Pespectives in Mathematical Sciences

Due: Tuesday, July 7, 2020, on NUCT.

Problem 1. Let R be a commutative ring and let $\mathfrak{p} \subset R$ be a proper ideal. Show that the following statements are equivalent.

(i) For all elements $a, b \in R$, $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.
(ii) For all ideals $\mathfrak{a}, \mathfrak{b} \subset R$, $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ implies $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.

An ideal $\mathfrak{p} \subset R$ for which the equivalent statements (i)–(ii) hold is called a *prime ideal.* We read (ii) as "if $\mathfrak{p}$ divides $\mathfrak{a}\mathfrak{b}$, then $\mathfrak{p}$ divides $\mathfrak{a}$ or $\mathfrak{p}$ divides $\mathfrak{b}$."

[Hint: To prove that (i) implies (ii), note that (ii) is equivalent to the statement that if $\mathfrak{a} \not\subset \mathfrak{p}$ and $\mathfrak{b} \not\subset \mathfrak{p}$, then $\mathfrak{a}\mathfrak{b} \not\subset \mathfrak{p}$.]

Problem 2. Let R be a commutative ring. Show that the following statements are equivalent.

(i) For all $a, b \in R$, if $ab = 0$, then $a = 0$ or $b = 0$.
(ii) The ring R is a subring of a field K.

A commutative ring R for which the equivalent statements (i)–(ii) hold is called an *integral domain.*

[Hint: To prove that (i) implies (ii), observe that $S = R \smallsetminus \{0\} \subset R$ is a multiplicative subset, and let K be the localization $S^{-1}R$.]