

Algebra III/Introduction to Algebra III: Representation Theory

Due: Please upload solutions to NUCT by Tuesday, May 26, 2020.

Problem 1. Let $\pi: G \rightarrow \mathrm{GL}(V)$ be a real representation of finite dimension d . Show that if π is irreducible and if d is odd, then the complexification

$$\pi_{\mathbb{C}}: G \rightarrow \mathrm{GL}(V_{\mathbb{C}})$$

is an irreducible complex representation.

Problem 2. Let $f: k \rightarrow k'$ be an extension of fields, let $f^*: \mathrm{Vect}_k \rightarrow \mathrm{Vect}_{k'}$ be extension of scalars along f , and let $f_*: \mathrm{Vect}_{k'} \rightarrow \mathrm{Vect}_k$ be restriction of scalars along f . Given k' -vector space V'_1 and V'_2 , we define

$$f_*(V'_1) \otimes_k f_*(V'_2) \xrightarrow{\mu} f_*(V'_1 \otimes_{k'} V'_2)$$

be the k -linear map given by $\mu(\mathbf{x}'_1 \otimes_k \mathbf{x}'_2) = \mathbf{x}'_1 \otimes_{k'} \mathbf{x}'_2$.

(a) Give an example to show that, in general, the map μ is *not* an isomorphism.

Given k -vector spaces V_1 and V_2 , the composite k -linear map

$$V_1 \otimes_k V_2 \xrightarrow{\eta \otimes \eta} f_* f^*(V_1) \otimes_k f_* f^*(V_2) \xrightarrow{\mu} f_*(f^*(V_1) \otimes_{k'} f^*(V_2))$$

determines a unique k' -linear map

$$f^*(V_1 \otimes_k V_2) \xrightarrow{\tilde{\mu}} f^*(V_1) \otimes_{k'} f^*(V_2).$$

(b) Calculate that for $\mathbf{x}_1 \in V_1$, $\mathbf{x}_2 \in V_2$, and $b \in k'$,

$$\tilde{\mu}((\mathbf{x}_1 \otimes_k \mathbf{x}_2) \otimes_k b) = (\mathbf{x}_1 \otimes_k b) \otimes_{k'} (\mathbf{x}_2 \otimes_k 1) = (\mathbf{x}_1 \otimes_k 1) \otimes_{k'} (\mathbf{x}_2 \otimes_k b).$$

(c) Show that the map $\tilde{\mu}$ is an isomorphism. [Hint: Write down a k' -linear map

$$f^*(V_1) \otimes_{k'} f^*(V_2) \longrightarrow f^*(V_1 \otimes_k V_2)$$

and show that it is inverse to $\tilde{\mu}$.]