset G$.

<u>Cor</u> If $k = \bar{k}$, then the functor

$$\begin{pmatrix} \text{1-dim'l reps.} \\ \text{of } G \end{pmatrix} \longrightarrow \begin{pmatrix} \text{irred. reps.} \\ \text{of } G^{ab} \end{pmatrix}$$

$$\pi \longmapsto \pi^{ab}$$

is an equivalence of categories.

<u>Pf</u> If $\dim_k(V) = 1$, then $GL(V)$ is abelian, so we have a unique factorization of every repr.

$$\begin{tikzcd} G \arrow[rr, "\pi"] \arrow[dr, "p"'] & & GL(V) \\ & G^{ab} \arrow[ur, "\pi^{ab}"'] & \end{tikzcd}$$

Since $k = \bar{k}$, a repr. of G^{ab} is irred. if and only if it is 1-dimensional. //

<u>Ex</u> The abelianization of the symm. group Σ_n is $\text{sgn} \colon \Sigma_n \to \{\pm 1\}$. So, up to isom., the only 1-dim'l repr. of Σ_n over an alg. closed

field k are the trivial repr. and the sign repr. //

Exterior tensor product:

<u>Thm. 6</u> Let $k = \bar{k}$ and let $\pi_1 : G_1 \to GL(V_1)$ and $\pi_2 : G_2 \to GL(V_2)$ be f.-d. k-lin. repr. Then the exterior tensor prod.

$$G_1 \times G_2 \xrightarrow{\pi_1 \boxtimes \pi_2} GL(V_1 \otimes V_2)$$

is irreducible.

<u>Pf</u> Let $\{0\} \neq W \subset V_1 \otimes V_2$ be $\pi_1 \boxtimes \pi_2$-inv. We must show that $W = V_1 \otimes V_2$. Note that

$$(\pi_1 \boxtimes \pi_2)(g_1, e) = (\pi_1 \otimes I)(g_1)$$

$$(\pi_1 \boxtimes \pi_2)(e, g_2) = (I \otimes \pi_2)(g_2)$$

so $W \subset V_1 \otimes V_2$ is both $\pi_1 \otimes I$-inv. as a repr. of $G_1 \times \{e\} \subset G_1 \times G_2$ and $I \otimes \pi_2$-inv. as a repr. of $\{e\} \times G_2 \subset G_1 \times G_2$. Hence, by Thm. 4, there exists subspaces $U_2 \subset V_2$ and $U_1 \subset V_1$, so that both the (injective) induced maps

$$V_1 \otimes U_2 \hookrightarrow V_1 \otimes V_2 \hookleftarrow U_1 \otimes V_2$$

have image W. So the square

$$\begin{tikzcd} U_1 \otimes U_2 \arrow[r, hook] \arrow[d, hook] & V_1 \otimes U_2 \arrow[d, hook, "\sim"] \\ U_1 \otimes V_2 \arrow[r, hook, "\sim"] & W \end{tikzcd}$$

is cocartesian, and the right vertical and bottom horizontal maps are isomorphisms. So

$$(V_1/U_1) \otimes U_2 \simeq 0 \simeq U_1 \otimes (V_2/U_2),$$

which implies $U_1 = V_1$ and $U_2 = V_2$. So $W = V_1 \otimes V_2$ as desired. //

Ex In Thm. 6, the assumption $k = \bar{k}$ is necessary. Indeed, consider

$$G = (\mathbb{R}, +) \xrightarrow{\pi} GL(\mathbb{R}^2)$$

$$t \longmapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Then π is irred., but we claim that $\pi \boxtimes \pi$ is not. If it were, then $(\pi \boxtimes \pi)_{\mathbb{C}}$ would either also be irred. or a direct sum of two irred. repr. But

$$\dim_{\mathbb{C}}((\pi \boxtimes \pi)_{\mathbb{C}}) = \dim_{\mathbb{R}}(\pi \boxtimes \pi) = 4$$

and $G \times G$ is abelian, so this is impossible. //

Matrix coefficients: Recall

$$k[G] = \{\text{all functions } f : G \to k\}$$

and the left and right regular representations

$$G \xrightarrow{L,R} GL(k[G])$$

defined by

$$L(g)(f)(h) = f(g^{-1}h)$$
$$R(g)(f)(h) = f(hg).$$

Note that

$$R(g_1) \circ L(g_2) = L(g_2) \circ R(g_1)$$

so we get a representation

$$G \times G \xrightarrow{Reg} GL(k[G])$$

defined by

$$Reg(g_1, g_2) = R(g_1) \circ L(g_2)$$
$$= L(g_2) \circ R(g_1).$$

We call this the two-sided regular representation. It is given by

$$\mathrm{Reg}(g_1, g_2)(f)(h) = f(g_2^{-1} h g_1).$$

Now, given a representation

$$G \xrightarrow{\pi} GL(V),$$

the k-linear map

$$V \otimes V^* \xrightarrow{\mu} k[G]$$

defined by

$$\mu(x \otimes f)(h) = f(\pi(h)(x))$$

is intertwining between $\pi \boxtimes \pi^*$ and Reg. Its image,

$$M(\pi) \subset k[G],$$

is called the subspace of matrix coefficients (or matrix elements). The reason is as follows. Suppose that $(e_1, \ldots, e_n)$ is a basis of V, and let $(e_1^*, \ldots, e_n^*)$ be the dual basis of V^*. Let

$$A(h) = (a_{ij}(h)) \in M_n(k)$$

be the matrix that represents

$\pi(h)$ with respect to $(e_1, \dots, e_n)$. Then

$$a_{ij}(h) = \mu(e_j \otimes e_i^*)(h).$$

So we have

$$M(\pi) = \mathrm{span}(a_{ij}) \subset k[G].$$

The reason that we do not define $M(\pi) \subset k[G]$ by this formula is that it is not immediately clear that $\mathrm{span}(a_{ij}) \subset k[G]$ not depend on the choice of basis.

<u>Thm. 7</u> Suppose that $k = \bar{k}$ and let

$$G \xrightarrow{\pi} GL(V)$$

be a f.-d. irred. repr. Then

$$V \otimes V^* \xrightarrow{\mu} M(\pi)$$

is an isomorphism.

Pf The map μ is surjective by definition. It is also intertwining with respect to the repr. $\pi \boxtimes \pi^*$ and $\mathrm{Reg}_{M(\pi)}$ of $G \times G$. Since π is irred., so is π^* and, by Thm. 6, $\pi \boxtimes \pi^*$. So $\ker(\mu)$ is either zero or all of $V \otimes V^*$. It is easy to see that $\ker(\mu) \neq \{0\}$. //

Some consequences of Thm. 7:

1) If $k = \bar{k}$ and π irred. of fin. dim. n, then

$$\dim_k M(\pi) = n^2.$$

2) For π as in 1),

$$R_{M(\pi)} \cong \pi \oplus \cdots \oplus \pi \quad (n \text{ times})$$

$$L_{M(\pi)} \cong \pi^* \oplus \cdots \oplus \pi^* \quad (n \text{ times})$$

3) If $k = \bar{k}$ and π_1, π_2 are f.-d. irred. repr. of G, then

$$\mathrm{Reg}_{M(\pi_1)} \cong \mathrm{Reg}_{M(\pi_2)} \Rightarrow \pi_1 \cong \pi_2$$

4) If $k = \bar{k}$ and $\pi_1, \dots, \pi_m$ are pairwise non-isom. repr. of G, then the canonical map

$$M(\pi_1) \oplus \cdots \oplus M(\pi_m) \longrightarrow k[G]$$

is injective.

Our final application of Schur's lemma concerns unitary repr. Recall that a repr.

$$G \xrightarrow{\pi} GL(V)$$

on a f.d. complex vector space V is unitary if there exists an hermitian inner product $\langle -,- \rangle$ on V that is π-invariant in the sense that for all $g \in G$ and $x, y \in V$,

$$\langle \pi(g)(x), \pi(g)(y) \rangle = \langle x, y \rangle.$$

Equivalently, $\langle -,- \rangle$ is π-inv. if the induced isomorphism

$$V \xrightarrow{b} V^*$$

given by $b(x)(y) = \langle x, y \rangle$ is intertwining w.r.t. π and π^*.

<u>Thm 8</u> Let $\pi : G \to GL(V)$ be an irred. f.d. unitary repr., and let $\langle -,- \rangle_1$ and $\langle -,- \rangle_2$ be two π-inv. hermitian inner products on V. Then there exists a scalar $\lambda \in \mathbb{C}$ s.t.

$$\langle x, y \rangle_2 = \lambda \cdot \langle x, y \rangle_1$$

for all $x, y \in V$.

<u>Pf</u> The composite isom.

$$\begin{tikzcd} V \arrow[r, "b_2"] \arrow[rr, bend right, "h"'] & V^* \arrow[r, "b_1^{-1}"] & V \end{tikzcd}$$

is intertwining w.r.t. π, since b_1 and b_2, by assumption, are intertwining w.r.t. π and π^*. Since π is irred., Schur's lemma shows

$$h = \lambda \cdot \mathrm{id}_V$$

for some $\lambda \in \mathbb{C}$. We are done. //