Today: Induced representations

Let $\pi : G \to GL(V)$ be a k-linear representation of a group G. If $f: G' \to G$ is a group homomorphism from a group G', then the composite group homomorphism

$$G' \xrightarrow{f} G \xrightarrow{\pi} GL(V)$$

is a k-linear representation of G'. We write $f^*(\pi)$ for this representation and call it the restriction of π along f. We will that there are two ways to produce a repr. of G given a repr. π' of G'. We write $f_!(\pi')$ and $f_*(\pi')$ for these two k-lin. repr. of G and call them the coinduction of π' along f and the induction of π' along f, respectively. However, to define and understand these, it is better to first generalize our notion of representation. So in this lecture, I will assume some familiarity with categories, functors, and natural transformations. We have already encountered these in Lecture 5, when we discussed extension / restriction of scalars.

If G is a group, then we define a category BG to have a single object "1" and to have morphisms

$$\mathrm{Hom}_{BG}(1, 1) = G.$$

We define the composition of morphisms in BG to be the product in the group G,

$$g \circ h := g \cdot h,$$

and we define the identity morphism of the object "1" to be the identity element

$$\mathrm{id}_1 := e$$

in the group G. Let k be a field, and let Vect_k be the category, whose objects are the (right) k-vector spaces, and whose morphisms are the k-linear maps. Composition of morphisms is defined to be the composition of maps, and the identity morphism of the object V is defined to be the identity map id_V.

Now, a k-linear repr. (V, π) of G determines and is determined by the functor

$$BG \xrightarrow{\pi} \mathrm{Vect}_k$$

$$\begin{array}{ccc} 1 & \longmapsto & V \\ \circlearrowleft g & & \circlearrowleft \pi(g) \end{array}$$

It is a functor because (V, π) is a representation. Indeed,

$$\pi(g \circ h) := \pi(g \cdot h) = \pi(g) \circ \pi(h)$$

$$\pi(\mathrm{id}_1) := \pi(e) = \mathrm{id}_V .$$

We generalize this as follows. Let (X, μ) be a pair of a set X and a group homomorphism

$$G \xrightarrow{\mu} \mathrm{Aut}(X) .$$

We say that (X, μ) is a "set with left G-action" or a "left G-set." We also abuse notation and write X instead of (X, μ), and

$$g \cdot x := \mu(g)(x) .$$

If X is a left G-set, then we define a category

$$[G\backslash X]$$

called the "translation groupoid of X" as follows. The set of objects is

$$\mathrm{ob}([G\backslash X]) = X,$$

and the set of morphisms is

$$\mathrm{mor}([G\backslash X]) = G \times X.$$

Here $(g, x) \in G \times X$ is a morphism from $x \in X$ to $g \cdot x \in X$. (Note that $g \cdot x = x$ if $g \in G_x$.) The identity morphism of $x \in X$ is $(e, x) \in G \times X$, and the composition of $(g_1, g_0 x)$ and (g_0, x) is $(g_1 g_0, x)$:

$$x \xrightarrow{(g_0, x)} g_0 x \xrightarrow{(g_1, g_0 x)} g_1 g_0 x$$

$$x \xrightarrow{(g_1 g_0, x)} g_1 g_0 x$$

We now define representations of this category in right k-vector

$$BG = [G \backslash \{1\}]$$

spaces as before, namely, as functors

$$[G \backslash X] \xrightarrow{\pi} \mathrm{Vect}_k$$

$$\begin{array}{ccc} x & & \pi(x) \\ \big\downarrow (g,x) & \longmapsto & \big\downarrow \pi(g,x) \\ gx & & \pi(gx) \end{array}$$

However, we do not call such a functor a representation. Instead, we say that π is a "quasi-coherent sheaf on $[G \backslash X]$," and we write

$$\mathrm{QCoh}([G \backslash X]) := \mathrm{Fun}([G \backslash X], \mathrm{Vect}_k)$$

for the category, whose objects are functors $\pi : [G \backslash X] \to \mathrm{Vect}_k$, and whose morphisms are natural transformations between such functors. So a morphism

$$\pi \xrightarrow{h} \pi'$$

is a family $(h_x)_{x \in X}$ of k-lin. maps

$$\pi(x) \xrightarrow{h_x} \pi'(x)$$

such that for all $(g,x) \in G \times X$, the diagram

$$\begin{array}{ccc} \pi(x) & \xrightarrow{h_x} & \pi'(x) \\ \downarrow \pi(g,x) & & \downarrow \pi'(g,x) \\ \pi(gx) & \xrightarrow{h_{gx}} & \pi'(gx) \end{array}$$

commutes. Note that

$$\mathrm{Rep}_k(G) = \mathrm{QCoh}([G \backslash \{1\}])$$

is the category of k-linear repr. of G and intertwining k-linear maps.

It happens rarely that categories are equal or even that they are isomorphic. Being equal and being isomorphic are not good notions for categories. (In fact, they are so-called evil notions.) Instead, the notion of equivalence is a good notion. A functor

$$\mathcal{C} \xrightarrow{F} \mathcal{C}'$$

from a category $\mathcal{C}$ to a category $\mathcal{C}'$

is defined to be an equivalence if there exists a functor

$$\mathcal{C}' \xrightarrow{H} \mathcal{C}$$

and natural transformations

$$F \circ H \xrightarrow{\varepsilon} \mathrm{id}_{\mathcal{C}'}$$

$$\mathrm{id}_{\mathcal{C}} \xrightarrow{\eta} H \circ F$$

such that for all objects $c' \in \mathcal{C}'$ and $c \in \mathcal{C}$, the morphisms

$$\begin{array}{ccc} (F \circ H)(c') & \xrightarrow{\varepsilon_{c'}} & \mathrm{id}_{\mathcal{C}'}(c') \\ \shortparallel & & \shortparallel \\ F(H(c')) & & c' \end{array}$$

$$\begin{array}{ccc} \mathrm{id}_{\mathcal{C}}(c) & \xrightarrow{\eta_c} & (H \circ F)(c) \\ \shortparallel & & \shortparallel \\ c & & H(F(c)) \end{array}$$

are isomorphisms. (We then say that ε and η are natural isomorphisms.)

In this situation, we say that H is a quasi-inverse of F. Note that H is <u>not</u> uniquely determined by F.

Rmk If $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories, then it is always possible to choose $H : \mathcal{C}' \to \mathcal{C}$ and $\varepsilon : F \circ H \to \mathrm{id}_{\mathcal{C}'}$ and $\eta : \mathrm{id}_{\mathcal{C}} \to H \circ F$ such that the diagrams

$$\begin{array}{ccc} F & \xrightarrow{F \circ \eta} & F \circ H \circ F \\ & \searrow^{=} \quad \swarrow_{\varepsilon \circ F} & \\ & F & \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{\eta \circ H} & H \circ F \circ H \\ & \searrow^{=} \quad \swarrow_{H \circ \varepsilon} & \\ & H & \end{array}$$

commute. In this case, we say that $(F, H, \varepsilon, \eta)$ is an adjoint equivalence from $\mathcal{C}$ to $\mathcal{C}'$. //

Prop Let G be a group, and let X be a transitive left G-set. Let $x \in X$, and let $G_x \subset G$ be the isotropy subgroup. Then the canonical inclusion functor

$$BG_x = [G_x \backslash \{x\}] \xrightarrow{\ i\ } [G \backslash X]$$

is an equivalence.

Pf To define a quasi-inverse

$$[G \backslash X] \xrightarrow{\ H\ } [G_x \backslash \{x\}],$$

we choose for all $y \in X$, an element $h_y \in G$ s.t. $h_y \cdot x = y$, and let H be the functor defined by

$$\begin{array}{ccc} y_0 & & x \\ \downarrow (g, y_0) & \overset{H}{\longmapsto} & \downarrow (h_{y_1}^{-1} g h_{y_0}, x) \\ y_1 & & x \end{array}$$

We define $\varepsilon : i \circ H \longrightarrow id_{[G \backslash X]}$ by

$$\begin{array}{ccc} (i \circ H)(y) & \xrightarrow{\varepsilon_y} & y \\ \| & & \| \\ x & \xrightarrow{(h_y, x)} & y \end{array}$$

and $\eta : id_{[G \backslash \{x\}]} \longrightarrow H \circ i$ by

$$\begin{array}{ccc} x & \xrightarrow{\eta_x} & (H \circ i)(x) \\ \| & & \| \\ x & \xrightarrow{(h_x^{-1}, x)} & x \end{array}$$

Now, ε is a natural transf., since

$$\begin{array}{ccc} x & \xrightarrow{(h_{y_0}, x)} & y_0 \\ \downarrow (h_{y_1}^{-1} g h_{y_0}, x) & & \downarrow (g, y_0) \\ x & \xrightarrow[(h_{y_1}, x)]{} & y_1 \end{array}$$

commutes, and η is a natural transformation, since

$$\begin{array}{ccc} x & \xrightarrow{(h_x^{-1},\, x)} & x \\ \downarrow{\scriptstyle (g,x)} & & \downarrow{\scriptstyle (h_x^{-1} g h_x,\, x)} \\ x & \xrightarrow{(h_x^{-1},\, x)} & x \end{array}$$

commutes. Both ε and η are automatically natural isomorphisms, since all morphisms in $[G\backslash X]$ and $[G\backslash\{x\}]$ are invertible. //

<u>Cor</u> In the situation of the proposition, the restriction along i,

$$\mathrm{QCoh}([G\backslash X]) \xrightarrow{i^*} \mathrm{Rep}_k(G_x)$$
$$\pi \longmapsto \pi \circ i$$

is an equivalence.

<u>Pf</u> If H is a quasi-inverse of i, then H^* is a quasi-inverse of i^*. //

<u>Rmk</u> Something better is true: As opposed to the equivalence i, the equivalence i^* has a <u>canonical</u>

quasi-inverse $i_! \simeq i_*$ given by the left or right Kan extension along i. Explicitly, the functors $i_!$ and i_* are given by

$$i_!(\pi)(y) = \operatorname{colim}\left(\pi \mid BG_x \times_{[G\backslash X]} [G\backslash X]_{/y}\right)$$
$$= \Big(\bigoplus_{x \xrightarrow{(h,x)} y} \pi(x)\Big) / G_x$$

$$i_*(\pi)(y) = \lim\left(\pi \mid BG_x \times_{[G\backslash X]} [G\backslash X]_{y/}\right)$$
$$= \Big(\prod_{y \xrightarrow{(h,x)} x} \pi(x)\Big)^{G_x}$$

It is the possibility of forming sum and product in Vect_k (which we cannot do in $[G\backslash X]$) that make it possible to define these functors. //

<u>Ex</u> If $H \subset G$ is a subgrp., then left multiplication by G on G/H makes $X = G/H$ a transitive left G-set. Choosing $x = H = eH \in G/H$, we obtain an equivalence

$$\mathrm{QCoh}([G\backslash(G/H)]) \xrightarrow{i^*} \mathrm{Rep}_k(H).$$
//

Let X be any left G-set, and let

$$X \xrightarrow{p} G \backslash X$$

be the canonical projection onto the set of orbits. (We note that

$$G\backslash X = \pi_0([G\backslash X])$$

is the set of isomorphism classes of objects in $[G\backslash X]$.) If we choose an element

$$x = s(\bar{x}) \in \bar{x} = G\cdot x \in G\backslash X$$

in each orbit, then we obtain an isomorphism of left G-sets

$$\coprod_{\bar{x} \in G\backslash X} G/G_x \longrightarrow X$$

$$gG_x \longmapsto g\cdot x$$

It induces an equivalence

$$\coprod_{\bar{x} \in G\backslash X} [G\backslash (G/G_x)] \longrightarrow [G\backslash X],$$

and hence,

$$\coprod_{\bar{x} \in G \backslash X} BG_x \longrightarrow [G \backslash X]$$

Taking functors into Vect_k, we get:

Prop Let G be a group, and let X be a left G-set. A choice of representatives $x \in \bar{x} \in G \backslash X$ of each orbit determines an equivalence

$$\mathrm{QCoh}([G \backslash X]) \longrightarrow \prod_{\bar{x} \in G \backslash X} \mathrm{Rep}_k(G_x).$$ //

In the proposition, the (big) advantage of the left-hand side compared to the right-hand side is that it only depends on the left G-set X and not on further choices. We are now ready to define coinduction and induction.

So let G be a group, and let $f \colon X \to Y$ be a G-equivariant map between left G-sets. We do not assume that G, X, or Y is finite. The map f induces a functor

$$[G\backslash X] \xrightarrow{f} [G\backslash Y]$$

$$\begin{array}{ccc} x_0 & & f(x_0) \\ \downarrow (g, x_0) & \longmapsto & \downarrow (g, f(x_0)) \\ x_1 & & f(x_1) \end{array}$$

which, by abuse of notation, we also denote by f. This functor induces a functor

$$\mathrm{QCoh}([G\backslash Y]) \xrightarrow{f^*} \mathrm{QCoh}([G\backslash X]).$$

$$\tau \longmapsto \tau \circ f$$

The functor f^* has a left adjoint functor $f_!$ given by left Kan extension along f and a right adjoint functor f_* given by right Kan extension along f. We spell these out in detail. The functor

$$\mathrm{QCoh}([G\backslash X]) \xrightarrow{f_!} \mathrm{QCoh}([G\backslash Y])$$

is given by

$$\begin{array}{ccc} f_!(\pi)(y) & = & \bigoplus_{f(x)=y} \pi(x) \\ \downarrow f_!(\pi)(g,y) & & \downarrow \bigoplus \pi(g,x) \\ f_!(\pi)(gy) & = & \bigoplus_{f(x)=y} \pi(gx) \end{array}.$$

Here the sums are indexed by

$$f^{-1}(y) = \{x \in X \mid f(x) = y\},$$

and we use that, since f is G-equivariant,

$$\bigoplus_{f(x')=gy} \pi(x') = \bigoplus_{f(x)=y} \pi(gx).$$

We define natural transformations

$$f_! f^*(\tau) \xrightarrow{\varepsilon_\tau} \tau$$

$$\pi \xrightarrow{\eta_\pi} f^* f_!(\pi)$$

as follows: The map

$$\begin{array}{ccc} f_! f^*(\tau)(y) & \xrightarrow{\varepsilon_{\tau,y}} & \tau(y) \\ \| & & \| \\ \bigoplus_{f(x)=y} \tau(f(x)) & \xrightarrow{\nabla} & \tau(y) \end{array}$$

is the fold map (or co-diagonal), whose restriction to each summand is the identity map of $\tau(y)$.

The map

$$\begin{array}{ccc} \pi(x) & \xrightarrow{\eta_{\pi,x}} & f^* f_! (\pi)(x) \\ \| & & \| \\ \pi(x) & \xrightarrow{i_x} & \bigoplus\limits_{f(x')=f(x)} \pi(x') \end{array}$$

is the inclusion of the summand indexed by x. One verifies that ε and η are well-defined natural transformations, and that the triangle diagrams

$$\begin{array}{ccc} f_! & \xrightarrow{f_!\eta} & f_! f^* f_! \\ & \searrow^{=} & \downarrow \varepsilon f_! \\ & & f_! \end{array} \qquad \begin{array}{ccc} f^* & \xrightarrow{\eta f^*} & f^* f_! f^* \\ & \searrow^{=} & \downarrow f^*\varepsilon \\ & & f^* \end{array}$$

commute. As explained in Lecture 5, this (immediately) implies:

Thm (Frobenius reciprocity – I) In the situation above, the maps

$$\mathrm{Hom}(f_!(\pi), \tau) \rightleftarrows \mathrm{Hom}(\pi, f^*(\tau))$$

$$f_!(\pi) \xrightarrow{h} \tau \quad \longmapsto \quad \pi \xrightarrow{\eta_\pi} f^* f_!(\tau) \xrightarrow{f^*(h)} f^*(\tau)$$

$$f_!(\pi) \xrightarrow{f_!(k)} f_! f^*(\tau) \xrightarrow{\varepsilon_\tau} \tau \quad \longleftarrow\!\shortmid \quad \pi \xrightarrow{k} f^*(\tau)$$

are each others inverses. //

Similarly, the functor

$$\mathrm{QCoh}([G\backslash X]) \xrightarrow{f_*} \mathrm{QCoh}([G\backslash Y])$$

is given by

$$\begin{array}{ccc} f_*(\pi)(y) & = & \prod_{f(x)=y} \pi(x) \\ \Big\downarrow f_*(\pi)(g,y) & & \Big\downarrow \prod \pi(g,x) \\ f_*(\pi)(y) & = & \prod_{f(x)=y} \pi(gx) \end{array}$$

and the natural transformations

$$f^*f_*(\pi) \xrightarrow{\varepsilon_\pi} \pi$$

$$\tau \xrightarrow{\eta_\tau} f_*f^*(\tau)$$

are defined by the projection

$$\begin{array}{ccc} f^*f_*(\pi)(x) & \xrightarrow{\varepsilon_{\pi,x}} & \pi(x) \\ \| & & \| \\ \prod_{f(x')=f(x)} \pi(x') & \xrightarrow{\mathrm{pr}_x} & \pi(x) \end{array}$$

on the factor $x'=x$, and the diagonal

$$\begin{array}{ccc} \tau(y) & \xrightarrow{\eta_{\tau,y}} & f_*f^*(\tau)(y) \\ \| & & \| \\ \tau(y) & \xrightarrow{\Delta} & \prod_{f(x)=y} \tau(f(x)) \end{array}$$

respectively. Again,

$$\begin{array}{ccc} f^* & \xrightarrow{f^*\eta} & f^*f_*f^* \\ & \searrow^{\Vert} \quad \swarrow_{\varepsilon f^*} & \\ & f^* & \end{array} \qquad \begin{array}{ccc} f_* & \xrightarrow{\eta f_*} & f_*f^*f_* \\ & \searrow^{\Vert} \quad \swarrow_{f_*\varepsilon} & \\ & f_* & \end{array}$$

commute, so we obtain:

<u>Thm</u> (Frobenius reciprocity - II) In the situation above, the maps

$$\mathrm{Hom}(f^*(\tau), \pi) \rightleftarrows \mathrm{Hom}(\tau, f_*(\pi))$$

$$f^*(\tau) \xrightarrow{h} \pi \quad \longmapsto \quad \tau \xrightarrow{\eta_\tau} f_*f^*(\tau) \xrightarrow{f_*(h)} f_*(\pi)$$

$$f^*(\tau) \xrightarrow{f^*(k)} f^*f_*(\pi) \xrightarrow{\varepsilon_\pi} \pi \quad \longleftarrow\!\!\mid \quad \tau \xrightarrow{k} f_*(\pi)$$

are each others inverses. //

There is a canonical natural transformation

$$f_! \xrightarrow{Nm_f} f_*$$

called the norm map. In our description of $f_!$ and f_*, it is given by the canonical inclusion

$$\begin{array}{ccc} f_!(\pi)(y) & \longrightarrow & f_*(\pi)(y) \\ \| & & \| \\ \bigoplus_{f(x)=y} \pi(x) & \longrightarrow & \prod_{f(x)=y} \pi(x) \end{array}$$

of the sum in the product. (But a better definition is given in Lurie's Higher Algebra, Section 6.1.6.)
We will say that the map

$$X \xrightarrow{f} Y$$

is finite, if for all $y \in Y$, the inverse image $f^{-1}(y) \subset X$ is finite.

__Thm__ If $f: X \to Y$ is finite, then

$$f_! \xrightarrow{\mathrm{Nm}_f} f_*$$

is a natural isomorphism.

__Pf__ Indeed, finite sums and finite products of k-vector spaces agree. //

Finally, we will prove an important result called the base change thm.
A commutative square of left

G-sets and G-equivariant maps

$$\begin{array}{ccc} X' & \xrightarrow{h'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{h} & Y \end{array} \qquad (\dagger)$$

induces a commutative square of categories and functors

$$\begin{array}{ccc} QCoh([G\backslash X']) & \xleftarrow{h'^*} & QCoh([G\backslash X]) \\ \uparrow{\scriptstyle f'^*} & & \uparrow{\scriptstyle f^*} \\ QCoh([G\backslash Y']) & \xleftarrow{h^*} & QCoh([G\backslash Y]) \end{array}.$$

The diagram $(\dagger)$ is said to be cartesian if the map

$$X' \xrightarrow{(h', f')} X \times_Y Y'$$
$$\| $$
$$\{(x, y') \in X \times Y' \mid f(x) = h(y')\}$$

is a bijection. In this case, also

$$\begin{array}{ccc} QCoh([G\backslash X']) & \xleftarrow{h'^*} & QCoh([G\backslash X]) \\ \downarrow{\scriptstyle f'_! \text{ (resp. } f'_*)} & & \downarrow{\scriptstyle f_! \text{ (resp. } f_*)} \\ QCoh([G\backslash Y']) & \xleftarrow{h^*} & QCoh([G\backslash Y]) \end{array}$$

commute. More precisely:

<u>Thm</u> (Base change) Given a cartesian square (†) of G-sets, the following hold:

(1) The composite natural transf.

$$\begin{array}{ccc} f'_! h'^* & \longrightarrow & h^* f_! \\ \big\downarrow {\scriptstyle f'_! h'^* \eta} & & \big\uparrow {\scriptstyle \varepsilon h^* f_!} \\ f'_! h'^* f^* f_! & = & f'_! f'^* h^* f_! \end{array}$$

is a natural isomorphism.

(2) The composite nat. transf.

$$\begin{array}{ccc} h^* f_* & \longrightarrow & f'_* h'^* \\ \big\downarrow {\scriptstyle \eta h^* f_*} & & \big\uparrow {\scriptstyle f'_* h'^* \varepsilon} \\ f'_* f'^* h^* f_* & = & f'_* h'^* f^* f_* \end{array}$$

is a natural isomorphism.

<u>Pf</u> We prove (2); the proof of (1) is analogous. Let $\pi \in QCoh([G \backslash X])$, and let $y' \in Y'$. Then

$$h^* f_*(\pi)(y') = f_*(\pi)(h(y'))$$

$$= \prod_{f(x)=h(y')} \pi(x)$$

$$f'_* h'^*(\pi)(y') = \prod_{f'(x')=y'} h'^*(\pi)(x')$$

$$= \prod_{f'(x')=y'} \pi(h'(x')),$$

and since (†) is cartesian, the two products agree. One checks that the map in the statement takes the factor indexed by (x, y') with $f(x) = h(y')$ to the factor indexed by the unique $x' \in X'$ s.t. $f'(x') = y'$ and $h'(x') = x$ by the identity map

$$\pi(x) \xrightarrow{\text{id}} \pi(h'(x')).$$

So it is an isom. as stated. //

Next time, we will specialize to the special case, where G is a finite group, $H, K < G$ are two subgroups, and (†) is the cartesian square

$$\begin{array}{ccc} G/H \times G/K & \xrightarrow{h'} & G/H \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ G/K & \xrightarrow{h} & G/G \end{array}$$

where f and h are the unique maps (G/G has only one element) and h' and f' are the two can. projections. Note that while G/H and G/K are transitive left G-sets, $G/H \times G/K$ is not transitive (unless $H=G$ or $K=G$). Decomposing $G/H \times G/K$ into orbits is rather complicated!

<u>Rmk</u> The formulas for $f_!$ and f_* are based on the fact that the square of ∞-groupoids

$$\begin{array}{ccc} X & \longrightarrow & [G \backslash X] \\ \downarrow & & \downarrow \\ Y & \longrightarrow & [G \backslash Y] \end{array}$$

is cartesian. /