Pespectives in Mathematical Sciences

Due: Tuesday, June 16, 2020, on NUCT.

Problem 1. Let R be a ring, and let R^{op} be the opposite ring defined in Remark 2.6. Let $M_n(R)$ be the ring of $n \times n$-matrices with entries in R, and let

$$(-)^t \colon M_n(R)^{\mathrm{op}} \to M_n(R^{\mathrm{op}})$$

be the map that to a matrix $A = (a_{ij})$ assigns its transpose $A^t = (a_{ji})$.

(a) Show that $(-)^t$ is a ring homomorphism.
(b) Show that $(-)^t$ is a ring isomorphism.

Problem 2. Let D be a division ring, and let $R = M_n(D)$ be the matrix ring. The set $S = M_{n,1}(D)$ of column vectors has both a structure of left R-module and of right D-module with sum given by matrix sum and scalar multiplication given by matrix product. Moreover, for all $A \in R$, $\boldsymbol{x} \in S$, and $a \in D$, $(A \cdot \boldsymbol{x}) \cdot a = A \cdot (\boldsymbol{x} \cdot a)$, by the associativity of matrix product. Show that the map

$$D^{\mathrm{op}} \xrightarrow{\rho} \mathrm{End}_R(S)$$

defined by $\rho(a)(\boldsymbol{x}) = \boldsymbol{x} \cdot a$ is a ring isomorphism.