

Algebra III/Introduction to Algebra III: Representation Theory

Due: Please upload solutions to NUCT by Tuesday, August 4, 2020.

Problem 1. Let $G = ((G, \mathcal{O}_G), \mu, \iota, e)$ be a Lie group, that is, a group object in the category of smooth manifolds and morphisms of smooth manifolds, and let

$$\mathfrak{g} = T(G, \mathcal{O}_G)_e$$

be the real vector space given by the tangent space at the identity element.¹ This problem will show that the fact that the inversion ι is a morphism of smooth manifolds is a consequence of the facts that the multiplication μ is a morphism of smooth manifolds and that (G, μ, ι) is a group.

- (a) Let $p_1, p_2: G \times G \rightarrow G$ be the two projections, and let $i_1, i_2: G \rightarrow G \times G$ be the maps defined by $i_1(g) = (g, e)$ and $i_2(g) = (e, g)$. Show that the diagram

$$T(G, \mathcal{O}_G)_e \xleftarrow[\text{di}_{1,e}]{dp_{1,e}} T(G \times G, \mathcal{O}_{G \times G})_{(e,e)} \xleftarrow[\text{di}_{2,e}]{dp_{2,e}} T(G, \mathcal{O}_G)_e$$

is a biproduct in the category of real vector spaces.²

- (b) Show that the composite map

$$\mathfrak{g} \oplus \mathfrak{g} \xrightarrow{di_{1,e} + di_{2,e}} T(G \times G, \mathcal{O}_{G \times G})_{(e,e)} \xrightarrow{d\mu_{(e,e)}} \mathfrak{g}$$

maps (v, w) to the vector space sum $v + w$.

Hence, to first order at $e \in G$, all group structures on $(G, \mathcal{O}_G)$ look the same! Let

$$(G \times G, \mathcal{O}_{G \times G}) \xrightarrow{\sigma} (G \times G, \mathcal{O}_{G \times G})$$

be the “shearing map” defined by $\sigma(g, h) = (g, \mu(g, h))$. It is a morphism of smooth manifolds, because μ is a morphism of smooth manifolds, and it is a bijection, because (G, μ, ι) is a group.

- (c) Show that the composite map

$$\begin{array}{ccc} \mathfrak{g} \oplus \mathfrak{g} & \xrightarrow{di_{1,e} + di_{2,e}} & T(G \times G, \mathcal{O}_{G \times G})_{(e,e)} \\ & & \downarrow d\sigma_{(e,e)} \\ & & T(G \times G, \mathcal{O}_{G \times G})_{(e,e)} \xrightarrow{(dp_{1,e}, dp_{2,e})} \mathfrak{g} \oplus \mathfrak{g} \end{array}$$

maps (v, w) to $(v, v + w)$, and conclude that it is an isomorphism.

- (d) Show that σ is both an immersion and submersion, and conclude that it is a diffeomorphism. [Hint: Use left translation and right translation to reduce to what was proved in (c).]

This shows that $\iota = p_2 \circ \sigma^{-1} \circ i_1$ is automatically a morphism of smooth manifolds.

¹ The group structure on G gives rise to a structure of real Lie on $\mathfrak{g}$, but we will not need this fact here.

² If you do not know what this means, then either look up the definition in MacLane’s book, *Categories for the Working Mathematician*, or in Wikipedia.