Pespectives in Mathematical Sciences

Due: Tuesday, June 23, 2020, on NUCT.

Problem 1. Let D be a division ring, and let $R = M_n(D)$ be the matrix ring. The set $S = M_{n,1}(D)$ of column vectors has both a structure of left R-module and of right D-module with sum given by matrix sum and scalar multiplication given by matrix product. Moreover, for all $A \in R$, $\boldsymbol{x} \in S$, and $a \in D$,

$$(A \cdot \boldsymbol{x}) \cdot a = A \cdot (\boldsymbol{x} \cdot a),$$

by the associativity of matrix product.

(a) Show that the family $(\boldsymbol{v})$ consisting of the single vector

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

generates the left R-module S.

(b) Show that if $n \geq 2$, then the family $(\boldsymbol{v})$ is *not* a linearly independent family in the left R-module S.

(c) Find $P \in R$ such that $P\boldsymbol{v} = \boldsymbol{v}$ and such that $PS = \boldsymbol{v}D \subset S$.