Today: Chap. 5 (5.1 – 5.4)

Assume G finite, $k = \bar{k}$, $\mathrm{char}(k) = 0$.
Since G is finite, a basis of $k[G]$ is given by the family $(\delta_x)_{x \in G}$,

$$\delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

Recall that for $\pi : G \to GL(V)$ a f.-d. k-lin. repr., we defined the space of matrix coeff.

$$M(\pi) \subset k[G]$$

to be the image of the map

$$V \otimes V^* \xrightarrow{\mu} k[G]$$

$$\mu(x \otimes f)(h) = f(\pi(h)(x)).$$

It is invariant for the two-sided regular repr.

$$G \times G \xrightarrow{Reg} GL(k[G])$$

$$Reg(g_1, g_2)(f)(h) = f(g_2^{-1} h g_1).$$

Using Schur's lemma, we proved:

(1) π irred. $\Rightarrow$ $Reg_{M(\pi)} \cong \pi \boxtimes \pi^*$

(2) If π_1, π_2 are irred., then

$$\pi_1 \cong \pi_2 \iff M(\pi_1) = M(\pi_2)$$

(3) If $\pi_1, \ldots, \pi_m$ are pairwise non-isom. irred. repr., then the can. map

$$M(\pi_1) \oplus \cdots \oplus M(\pi_m) \longrightarrow k[G]$$

is injective. ✓

<u>Thm 1</u> If G is finite and $k = \bar{k}$, then G has at most $|G|$ pairwise non-isom. f.d., irred. k-linear representations.

<u>Pf</u> This follows from (3), since

$$\begin{array}{c} \dim_k (M(\pi_1) \oplus \cdots \oplus M(\pi_q)) \geq q \\ \wedge\!\!\!| \\ \dim_k (k[G]) = |G|. \quad // \end{array}$$

<u>Thm 2</u> Suppose G finite, $k = \bar{k}$, $\mathrm{char}(k) = 0$. If $\pi_1, \ldots, \pi_q$ are representatives of the isomorphism classes of f.d. irred. k-lin. repr., then the canonical map

$$M(\pi_1) \oplus \cdots \oplus M(\pi_q) \longrightarrow k[G]$$

is an isomorphism.

<u>Pf</u> We have already seen that the map is injective, so we must prove that it is surjective. Let

$R : G \to GL(k[G])$ be the right reg. repr., $R(g)(f)(h) = f(hg)$. Claim that $M(R) = k[G]$. Indeed, let $\varepsilon : k[G] \to k$ be the k-linear map defined by $\varepsilon(f) = f(e)$. So $\varepsilon \in k[G]^*$ and for $f \in k[G]$, the calculation

$$\mu(f \otimes \varepsilon)(h) = \varepsilon(R(h)(f))$$
$$= f(e \cdot h) = f(h)$$

shows that

$$f = \mu(f \otimes \varepsilon) \in M(R).$$

Now, since G is finite and since $\mathrm{char}(k) \nmid |G|$, Maschke's thm. shows that every f.-d. k-lin. repr. is completely reducible. So

$$R \simeq \pi_1^{n_1} \oplus \cdots \oplus \pi_q^{n_q}$$

But for any f.-d. k-lin. repr. ρ, τ,

$$M(\rho \oplus \tau) = M(\rho) + M(\tau) \subset k[G],$$

so we conclude

$$\begin{array}{c} M(R) \subset M(\pi_1) + \cdots + M(\pi_q) \subset k[G] \\ \parallel \\ k[G] \end{array}$$

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For $\pi_i : G \to GL(V_i)$ as in Thm. 3, set

$$n_i = \dim_k(V_i).$$

Cor In the situation of Thm. 3,

$$|G| = n_1^2 + \cdots + n_q^2 .$$

Pf By Thm. 3,

$$R \simeq R_{M(\pi_1)} \oplus \cdots \oplus R_{M(\pi_q)}$$

and

$$R_{M(\pi_i)} \simeq \pi_i^{n_i} . \qquad //$$

Ex 1) A finite abelian group A has $|A|$ non-isom. irred. repr., all of which are 1-dim'l.

2) $G = \Sigma_3$. We have found 3 non-isom. $\mathbb{C}$-lin. repr.:

"1" = I : 1-dim'l trivial repr.

sgn : 1-dim'l sign repr.:

$$\mathrm{sgn}(\sigma)(z) = \mathrm{sgn}(\sigma) \cdot z .$$

π : 2-dim'l repr. on

$$V = \{ z \in \mathbb{C}^3 \mid z_1 + z_2 + z_3 = 0 \} \subset \mathbb{C}^3$$

$$\pi(\sigma)(z_1, z_2, z_3) = (z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}).$$

So, up to isom., these are all f.d. irred. $\mathbb{C}$-lin. repr., since

$$|\Sigma_3| = 6 = 1^2 + 1^2 + 2^2. \qquad //$$

We next prove a lemma concerning matrix coefficients. So let (π, V) be a f.d. k-lin. repr. of a group G, and recall that $M(\pi) \subset k[G]$ is defined to be the image of

$$V \otimes V^* \xrightarrow{\mu} k[G]$$

$$\mu(x \otimes f)(g) = f(\pi(g)(x)).$$

<u>Lemma</u> In this situation, there is a commutative diagram

$$\begin{tikzcd} \mathrm{End}_k(V) \arrow[rr, "\mu'"] & & k[G] \\ & V \otimes V^* \arrow[ul, "\alpha"] \arrow[ur, "\mu"'] & \end{tikzcd}$$

where α and μ' are defined by

$$\mu'(h)(g) = \mathrm{tr}(\pi(g) \circ h)$$

$$\alpha(x \otimes f)(y) = x \cdot f(y).$$

Moreover, the map α is an isom.

<u>Pf</u> Let $(e_1, \dots, e_n)$ be a basis for V, and let $(e_1^*, \dots, e_n^*)$ be the dual

basis of V^*. For $g \in G$, let

$$A(g) = (a_{ij}(g))$$

be the matrix that repr. $\pi(g)$ w.r.t. the basis $(e_1, \dots, e_n)$. As we have calculated before,

$$\mu(e_j \otimes e_i^*)(g) = a_{ij}(g).$$

Now, we have $\alpha(e_j \otimes e_i^*) = e_j \cdot e_i^*$, and

$$(\pi(g) \circ e_j \cdot e_i^*)(e_k) = \begin{cases} \pi(g)(e_j) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

so

$$\begin{aligned} &(\mu' \circ \alpha)(e_j \otimes e_i^*)(g) \\ &= \mathrm{tr}(\pi(g) \circ e_j \cdot e_i^*) = a_{ij}(g), \end{aligned}$$

as required. Finally, the map α is an isomorphism, since it maps the basis $(e_j \otimes e_i^*)_{1 \le i,j \le n}$ of $V \otimes V^*$ to the basis $(e_j \cdot e_i^*)_{1 \le i,j \le n}$ of $\mathrm{End}_k(V)$. //

It follows that $M(\pi) \subset k[G]$ is also the image of the map

$$\mathrm{End}_k(V) \xrightarrow{\mu'} k[G].$$

Note that $\mu = \mu_\pi$ and $\mu' = \mu'_\pi$

depend on π, whereas $\alpha = \alpha_V$ only depends on V.

<u>Def</u> If (V, π) is a f.d. k-lin. repr. of a group G, then its character

$$\chi_\pi \in k[G]$$

is the fct. $\chi_\pi : G \to k$ defined by

$$\chi_\pi(g) = \mathrm{tr}(\pi(g)).$$ //

Note that

$$\chi_\pi = \mu'_\pi(\mathrm{id}_V) \in M(\pi) \subset k[G].$$

A main result for G finite and $k = \bar{k}$ of char. 0 (e.g. $k = \mathbb{C}$) is that χ_π determines π, up to isom. First, record some properties:

1) $\pi_1 \simeq \pi_2 \implies \chi_{\pi_1} = \chi_{\pi_2}$

Pf: $\pi_2(g) = h\,\pi_1(g)\,h^{-1}$, some $h : V_1 \xrightarrow{\sim} V_2$.

2) $\chi_{\pi^*}(g) = \chi_\pi(g^{-1})$

Pf: $\pi^*(g) = \pi(g^{-1})^*$ and $\mathrm{tr}(f) = \mathrm{tr}(f^*)$.

3) $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$

Pf: $\operatorname{tr}(f_1 \oplus f_2) = \operatorname{tr}(f_1) + \operatorname{tr}(f_2)$.

4) $\chi_{\pi_1 \otimes \pi_2} = \chi_{\pi_1} \cdot \chi_{\pi_2}$

Pf: More generally,

$$\chi_{\pi_1 \boxtimes \pi_2}(g_1, g_2) = \chi_{\pi_1}(g_1) \cdot \chi_{\pi_2}(g_2),$$

because $\operatorname{tr}(f_1 \otimes f_2) = \operatorname{tr}(f_1) \cdot \operatorname{tr}(f_2)$.

5) $\chi_\pi(ghg^{-1}) = \chi_\pi(h)$.

Pf: $\operatorname{tr}(f_1 \circ f_2) = \operatorname{tr}(f_2 \circ f_1)$.

<u>Def</u> A function $f : G \to k$ is central if for all $g, h \in G$,

$$f(ghg^{-1}) = f(h).$$

I will write

$$Z(k[G]) \subset k[G]$$

for the subspace of central fct. (Book writes $k[G]^\#$.)

<u>Rmk</u> Explaination: Define the convolution product $*$ on $k[G]$ by

$$(f_1 * f_2)(g) = \sum_{h_1 h_2 = g} f_1(h_1) f_2(h_2).$$

The map $\mu' : \mathrm{End}_k(V) \to k[G]$ satisfies

$$\mu'(f_1 \circ f_2) = \mu'(f_1) * \mu'(f_2).$$

Now, $Z(k[G]) \subset k[G]$ is the center of the ring $(k[G], +, *)$:

$$f \in Z(k[G]) \iff f * h = h * f, \ \forall h \in k[G].$$

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<u>Lemma</u> Suppose G finite, $k = \bar{k}$, and $\mathrm{char}(k) = 0$. If π is a f.d. irred. k-lin. repr. of G, then

$$Z(k[G]) \cap M(\pi) = \mathrm{span}(\chi_\pi).$$

<u>Pf</u> We use that, by previous lemma,

$$M(\pi) = \{ \mu'_\pi(f) \mid f \in \mathrm{End}_k(V) \}$$

where $\mu'_\pi(f)(g) = \mathrm{tr}(\pi(g) \circ f)$. Since π is irred., $\mu = \mu_\pi$, and hence $\mu' = \mu'_\pi$ is injective, so

$$\mathrm{End}_k(V) \xrightarrow{\mu'_\pi} M(\pi)$$

is an isomorphism. Now

$$\mu'_\pi(f)(ghg^{-1}) = \mathrm{tr}(\pi(ghg^{-1}) \circ f)$$

$$= \operatorname{tr}(\pi(g) \circ \pi(h) \circ \pi(g)^{-1} \circ f)$$

$$= \operatorname{tr}(\pi(h) \circ \pi(g)^{-1} \circ f \circ \pi(g))$$

$$= \mu_\pi'(\pi(g)^{-1} \circ f \circ \pi(g))(h),$$

so $\mu'(f)$ is central if and only if for all $g \in G$,

$$f = \pi(g)^{-1} \circ f \circ \pi(g).$$

So we have proved that

$$\mu_\pi'(f) \text{ central} \iff f \ \pi\text{-inv.}$$

By Schur's lemma, f is π-inv. if and only if $f = c \cdot \mathrm{id}_V$, for some $c \in k$. But

$$\mu_\pi'(c \cdot \mathrm{id}_V) = c \cdot \mu_\pi'(\mathrm{id}_V)$$

$$= c \cdot \chi_\pi .$$

//

<u>Thm 3</u> Suppose G finite, $k = \bar{k}$, and char$(k) = 0$. Let $\pi_1, \dots, \pi_q$ be representative of the isom. classes of f.d. irred. k-lin. repr. of G. Then $(\chi_{\pi_1}, \dots, \chi_{\pi_q})$ is a basis of $Z(k[G])$.

Pf This follows from Thm. 2 and from the lemma. //

Corollaries of Thm. 3:

1) $\dim_k Z(k[G])$

$= \# \{$conj. classes of elem. in $G\}$

$= \# \{$eq. classes of f.d. irrep. of $G\}$.

2) The isom. class of any f.d. k-lin. repr. (not nec. irred.) π of G is determined by χ_π.

Pf: Write $\pi \simeq \pi_1^{m_1} \oplus \cdots \oplus \pi_q^{m_q}$. Then

$$\chi_\pi = m_1 \chi_{\pi_1} + \cdots + m_q \chi_{\pi_q},$$

and since $\mathrm{char}(k) = 0$, this determines $m_1, \ldots, m_q \in \mathbb{Z}$. //

(Recall: $\mathrm{char}(k) = 0 \Leftrightarrow \mathbb{Z} \hookrightarrow k$.)

Example $G = \Sigma_4$, $|G| = 24$.

Lemma The conjugacy classes of elements in Σ_n are determined by their cycle types.

(The cycle type of a permutation is the partition of n obtained by counting the elements in cycles.)

Pf Conjugation of a permutation corresponds to relabelling the elements in $\{1, 2, \dots, n\}$. //

For $n=4$, there are 5 cycle types: $1+1+1+1$, $2+1+1$, $2+2$, $3+1$, and 4, and representatives of the corresponding conjugacy classes of permutations are:

$$e, \ (12), \ (12)(34), \ (123), \ (1234).$$

So $\dim_{\mathbb{C}} Z(\mathbb{C}[G]) = 5$ and there are 5 isom. classes of f.-d. irred. complex repr. of G. We know 3 of these already:

$\pi_1 = $ "1" : trivial 1-dim'l repr.

$\pi_2 = \operatorname{sgn}$ (1-dim'l)

π_3 : 3-dim'l permutation repr. on

$$V = \{ z \in \mathbb{C}^4 \mid z_1 + z_2 + z_3 + z_4 = 0 \}.$$

The dimensions of the remaining two representations π_4 and π_5, say n_4 and n_5, satisfy

$$1^2 + 1^2 + 3^2 + n_4^2 + n_5^2 = 24,$$

so $n_4 = 3$ and $n_5 = 2$

$$\pi_4 \simeq \text{sgn} \otimes \pi_3 = \pi_2 \otimes \pi_3$$

Pf: Must show that it is irred. and that it is not isom. to π_3. It is irred., since

$$\text{sgn} \otimes \text{sgn} \otimes \pi_3 \simeq \pi_3$$

and since π_3 is irreducible. To show that $\text{sgn} \otimes \pi_3$ is not isom. to π_3, it suffices to show that

$$\chi_{\text{sgn} \otimes \pi_3} = \text{sgn} \cdot \chi_{\pi_3} \neq \chi_{\pi_3}.$$

So we must find $\sigma \in G$ s.t. $\text{sgn}(\sigma) = -1$ and $\chi_{\pi_3}(\sigma) \neq 0$. Consider $\pi = \pi_1 \oplus \pi_3$. We have

$$\chi_\pi = \chi_{\pi_1} + \chi_{\pi_3} = 1 + \chi_{\pi_3},$$

since π_1 is the trivial 1-dim'l repr. Now, π is the standard repr. of $G = \Sigma_4$ on $\mathbb{C}^4$ given by

$$\pi(\sigma)(z_1, \dots, z_4) = (z_{\sigma(1)}, \dots, z_{\sigma(4)}).$$

So the matrix that represents $\pi((12))$ with respect to the standard basis $(e_1, \dots, e_4)$ of $\mathbb{C}^4$ is

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

so $\chi_\pi((12)) = 2$, and hence $\chi_{\pi_3}((12)) = 1 \neq 0$. //

$\pi_5 \simeq$? (See Example 2, p. 42)

One can show that

$$N = \{e, (12)(34), (13)(24), (14)(23)\}$$

is a normal subgroup of G and that $G/N \simeq \Sigma_3$. So every irred. repr. π of Σ_3 gives an irred. repr. of $G = \Sigma_4$ defined by

$$G \xrightarrow{p} G/N \xrightarrow{\pi} GL(V).$$

In particular, the 2-dim'l irred. repr. of Σ_3 gives a 2-dim'l irred. repr. of G. This is π_5.