Algebra III/Introduction to Algebra III: Representation Theory

Due: Please upload solutions to NUCT by Tuesday, June 23, 2020.

Problem 1. Let $G = \Sigma_4$ be the symmetric group on 4 letters, and recall that G has 5 conjugacy classes of elements with representatives $x_1 = e$, $x_2 = (12)$, $x_3 = (12)(34)$, $x_4 = (123)$, and $x_5 = (1234)$. The number $\ell_i = [G : G_{x_i}]$ of elements in the conjugacy class containing x_i is $\ell_1 = 1$, $\ell_2 = 6$, $\ell_3 = 3$, $\ell_4 = 8$, and $\ell_5 = 6$. In Lecture 7, we defined 5 pairwise non-isomorphic irreducible finite dimensional complex representations of G, namely, the trivial 1-dimensional representation π_1, the 1-dimensional sign representation π_2, the 3-dimensional standard representation π_3, the 3-dimensional representation $\pi_4 = \pi_2 \otimes \pi_3$, and a 2-dimensional representation π_5 obtained from the standard representation $\overline{\pi}_3$ of $\overline{G} = \Sigma_3$.[1] Let χ_i be the character of π_i. By using the definition of π_1, π_2, and π_3 it is (rather) easy to find the values of χ_1, χ_2, and χ_3 on the $x_1, \dots, x_5$:

x_i	e	(12)	$(12)(34)$	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	-1	0	-1
χ_4	3	?	?	?	?
χ_5	2	?	?	?	?
ℓ_i	1	6	3	8	6

Please answer the following questions:

(a) Find the values of χ_4 on $x_1, \dots, x_5$.
[Hint: Use that $\chi_{\pi \otimes \tau} = \chi_\pi \cdot \chi_\tau$.]

(b) Find the values of χ_5 on $x_1, \dots, x_5$.
[Hint: Use that $(\chi_1, \chi_2, \chi_3, \chi_4, \chi_5)$ is a basis of $Z(\mathbb{C}[G])$ and orthonormal with respect to the Schur inner product to produce a system of 4 linear equations in the 4 variables $\chi_5(x_2)$, $\chi_5(x_3)$, $\chi_5(x_4)$, and $\chi_5(x_5)$ and solve it.]

(c) Find the non-negative integers $m_1, \dots, m_5$ such that

$$\pi_3 \otimes \pi_3 \simeq \pi_1^{m_1} + \pi_2^{m_2} + \pi_3^{m_3} + \pi_4^{m_4} + \pi_5^{m_5}.$$

Here π^m denotes the direct sum of m copies of π.
[Hint: Use that $(\chi_1, \chi_2, \chi_3, \chi_4, \chi_5)$ is a basis of $Z(\mathbb{C}[G])$ that is orthonormal with respect to the Schur hermitian inner product.]

[1] More precisely, the subgroup $N = \{e, (12)(34), (13)(24), (14)(23)\} \subset G$ is normal, and each coset $gN \in G/N$ contains a unique element $g' \in gN$ such that $g'(4) = 4$. Therefore, we can define a map $p\colon G = \Sigma_4 \to \overline{G} = \Sigma_3$ by sending $g \in G$ to $p(g) = g'|_{\{1,2,3\}}$. This map is a group homomorphism, and $\pi_5 = \overline{\pi}_3 \circ p\colon G \to \mathrm{GL}(\mathbb{C}^2)$.