Algebra III/Introduction to Algebra III: Representation Theory

Due: Please upload solutions to NUCT by Tuesday, June 2, 2020.

Problem 1. We consider the additive group $G = (\mathbb{R}, +)$ of real numbers and the represenation $\pi\colon G \to \mathrm{GL}(V)$ on $V = \mathbb{R}^2$ defined by

$$\pi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Since π is irreducible, the ring $D = \mathrm{End}(\pi) \subset \mathrm{End}_{\mathbb{R}}(V)$ is a real division algebra. Determine the structure of D.

Problem 2. Let G be an abelian group, and let $\pi\colon G \to \mathrm{GL}(V)$ be an irreducible real representation. Show that either $\dim_{\mathbb{R}}(V) = 1$ or $\dim_{\mathbb{R}}(V) = 2$.

Problem 3. Let $k = \overline{k}$ be an algebraically closed field,[1] let $\pi\colon G \to \mathrm{GL}(V)$ be a finite dimensional irreducible k-linear representation, and let

$$A = \mathrm{span}(\pi(g) \mid g \in G\,) \subset \mathrm{End}_k(V).$$

Show that $A = \mathrm{End}_k(V)$.

[Hint: Consider the representation $\rho\colon G \times G \to \mathrm{GL}(\mathrm{End}_k(V))$ defined by

$$\rho(g_1, g_2)(f)(\boldsymbol{x}) = \pi(g_1)(f(\pi(g_2^{-1})(\boldsymbol{x}))).$$

Show that $A \subset \mathrm{End}_k(V)$ is ρ-invariant and that $\rho \simeq \pi \boxtimes \pi^*$. Then use a theorem.]

[1] You are welcome to assume that $k = \mathbb{C}$.