4. Representations of groups

Let k be a field, and let G be a group. We recall that the group ring $k[G]$ is defined to be the free k-module with basis $(g)_{g \in G}$ and with multiplication given by the convolution product

$$(\sum_{g \in G} a_g g) * (\sum_{g \in G} b_g g) = \sum_{g \in G} (\sum_{hk = g} a_h b_k) g.$$

A left $k[G]$-module V is also said to be a k-linear representation of G. It determines and is determined by the ring homomorphism

$$\rho \colon k[G] \to \operatorname{End}_k(V)$$

defined by

$$\rho(\sum_{g \in G} a_g g)(\boldsymbol{v}) = (\sum_{g \in G} a_g g) \cdot \boldsymbol{v}.$$

In particular, this map assigns to every element $g \in G$ the k-linear endomorphism $\rho(g) \colon V \to V$ of the k-vector space V, so, in this way, the elements of the (abstract) group G become represented by (concrete) endomorphisms of the vector space V, whence the name. In fact, the k-linear endomorphism $\rho(g) \colon V \to V$ is a k-linear automorphism with inverse $\rho(g^{-1}) \colon V \to V$.

We have proved Maschke's theorem, which states that if the group G is finite and if its order is invertible in k, then the group ring $k[G]$ is a semi-simple ring. This implies that every finitely generated left $k[G]$-module is semi-simple.

Example 4.1. Let C_n be a cyclic group of order n. Maschke's theorem shows that the group ring $\mathbb{C}[C_n]$ is semi-simple, and we now determine its structure. We choose a generator $g \in C_n$ and a primitive nth root of unity $\zeta_n \in \mathbb{C}$. For $0 \leqslant k < n$, we define the left $\mathbb{C}[C_n]$-module $\mathbb{C}(\zeta_n^k)$ to be the sub-$\mathbb{C}$-vector space of $\mathbb{C}$ generated by the family $(\zeta_n^{jk})_{0 \leqslant j < n}$ with the left $\mathbb{C}[C_n]$-module structure given by

$$(\sum_{j=0}^{n-1} a_j g^j) \cdot z = \sum_{j=0}^{n-1} a_j \zeta_n^{jk} z.$$

The left $\mathbb{C}[C_n]$-module $\mathbb{C}(\zeta_n^k)$ is simple. For as a $\mathbb{C}$-vector space, $\mathbb{C}(\zeta_n^k) = \mathbb{C}$, and therefore has no non-trivial proper submodules. Suppose that $f \colon \mathbb{C}(\zeta_n^k) \to \mathbb{C}(\zeta_n^l)$ is a $\mathbb{C}[C_n]$-linear isomorphism. Then we have

$$\zeta_n^k f(1) = f(\zeta_n^k) = f(g \cdot 1) = g \cdot f(1) = \zeta_n^l f(1),$$

where the first and third equalities follows from $\mathbb{C}[C_n]$-linearity. Since $f(1) \neq 0$, we conclude that $k = l$. So the n simple left $\mathbb{C}[C_n]$-modules $\mathbb{C}(\zeta_n^k)$, $0 \leqslant k < n$, are pairwise non-isomorphic, and therefore, Theorem 3.5 shows that the map

$$f \colon \bigoplus_{k=0}^{n-1} \mathbb{C}(\zeta_n^k) \to \mathbb{C}[C_n]$$

that to $\boldsymbol{w} = (w_k)_{0 \leqslant k < n}$ assigns $\boldsymbol{z} = f(\boldsymbol{w}) = \sum_{0 \leqslant k < n} w_k \boldsymbol{e}_k$, where

$$\boldsymbol{e}_k = \sum_{j=0}^{n-1} \zeta_n^{-jk} g^j \in \mathbb{C}[C_n],$$

is an isomorphism of left $\mathbb{C}[C_n]$-modules. The ring $\operatorname{End}_{\mathbb{C}[C_n]}(\mathbb{C}(\zeta_n^k))$ is isomorphic to the field $\mathbb{C}$ for all $0 \leqslant k < n$.

REMARK 4.2. The identification in Example 4.1 of the complex group ring $\mathbb{C}[C_n]$ is called the *discrete Fourier transform* and is of great practical importance. Indeed, it is the foundation for the digital representation of various signals, including audio and video signals. Let us define a signal to be a function $\sigma\colon \mathbb{R} \to \mathbb{C}$. We call $\sigma(t) \in \mathbb{C}$ the signal at time $t \in \mathbb{R}$.[1] To digitally represent the signal in the time interval $[0,1)$, we choose some (large) sampling rate n and record the signal at the times $t = {}^{j}\!/_{n}$ with $0 \leqslant j < n$. That is, we record the vector

$$\boldsymbol{z} = \sum_{j=0}^{n-1} \sigma({}^{j}\!/_{n}) g^j \in \mathbb{C}[C_n]$$

and think of this as an approximation of the signal in the time interval $[0,1)$. We wish to determine the unique vector

$$\boldsymbol{w} = (w_k)_{0 \leqslant k < n} \in \bigoplus_{k=0}^{n-1} \mathbb{C}(\zeta_n^k)$$

such that $f(\boldsymbol{w}) = \boldsymbol{z}$. We interpret the component w_k as the contribution to the signal with frequency ${}^{k}\!/_{n}$. We think of $\boldsymbol{w}$ as a better representation of the signal than $\boldsymbol{z}$; if we compress the signal before e.g. transmitting it or storing it, then we loose less essential information by compressing $\boldsymbol{w}$ instead of $\boldsymbol{z}$. We let

$$\boldsymbol{e}'_k \in \bigoplus_{l=0}^{n-1} \mathbb{C}(\zeta_n^l)$$

be the unique vector such that $f(\boldsymbol{e}'_k) = \boldsymbol{e}_k$. So $\boldsymbol{e}'_k$ has lth component $\delta_{kl} \in \mathbb{C}(\zeta_n^l)$, where we use Kronecker's symbol. Now, the matrix $P_n \in M_n(\mathbb{C})$ that represents

$$f\colon \bigoplus_{k=0}^{n-1} \mathbb{C}(\zeta_n^k) \to \mathbb{C}[C_n]$$

with respect to the basis $(\boldsymbol{e}'_k)_{0 \leqslant k < n}$ of the domain and the basis $(g^j)_{0 \leqslant j < n}$ of the codomain is given by

$$P_n = (\zeta_n^{-jk})_{0 \leqslant j,k < n} \in M_n(\mathbb{C}).$$

So the coordinates $\boldsymbol{x} = (w_k)_{0 \leqslant k < n}$ of $\boldsymbol{w}$ with respect to the basis $(\boldsymbol{e}'_k)_{0 \leqslant k < n}$ and the coordinates $\boldsymbol{y} = (\sigma({}^{j}\!/_{n}))_{0 \leqslant j < n}$ of $\boldsymbol{z}$ with respect to the basis $(g^j)_{0 \leqslant j < n}$ satisfy

$$\boldsymbol{y} = P_n \boldsymbol{x}.$$

It is easy to give a formula for the inverse matrix of P_n. Indeed, recall that if ζ_d is a primitive dth root of unity, then

$$\sum_{i=0}^{d-1} \zeta_d^i = \begin{cases} 1 & \text{if } d = 1, \\ 0 & \text{if } d > 1. \end{cases}$$

Hence, the (i,k)th entry in the matrix $P_n^* P_n$ is

$$\sum_{j=0}^{n-1} (\zeta_n^{-ij})^* \zeta_n^{-jk} = \sum_{j=0}^{n-1} \zeta_n^{ij} \zeta_n^{-jk} = \sum_{j=0}^{n-1} (\zeta_n^{i-k})^j = \begin{cases} n & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases}$$

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[1] Traditionally, one calls the modulus $|\sigma(t)| \in [0,\infty)$ and the argument $\arg(\sigma(t)) \in [0,2\pi)$ for the *amplitude* and the *phase* of the signal at time t, respectively.

since ζ_n^{k-i} is a primitive dth root of unity for some divisor d in n. So $P_n^* P_n = nI_n$, and since also $P_n P_n^* = nI_n$, we find that

$$P_n^{-1} = \frac{1}{n} P_n^* = \frac{1}{n} (\zeta_n^{jk})_{0 \leqslant j,k < n}.$$

A priori we need to calculate the n^2 complex numbers ζ_n^{jk} with $0 \leqslant j, k < n$ to determine the matrix P_n and its inverse. However, if $n = 2^m$ is a power of 2, then P_n can be calculated much more effectively. So let $n = 2r$, let T_n be the permutation matrix, whose first r columns are the odd columns of I_n, and whose last r columns are the even columns of I_n, and let D_r be the diagonal matrix

$$D_r = \operatorname{diag}(1, \zeta_n^{-1}, \zeta_n^{-2}, \dots, \zeta_n^{-(r-1)}).$$

Then, as first noted by Gauss in 1805 and, independently, by Cooley and Tukey in 1965, one has the matrix identity

$$P_n = \begin{pmatrix} I_r & D_r \\ I_r & -D_r \end{pmatrix} \begin{pmatrix} P_r & O \\ O & P_r \end{pmatrix} T_n,$$

which greatly reduces the amount of calculation necessary to determine P_n. The algorithm for calculating P_n for $n = 2^m$ based on this matrix identity is called the *fast Fourier transform*[2] and is one of the most important algorithms in terms of its myriad applications. So starting from $P_1 = \big(1\big)$, we find e.g.

$$\begin{aligned} P_2 &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ P_4 &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & i \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}. \end{aligned}$$

We leave it as an exercise to the reader to verify the matrix identity in general.

EXAMPLE 4.3. Let us also determine the real group ring $\mathbb{R}[C_n]$, where C_n is a cyclic group of order n. We again fix a generator $g \in C_n$ and a primitive nth root of unity $\zeta_n \in \mathbb{C}$. For $0 \leqslant k < n$, we define the left $\mathbb{R}[C_n]$-module $\mathbb{R}(\zeta_n^k)$ to be the sub-$\mathbb{R}$-vector space $\mathbb{R}(\zeta_n^k) \subset \mathbb{C}$ generated by the family of complex numbers $(\zeta_n^{jk})_{0 \leqslant j < n}$ and with the left $\mathbb{R}[C_n]$-module structure given by

$$\big(\sum_{j=0}^{n-1} a_j g^j\big) \cdot z = \sum_{j=0}^{n-1} a_j \zeta_n^{jk} z.$$

The left $\mathbb{R}[C_n]$-module $\mathbb{R}(\zeta_n^k)$ is simple. Indeed, if $z, z' \in \mathbb{R}(\zeta_n^k)$ are two non-zero elements, then there exists $\omega \in \mathbb{R}[C_n]$ with $\omega \cdot z = z'$. The dimension of $\mathbb{R}(\zeta_n^k)$ as

[2]The time required to compute P_n by means of FFT is $O(n \log n)$.

an $\mathbb{R}$-vector space is

$$\dim_{\mathbb{R}}(\mathbb{R}(\zeta_n^k)) = \begin{cases} 1 & \text{if } \zeta_n^k \in \mathbb{R}, \\ 2 & \text{if } \zeta_n^k \notin \mathbb{R}, \end{cases}$$

and the left $\mathbb{R}[C_n]$-modules $\mathbb{R}(\zeta_n^k)$ and $\mathbb{R}(\zeta_n^l)$ are isomorphic if and only if the complex numbers ζ_n^k and ζ_n^l are conjugate. Hence, by counting dimensions, we conclude from Theorem 3.5 that, as a left $\mathbb{R}[C_n]$-module,

$$\mathbb{R}[C_n] = \bigoplus_{k=0}^{\lfloor n/2 \rfloor} \mathbb{R}(\zeta_n^k),$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. For example,

$$\begin{aligned} \mathbb{R}[C_3] &= \mathbb{R}(\zeta_3^0) \oplus \mathbb{R}(\zeta_3^1), \\ \mathbb{R}[C_4] &= \mathbb{R}(\zeta_4^0) \oplus \mathbb{R}(\zeta_4^1) \oplus \mathbb{R}(\zeta_4^2). \end{aligned}$$

In general, for $n = 2r$ even, the 1-dimensional representation $\mathbb{R}(\zeta_n^r) = \mathbb{R}(-1)$ is called the sign representation, since the left $\mathbb{R}[C_n]$-module structure is given by

$$(\sum_{j=0}^{n-1} a_j g^j) \cdot x = \sum_{j=0}^{n-1} a_j (-1)^j x.$$

Finally, the endomorphism ring $\mathrm{End}_{\mathbb{R}[C_n]}(\mathbb{R}(\zeta_n^k))$ is isomorphic to $\mathbb{R}$, if $k = 0$ or if $n = 2r$ and $k = r$, and is isomorphic to $\mathbb{C}$, otherwise.

Remark 4.4. Representations of groups are also important in chemistry and physics. To wit, if a molecule has finite symmetry group G, then its wave function must be a representation of G on the Hilbert space $\mathfrak{h} = \bigotimes_a \mathfrak{h}_a$, where the tensor product ranges over the atoms in the molecule, where the wave function of atom a is a vector in $\mathfrak{h}_a$, and where G acts on the tensor product by permuting the tensor factors. In this way, the representation theory of finite groups determines the relative likelihood of the outcome of chemical reactions.

In the standard model of elementary particle physics, the elementary particles in the three generations (up-down, strange-charmed, top-bottom) of elementary particles correpond to simple representations of the group

$$G = U(1) \times SU(2) \times SU(3).$$

This group is not finite, but it has a natural topology with respect to which it is compact. In fact, the topological space G has the structure of a compact smooth manifold, and the maps $\mu \colon G \times G \to G$ and $\iota \colon G \to G$ given by multiplication and inversion in the group G, respectively, are smooth maps. We say that G is a Lie group. If V is a finite dimensional $\mathbb{C}$-vector space, then $\mathrm{Aut}_{\mathbb{C}}(V)$ also has a canonical Lie group structure, and by a representation of G on V we mean a map

$$\rho \colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

that is both smooth and a group homomorphism. We let $SU(5) \subset M_5(\mathbb{C})$ act on $\mathbb{C}^5 = M_{5,1}(\mathbb{C})$ by matrix multiplication. This defines a simple representation of $SU(5)$ that we call the standard representation. By functoriality, we obtain an induced action by the group $SU(5)$ on the exterior algebra

$$\Lambda_{\mathbb{C}}(\mathbb{C}^5) = \bigoplus_{0 \leqslant j \leqslant 5} \Lambda_{\mathbb{C}}^j(\mathbb{C}^5),$$

and this again is a representation of $SU(5)$. In fact, each exterior power $\Lambda^j_{\mathbb{C}}(\mathbb{C}^5)$ is simple as an $SU(5)$-representation. We now consider the map

$$f\colon G = U(1) \times SU(2) \times SU(3) \to SU(5)$$

given by

$$f(\alpha, A, B) = \begin{pmatrix} \alpha^3 A & 0 \\ 0 & \alpha^{-2} B \end{pmatrix},$$

which is both smooth and a group homomorphism.[3] Therefore, we can let G act on $\Lambda_{\mathbb{C}}(\mathbb{C}^5)$ by defining $g \cdot \omega$ to be $f(g) \cdot \omega$. As a G-representation, the exterior powers $\Lambda^j_{\mathbb{C}}(\mathbb{C}^5)$ with $0 < j < 5$ are no longer simple, but decompose as direct sums of simple representations. It turns out that the simple G-representations that appear in this decomposition of the G-representation $\Lambda_{\mathbb{C}}(\mathbb{C}^5)$ precisely enumerate the elementary fermions observed in nature; see the figure below, which is taken from [**1**]. This is a purely phenomenological fact; nobody understands the reason why this is so! In the figure, the basis of $\mathbb{C}^5$ is (u, d, r, g, b), which stands for up, down, red, green, and blue. For example, rgb, which corresponds to the right-handed electron e^-_R, is an abbreviations for $r \wedge g \wedge b$. The 1-dimensional subspace of $\Lambda^3_{\mathbb{C}}(\mathbb{C}^5)$ generated by $(r \wedge g \wedge b)$ is a simple G-representation. Similarly, the 3-dimensional subspace of $\Lambda^2_{\mathbb{C}}(\mathbb{C}^5)$ generated by $(d \wedge r, d \wedge g, d \wedge b)$ is a simple G-representation, which corresponds to the left-handed down-quark d^c_L. The left-handed electron e^-_L and the left-handed neutrino ν_L form the basis $(d \wedge r \wedge g \wedge b, u \wedge r \wedge g \wedge b)$ of a simple 2-dimensional representation of $\Lambda^4_{\mathbb{C}}(\mathbb{C}^5)$. Accordingly, these two particles are rather two states of a single particle. Finally, the left-handed anti-neutrino $\bar{\nu}_L$ and the right-handed neutrino ν_R correspond to the simple 1-dimensional G-representations $\Lambda^0_{\mathbb{C}}(\mathbb{C}^5)$ and $\Lambda^5_{\mathbb{C}}(\mathbb{C}^5)$, respectively. Both of these representations are trivial, which, in physics, means that $\bar{\nu}_L$ and ν_R do not interact through any of the three forces (electro-magnetic, weak, strong) described by the standard model. In fact, it is not known whether or not $\bar{\nu}_L$ and ν_R exist.

TABLE 4. Binary code for first-generation fermions, where $c = r, g, b$ and $\overline{c} = gb, br, rg$

The Binary Code for SU(5)

$\Lambda^0\mathbb{C}^5$	$\Lambda^1\mathbb{C}^5$	$\Lambda^2\mathbb{C}^5$	$\Lambda^3\mathbb{C}^5$	$\Lambda^4\mathbb{C}^4$ ← 5	$\Lambda^5\mathbb{C}^5$
$\overline{\nu}_L = 1$	$e^+_R = u$	$e^+_L = ud$	$e^-_R = rgb$	$e^-_L = drgb$	$\nu_R = udrgb$
	$\overline{\nu}_R = d$	$u^c_L = uc$	$\overline{u}^{\overline{c}}_R = d\overline{c}$	$\nu_L = urgb$	
	$d^c_R = c$	$d^c_L = dc$	$\overline{d}^{\overline{c}}_R = u\overline{c}$	$\overline{d}^{\overline{c}}_L = ud\overline{c}$	
		$\overline{u}^{\overline{c}}_L = \overline{c}$	$u^c_R = udc$		

References

[1] J. Baez and J. Huerta, *The algebra of grand unified theories*, Bull. Amer. Math. Soc. **47** (2010), 483–552.

[3] The map $f\colon G \to SU(5)$ is not quite injective; its kernel is a cyclic group of order 6.