Today: Rest of Chap. 5 + Chap. 6.

Let $G \times X \to X$, $(g,x) \mapsto g\cdot x$, be an action by a group G on a set X. For $x \in X$, the subgroup

$$G_x = \{ h \in G \mid h\cdot x = x \} \subset G$$

is called the isotropy subgroup (or stabilizer) at x, and the subset

$$G\cdot x = \{ g\cdot x \in X \mid g \in G \} \subset X$$

is called the orbit through x. The map

$$G/G_x \longrightarrow G\cdot x$$

$$gG_x \longmapsto g\cdot x$$

is well-defined and a bijection. It is also G-equivariant for the action by G on G/G_x by left mult. and the action by G on $G\cdot x \subset X$ obtained by restriction of the action by G on X. (Check this!)

If $x, y \in X$ belong to the same orbit, then the subgroups $G_x, G_y \subset G$ are conjugate. Indeed, if we choose $g \in G$ s.t. $y = g\cdot x$, then the map

$$\begin{aligned} G_x &\longrightarrow G_y \\ h &\longmapsto ghg^{-1} \end{aligned}$$

is a group isomorphism. (But note that this isomorphism depends on the choice of g!)

If $H \subset G$ is a subgroup, then we say that the subset

$$X^H = \{x \in X \mid \forall h \in H : h \cdot x = x\} \subset X$$

is the H-fixed subset of X. It is <u>not</u> a G-invariant subset, in general, but the action of the normalizer of H in G,

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\} \subset G,$$

on X restricts to an action by $N_G(H)$ on X^H. Indeed, if $g \in N_G(H)$, then for all $h \in H$, there exists some $h' = h'(g) \in H$ s.t. $h \cdot g = g \cdot h'$, so if $x \in X^H$, then

$$h \cdot g \cdot x = g \cdot h' \cdot x = g \cdot x,$$

which shows that also $g \cdot x \in X^H$.

We have $H \subset N_G(H)$, and, by definition, H acts trivially on X^H, so the action of $N_G(H)$ factors through an action by the quotient group

$$W_G(H) = N_G(H)/H,$$

which is called the Weyl group of H in G.

The set of orbits for the (left) action by G on X is denoted

$$G\backslash X = \{G \cdot x \in \mathcal{P}(X) \mid x \in X\}.$$

If there is only one orbit, i.e.

$$G\backslash X = \{X\},$$

then the action is called transitive.

<u>Ex 1)</u> The group $G = O(3)$ of orthogonal 3×3-matrices acts on

$$S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\} \subset \mathbb{R}^3$$

by left matrix multiplication. The

action is transitive, and the isotropy subgroup of the "north pole"

$$x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in S^2$$

is the subgroup of block matrices

$$G_x = \left\{ \begin{pmatrix} Q & \\ & 1 \end{pmatrix} \in O(3) \;\middle|\; Q \in O(2) \right\}$$

Hence, identifying $G_x = O(2)$ by abuse of notation, we have a bijection

$$O(3)/O(2) \xrightarrow{\sim} S^2.$$

This is, in fact, a homeomorphism.

2) Let k be a field. The action by $G = GL_m(k)$ on $X = M_{m,n}(k)$ by left multiplication is not transitive (except in trivial cases). The theorem in linear algebra, which we call Gauss elimination, gives a (non-canonical) bijection

$$\left\{ \begin{matrix} m\times n\text{-matrices} \\ \text{on reduced} \\ \text{echelon form} \end{matrix} \right\} \xrightarrow{\sim} GL_m(k) \backslash M_{m,n}(k)$$

$$A \longmapsto GL_m(k) \cdot A$$

Challenge exercise: Given $A \in M_{m,n}(k)$ on reduced echelon form, identify the isotropy subgroup

$$GL_m(k)_A \subset GL_m(k).$$

//

Now, let $H \subset G$ be a subgroup of a group G, and let k be a field. By analogy with the two-sided reg. repr. of $G \times G$ on $k[G]$, we have the k-linear repr.

$$W_G(H) \times G \xrightarrow{\bar{\rho}} GL(k[G/H])$$

$$\bar{\rho}(g_1 H, g_2)(\bar{f})(gH) = \bar{f}(g_2^{-1} g g_1 H).$$

Moreover, the two-sided regular repr. of $G \times G$ on $k[G]$ restricts to a repr.

$$W_G(H) \times G \xrightarrow{\rho} GL(k[G]^{H \times \{e\}})$$

$$\rho(g_1 H, g_2)(f)(g) = f(g_2^{-1} g g_1).$$

Here, we use the identification

$$W_{G \times G}(H \times \{e\}) = W_G(H) \times G.$$

<u>Lemma</u> Let $p: G \to G/H$ be the can. projection. The map

$$k[G/H] \xrightarrow{p^*} k[G]^{H \times \{e\}}$$
$$\bar{f} \longmapsto \bar{f} \circ p$$

is a k-linear isomorphism that is intertwining w.r.t. $\bar{\rho}$ and ρ.

Pf The RHS is the set of functions $f: G \to k$ s.t. for all $g \in G$ and $h \in H$, $f(gh) = f(g)$. Every such fct. is of the form $f = \bar{f} \circ p$ for a unique function $\bar{f}: G/H \to k$. So the map p^* is a bijection. It is clear that it is k-linear and intertwining w.r.t. $\bar{\rho}$ and ρ. //

If (V, π) is a k-linear repr. of G and $H \subset G$ a subgroup, then we write (V^H, π^H) for the induced k-lin. repr. of $W_G(H)$ on V^H,

$$W_G(H) \xrightarrow{\pi^H} GL(V^H)$$

$$\pi^H(g)(x) = \pi(g)(x).$$

char(k) = 0

<u>Thm. 4</u> Let G be a finite group, $H < G$ a subgroup, and $k = \bar{k}$ an alg. cl. field. Let $(V_1, \pi_1), \dots, (V_q, \pi_q)$ be representatives of the isomorphism classes of f.d. irred. k-linear repr. of G. Then the isomorphism

$$\pi_1 \boxtimes \pi_1^* \oplus \dots \oplus \pi_q \boxtimes \pi_q^* \xrightarrow[\sim]{\mu} \mathrm{Reg}$$

of k-lin. repr. of $G \times G$ induces an isomorphism

$$\pi_1^H \boxtimes \pi_1^* \oplus \dots \oplus \pi_q^H \boxtimes \pi_q^* \xrightarrow[\sim]{\mu^{H \times \{e\}}} \mathrm{Reg}^{H \times \{e\}}$$

of k-lin. repr. of $W_G(H) \times G$.

Pf This is clear: an intertwining isom. induces an isomorphism of fixed points //

We will use Thm. 4 to determine the structure of the left reg. repr.

$$G \xrightarrow{L} k[X]$$

for X a set with transitive left G-action. Recall that

$$L(g)(f)(x) = f(g^{-1}x).$$

char(k) = 0

<u>Cor</u> Let G be a finite ~~sub~~group, let X be a set with transitive G-action, let $x \in X$, and let $H = G_x \subset G$ be the isotropy subgroup. Let $k = \bar{k}$ be an alg. closed field, and let $(V_1, \pi_1), \dots, (V_q, \pi_q)$ be representatives of the isom. cl. of irred. f.-d. k-lin. repr. of G. Let $L : G \to GL(k[X])$ be the left regular repr. of G on $k[X]$. There is a non-can. isom.

$$L \simeq \bigoplus_{i=1}^{q} \pi_i^{m_i}$$

where $m_i = \dim_k(V_i^H)$.

<u>Pf</u> We identify $G/H \xrightarrow{\sim} X$ by the map $gH \mapsto gx$. Thm. 4 and the lemma give canonical isomorphisms

$$k[G/H] \xrightarrow[\sim]{p^*} k[G]^{H \times \{e\}} \xleftarrow[\sim]{u} \bigoplus_{i=1}^{q} V_i^H \otimes V_i^*$$

that are intertwining w.r.t. the respective repr. of $W_G(H) \times G$ on these vector spaces. In particular, they are also intertwining w.r.t. the representations of the subgroup

$$G = \{H\} \times G \subset W_G(H) \times G.$$

So the repr. $\pi_{L'}^*$ appears with multiplicity $\dim_k(V_{L'}^H)$ in L, or equivalently, the repr. $\pi_{L'}$ appears with mult. $\dim_k((V_{L'}^*)^H)$ in L. So to prove the corollary, we wish to prove that

$$\dim_k((V_{L'}^*)^H) = \dim_k(V_{L'}^H).$$

We will prove that for every f.d. k-lin. repr. (V, π) of G,

$$\dim_k((V^*)^H) = \dim_k(V^H).$$

The composite map

$$H \hookrightarrow G \xrightarrow{\pi} GL(V)$$

is a f.d. k-lin. repr. of the finite group H. Hence, it decomposes

$$\pi \simeq \rho_1^{m_1} \oplus \cdots \oplus \rho_r^{m_r}$$

as a direct sum of irred. k-lin. repr. of H. So

$$\pi^* \simeq (\rho_1^*)^{m_1} \oplus \cdots \oplus (\rho_r^*)^{m_r}.$$

But ρ_i is trivial if and only if ρ_i^* is trivial. So the equality of dimensions follows. //

Rmk The isom. in the corollary amounts to choosing a basis of $(V_i^*)^H$ for all $1 \leq i \leq q$. Therefore, it is non-can. //

Skip: Repr. theory of $A_5 \subset \Sigma_5$.

Schur orthogonality:
Let G be a finite grp. and $k = \mathbb{C}$. If $\pi : G \to GL(V)$ is a f.-d. $\mathbb{C}$-lin. repr., then recall that we have defined the space of matrix coeff.

$$M(\pi) \subset \mathbb{C}[G]$$

to be the images of the maps

$$\begin{tikzcd} \mathrm{End}_{\mathbb{C}}(V) \arrow[rr, "\mu'"] & & GL(\mathbb{C}[G]) \\ & V \otimes V^* \arrow[ul, "\alpha", "\sim"'] \arrow[ur, "\mu"'] & \end{tikzcd}$$

where $\mu(x \otimes f)(g) = f(\pi(g)(x))$ and $\mu'(h)(g) = \mathrm{tr}(\pi(g) \circ h)$. We also saw that if $(e_1, \dots, e_n)$ is a basis of V and $(e_1^*, \dots, e_n^*)$ the dual basis of V^*, then

$$M(\pi) = \mathrm{span}(\mu(e_j \otimes e_i^*))_{1 \le i,j \le n}.$$

If π is irred., then μ and μ' are isom. onto $M(\pi)$, so in this case, the family $(\mu(e_j \otimes e_i^*))_{1 \le i,j \le n}$ is a basis of $M(\pi)$.

Suppose that $\langle -,- \rangle$ is an hermitian inner product on V. I will use the convention that for $x, y \in V$ and $z, w \in \mathbb{C}$,

$$\langle x \cdot z, y \cdot w \rangle = \bar{z} \cdot \langle x, y \rangle \cdot w,$$

which is the opposite of the book's convention. It determines the $\mathbb{C}$-linear isomorphism

$$\bar{V} \xrightarrow{b} V^*$$

defined by $b(x)(y) = \langle x, y \rangle$ and vice versa. So we may also identify $M(\pi) \subset \mathbb{C}[G]$ with the image of the composite map

$$V \otimes \bar{V} \xrightarrow{\mathrm{id} \otimes b} V \otimes V^* \xrightarrow{\mu} \mathbb{C}[G],$$

which maps $x \otimes y$ to the fct.

$$\mu(x \otimes b(y))(g) = b(y)(\pi(g)(x))$$

$$= \langle y, \pi(g)(x) \rangle.$$

So if $(e_1, \dots, e_n)$ is a basis of V that is orthonormal w.r.t. $\langle -,- \rangle$, then the matrix

$$A(g) = (a_{ij}(g)) \in M_n(\mathbb{C})$$

that repr. $\pi(g) : V \to V$ w.r.t. the basis $(e_1, \dots, e_n)$ is given by

$$a_{ij}(g) = \langle e_i, \pi(g)(e_j) \rangle.$$

We also remark that the hermitian inner product $\langle -,- \rangle$ on V induces a hermitian inner product $\langle -,- \rangle_{\mathrm{Frob}}$ on $\mathrm{End}_{\mathbb{C}}(V)$ defined as follows. Given $h : V \to V$ $\mathbb{C}$-linear, the adjoint $h^* : V \to V$ is the unique $\mathbb{C}$-linear map s.t. for all $x, y \in V$,

$$\langle h^*(x), y \rangle = \langle x, h(y) \rangle.$$

(Equivalently, $h^* : V \to V$ is the unique $\mathbb{C}$-linear map s.t.

$$\begin{tikzcd}
\bar{V} \arrow[r, "b"] \arrow[d, "h^*"] & V^* \arrow[d, "h^*"] \\
\bar{V} \arrow[r, "b"] & V^*
\end{tikzcd}$$

commutes.)

Now, for $h_1, h_2 \in \mathrm{End}_{\mathbb{C}}(V)$, we define

$$\langle h_1, h_2 \rangle_{\mathrm{Frob}} = \mathrm{tr}(h_1^* \circ h_2).$$

It is called the Frobenius inner product.

<u>Def</u> The Schur inner product on $\mathbb{C}[G]$, where G is a finite group, is the hermitian inner product given by

$$\langle f_1, f_2 \rangle_{\mathrm{Sch}} = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)}\, f_2(g). \quad //$$

The Schur inner product is invariant w.r.t. the two-sided regular repr. of $G \times G$ on $\mathbb{C}[G]$, i.e. for all $(g_1, g_2) \in G \times G$ and $f_1, f_2 \in \mathbb{C}[G]$,

$$\langle \mathrm{Reg}(g_1,g_2)(f_1), \mathrm{Reg}(g_1,g_2)(f_2)\rangle_{Sch}$$
$$= \langle f_1, f_2\rangle_{Sch}.$$

<u>Thm</u> (Schur orthogonality) G = finite.

(a) If π_1, π_2 are non-isom. irred. f.d. $\mathbb{C}$-lin. repr. of G, then the subspaces

$$M(\pi_1), M(\pi_2) \subset \mathbb{C}[G]$$

are orthogonal w.r.t. $\langle -,-\rangle_{Sch}$.

(b) If (V,π) is an irred. f.d. $\mathbb{C}$-lin. repr. that is unitary w.r.t. an hermitian inner prod. $\langle -,-\rangle$ on V, then for all $h_1, h_2 \in \mathrm{End}_{\mathbb{C}}(V)$,

$$\langle \mu'_\pi(h_1), \mu'_\pi(h_2)\rangle_{Sch} = \frac{1}{n}\langle h_1, h_2\rangle_{Frob},$$

where $n = \dim_{\mathbb{C}}(V)$.

<u>Pf</u> (a) We wish to prove that the composition

$$M(\pi_1) \xrightarrow{i} \mathbb{C}[G] \xrightarrow{p} M(\pi_2)$$

of the canonical inclusion and the

orthogonal projection w.r.t. $\langle -,- \rangle_{Sch}$ is the zero map. But the composite map is intertwining between $\mathrm{Reg}_{M(\pi_1)}$ and $\mathrm{Reg}_{M(\pi_2)}$, and we have proved before that, as repr. of $G \times G$, $\mathrm{Reg}_{M(\pi_1)}$ and $\mathrm{Reg}_{M(\pi_2)}$ are irred. and non-isomorphic. So by Schur's lemma, the map is zero.

(b) The repr. $\pi : G \to GL(V)$ induces a repr. $\rho : G \times G \to \mathrm{End}_{\mathbb{C}}(V)$ given by

$$\rho(g_1, g_2)(h) = \pi(g_1) \circ h \circ \pi(g_2^{-1}),$$

and the map $\mu'_\pi : \mathrm{End}_{\mathbb{C}}(V) \to \mathbb{C}[G]$ is intertwining between ρ and Reg. For

$$\begin{aligned}
&\mu'_\pi(\rho(g_1, g_2)(h))(g) \\
&= \mathrm{tr}(\pi(g) \circ \pi(g_1) \circ h \circ \pi(g_2^{-1})) \\
&= \mathrm{tr}(\pi(g_2^{-1}) \circ \pi(g) \circ \pi(g_1) \circ h) \\
&= \mathrm{tr}(\pi(g_2^{-1} g g_1) \circ h) \\
&= \mathrm{Reg}(g_1, g_2)(\mu'_\pi(h))(g).
\end{aligned}$$

Since π is irreducible, μ'_π is injective, and hence, defines an isom.

$$\mathrm{End}_{\mathbb{C}}(V) \xrightarrow[\sim]{\mu_\pi'} M(\pi)$$

that is intertwining w.r.t. ρ and $\mathrm{Reg}_{M(\pi)}$. We have two hermitian inner products on $\mathrm{End}_{\mathbb{C}}(V)$, namely, the Frobenius inner product $\langle -,- \rangle_{\mathrm{Frob}}$ and the inner product

$$\langle h_1, h_2 \rangle_{\mathrm{Sch}}' := \langle \mu_\pi'(h_1), \mu_\pi'(h_2) \rangle_{\mathrm{Sch}},$$

and both are ρ-invariant. Since $\rho \cong \pi \boxtimes \pi^*$ is an irred. repr. of $G \times G$, we conclude from Thm. 8 on p. 52, which we proved in lecture 6, that there exists $c \in \mathbb{R}$, $c > 0$, s.t.

$$\langle h_1, h_2 \rangle_{\mathrm{Sch}}' = c \cdot \langle h_1, h_2 \rangle_{\mathrm{Frob}},$$

for all $h_1, h_2 \in \mathrm{End}_{\mathbb{C}}(V)$.

It remains to determine c. To this end, we let $(e_1, \dots, e_n)$ be a basis of V that is orthonormal w.r.t. the given inner prod. $\langle -,- \rangle$ on V. Since π is unitary w.r.t. $\langle -,- \rangle$, the matrix

$$A(g) = (a_{ij}(g)) = (\mu_\pi(e_j \otimes e_i^*)(g))$$

that represents $\pi(g) : V \to V$ w.r.t. the basis $(e_1, \ldots, e_n)$ is then a unitary matrix. Therefore,

$$\langle a_{ij}, a_{kl} \rangle_{Sch}$$
$$:= \frac{1}{|G|} \sum_{g \in G} \overline{a_{ij}(g)}\, a_{kl}(g)$$
$$= \frac{1}{|G|} \sum_{g \in G} a_{ji}(g^{-1})\, a_{kl}(g),$$

where the last identity holds, because $(a_{ij}(g))$ is unitary. This formula gives us the idea to consider the sum

$$\sum_{i=1}^{n} \langle \alpha(e_j \otimes e_i^*), \alpha(e_l \otimes e_i^*) \rangle'_{Sch}$$
$$= \sum_{i=1}^{n} \langle \mu_\pi(e_j \otimes e_i^*), \mu_\pi(e_l \otimes e_i^*) \rangle_{Sch}$$
$$= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{n} a_{ji}(g^{-1})\, a_{il}(g)$$
$$= \delta_{jl},$$

where the last identity uses that

$A(g^{-1}) = A(g)^{-1}$. By comparison,

$$\sum_{i=1}^{n} \langle \alpha(e_j \otimes e_i^*), \alpha(e_\ell \otimes e_i^*) \rangle_{\mathrm{Frob}}$$

$$= \sum_{i=1}^{n} \mathrm{tr}\left(\alpha(e_j \otimes e_i^*)^* \circ \alpha(e_\ell \otimes e_i^*)\right)$$

$$= \sum_{i=1}^{n} \mathrm{tr}\left(\alpha(e_i \otimes e_j^*) \circ \alpha(e_\ell \otimes e_i^*)\right)$$

$$= \sum_{i=1}^{n} \delta_{j\ell} = n \cdot \delta_{j\ell}.$$

So $c = \frac{1}{n}$ as stated. //

Cor The basis $(\chi_{\pi_1}, \ldots, \chi_{\pi_q})$ of $Z(\mathbb{C}[G])$ given by the characters of the irred. f.d. repr. of G is orthonormal w.r.t. the Schur inner product.

Pf Orthogonal by (a); orthonormal by (b): Since $\chi_{\pi_i} = \mu'_{\pi_i}(\mathrm{id}_{V_i})$, we have

$$\langle \chi_{\pi_i}, \chi_{\pi_i} \rangle_{\mathrm{Sch}} = \frac{1}{n_i} \langle \mathrm{id}_{V_i}, \mathrm{id}_{V_i} \rangle_{\mathrm{Frob}},$$

where $n_i = \dim_{\mathbb{C}}(V_i)$. But, by def.,

$$\langle \mathrm{id}_{V_i}, \mathrm{id}_{V_i} \rangle_{\mathrm{Frob}} = \mathrm{tr}(\mathrm{id}_{V_i}^* \circ \mathrm{id}_{V_i})$$

$$= \mathrm{tr}(\mathrm{id}_{V_i} \circ \mathrm{id}_{V_i}) = \mathrm{tr}(\mathrm{id}_{V_i}) = n_i. \quad //$$

Cor Let $\pi : G \to GL(V)$ be a f.d. $\mathbb{C}$-lin. repr. of a finite grp. G. Then there is a non-can. isom.

$$\pi \simeq \pi_1^{m_1} \oplus \cdots \oplus \pi_q^{m_q},$$

where $m_i = \langle \chi_\pi, \chi_{\pi_i} \rangle_{Sch}$. Here $\pi_1, \dots, \pi_q$ are representatives of the isom. classes of irred. f.d. $\mathbb{C}$-lin. repr. of G.

Pf By Cor. 2 of Thm. 3 from last time, the statement is equiv. to

$$\chi_\pi = m_1 \chi_{\pi_1} + \cdots + m_q \chi_{\pi_q},$$

so follows immediately from the previous corollary.