REPRESENTATION THEORY

Basic idea: If k is a field and V is a k-vector space, then the set of k-linear automorphisms of V form a group $\mathrm{GL}(V)$ under composition. The basic idea is to study a general group G by considering group homomorphisms

$$G \xrightarrow{\pi} \mathrm{GL}(V).$$

We think of $g \in G$ as being complicated and of $\pi(g) \in \mathrm{GL}(V)$ as being easier. Indeed, we can use the methods of linear algebra to study $\pi(g) \in \mathrm{GL}(V)$.

Textbook: E. B. Vinberg, Linear representations of groups, Translated from the 1985 Russian original by A. Iacob. Reprint of the 1989 translation. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2010.ISBN: 978-3-0348-0062-4.

Schedule: The plan is to cover one chapter in the textbook each week, beginning with Chapter 0.

Let k be a field (typically $k = \mathbb{R}$ or $k = \mathbb{C}$), and let V be a k-vector space. The group of k-linear automorphisms of V is given by the set

$$\mathrm{GL}(V) = \{f \colon V \to V \mid f \text{ is } k\text{-linear and an isomorphism}\}$$

with group structure given by the map

$$\mathrm{GL}(V) \times \mathrm{GL}(V) \xrightarrow{\circ} \mathrm{GL}(V)$$

that to (f, g) assigns $f \circ g$.

Definition 1. A k-linear representation of a group G is a pair (V, π) of a k-vector space V and a group homomorphism $\pi \colon G \to \mathrm{GL}(V)$.

So elements $g \in G$ are represented by k-linear operators $\pi(g) \colon V \to V$ such that

$$\pi(g \cdot h) = \pi(g) \circ \pi(h)$$

and such that

$$\pi(e) = \mathrm{id}_V .$$

Here $e \in G$ is the identity element.

Suppose that $\dim_k(V) = n < \infty$. A choice of a basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ of V determines an isomorphism of groups

$$\mathrm{GL}(V) \xrightarrow{\alpha} \mathrm{GL}_n(k)$$

that to the k-linear automorphism $f \colon V \to V$ assigns the invertible $n \times n$-matrix $\alpha(f) = (a_{ij})$, whose entries $a_{ij} \in k$ are the unique solutions to the equations

$$f(\boldsymbol{e}_j) = \boldsymbol{e}_1 a_{1j} + \boldsymbol{e}_2 a_{2j} + \cdots + \boldsymbol{e}_n a_{nj}$$

for $1 \leq j \leq n$. Hence, a k-linear representation $\pi \colon G \to \mathrm{GL}(V)$ determines and is determined by the composite group homomorphism

$$G \xrightarrow{\pi} \mathrm{GL}(V) \xrightarrow{\alpha} \mathrm{GL}_n(k).$$

We stress that the group isomorphism α depends on the choice of basis! We say that the composite map $\alpha \circ \pi \colon G \to \mathrm{GL}_n(k)$ is a matrix representation of G.

Definition 2. A matrix representation of a group G over a field k is a group homomorphism $\pi \colon G \to \mathrm{GL}_n(k)$.

In order to do calculations, it can be convenient to choose a basis of a vector space and calculate in coordinates. However, for theoretical considerations, it is always best to avoid making a choice of basis.

If we study mathematical objects given by sets equipped with some structure, then we should at the same time study the maps between such objects that preserve this structure. If V_1 and V_2 are k-vector spaces, then the maps $f \colon V_1 \to V_2$ that preserve the structure of a k-vector space are the k-linear maps. And if X_1 and X_2 are topological spaces, then the maps $f \colon X_1 \to X_2$ that preserve the structure of a topological space are the continuous maps. The maps that preserve the structure of a k-linear representation of a fixed group G, which we now define, are called the intertwining (or equivariant) maps.

Definition 3. If (V_1, π_1) and (V_2, π_2) are k-linear representations of G, then an intertwining map $f\colon (V_1, \pi_1) \to (V_2, \pi_2)$ is a k-linear map $f\colon V_1 \to V_2$ such that

$$f(\pi_1(g)(\boldsymbol{x})) = \pi_2(g)(f(\boldsymbol{x}))$$

for all $g \in G$ and $\boldsymbol{x} \in V_1$.

If (V, π) is a k-linear representation of G, then we will sometime abbreviate

$$g\boldsymbol{x} = \pi(g)(\boldsymbol{x}).$$

So if both (V_1, π_1) and (V_2, π_2) are representations of the same group G, then a k-linear map $f\colon V_1 \to V_2$ is intertwining if and only if

$$f(g\boldsymbol{x}) = gf(\boldsymbol{x})$$

for all $g \in G$ and $\boldsymbol{x} \in V_1$.

Definition 4. Let (V_1, π_1) and (V_2, π_2) be k-linear representations of a group G. An intertwining map $f\colon (V_1, \pi_1) \to (V_2, \pi_2)$ is an isomorphism, if there exists an intertwining map $g\colon (V_2, \pi_2) \to (V_1, \pi_1)$ such that $g \circ f = \mathrm{id}_{V_1}$ and $f \circ g = \mathrm{id}_{V_2}$.

We show in the problem set that an intertwining map $f\colon (V_1, \pi_1) \to (V_2, \pi_2)$ is an isomorphism if and only if the map $f\colon V_1 \to V_2$ is a bijection.

Remark 5. Let (V_1, π_1) and (V_2, π_2) be two k-linear representations of a group G. We say that (V_1, π_1) and (V_2, π_2) are isomorphic and write $(V_1, \pi_1) \simeq (V_2, \pi_2)$, if there exists an isomorphism $f\colon (V_1, \pi_1) \to (V_2, \pi_2)$. However, note that it is much better to know that "the map $f\colon (V_1, \pi_1) \to (V_2, \pi_2)$ is an isomorphism" than it is to know that "(V_1, π_1) and (V_2, π_2) are isomorphic." Indeed, in the former case, the given isomorphism $f\colon (V_1, \pi_1) \to (V_2, \pi_2)$ tells us *how* to translate between the two representations, whereas in the latter case, we only know that, in principle, such a translation is possible.

Examples

We consider examples of representations and begin with the group

$$G = (\mathbb{R}, +)$$

of real numbers under addition. Given $a \in \mathbb{R}$, the exponential function

$$G \xrightarrow{\pi_a} \mathrm{GL}_1(\mathbb{R})$$

defined by $\pi_a(t) = e^{at}$ is a matrix representation. Indeed, we have

$$\pi_a(t+u) = e^{a(t+u)} = e^{at+au} = e^{at}e^{au} = \pi_a(t)\pi_a(u)$$

and

$$\pi_a(0) = e^{a0} = e^0 = 1,$$

as required. This begs the question as to whether every 1-dimensional representation $\pi\colon G \to \mathrm{GL}_1(\mathbb{R})$ of G is of this form. The answer is "Yes," provided that we require the map π to be continuous. We prove the following weaker result:

Lemma 6. *Let $G = (\mathbb{R}, +)$ be the additive group of real numbers. For every differentiable 1-dimensional representation $\pi\colon G \to \mathrm{GL}_1(\mathbb{R})$, there exists a unique $a \in \mathbb{R}$ such that $\pi = \pi_a\colon G \to \mathrm{GL}_1(\mathbb{R})$, namely, $a = \pi'(0)$.*

Proof. We will assume the stronger hypothesis that π be differentiable instead of continuous. That π is a representation means that $\pi(0) = 1$ and that for all $t, u \in \mathbb{R}$,

$$\pi(t + u) = \pi(t) \cdot \pi(u).$$

We differentiate the latter equation with respect to u at $u = 0$, which gives the ordinary differential equation

$$\pi'(t) = \pi(t) \cdot \pi'(0).$$

Every solution to the ODE is of the form $\pi(t) = Ce^{at}$, where $a = \pi'(0) \in \mathbb{R}$, and the initial condition $\pi(0) = 1$ implies that $C = 1$. This proves the lemma in the case, where π is differentiable. □

We let $M_n(k)$ be the set of $n \times n$-matrices with entries in k, considered as a ring under matrix addition and matrix multiplication.[1] If $k = \mathbb{R}$ or $k = \mathbb{C}$, then we have the matrix exponential of $A \in M_n(k)$ defined by the series

$$e^A = \sum_{n \geq 0} \frac{1}{n!} A^n \in M_n(k),$$

which converges in operator norm, because

$$\| \frac{1}{n!} A^n \| \leq \frac{1}{n!} \|A\|^n \leq e^{\|A\|}.$$

If $AB = BA$, then $e^{A+B} = e^A e^B$, but this is generally *not* true without this assumption! In particular, the map

$$G = (\mathbb{R}, +) \xrightarrow{\pi_A} \mathrm{GL}_n(k)$$

defined by $\pi_A(t) = e^{tA}$ is a group homomorphism, and hence, an n-dimensional matrix representation of G, where $k = \mathbb{R}$ or $k = \mathbb{C}$.

Lemma 7. *If $\pi \colon G = (\mathbb{R}, +) \to \mathrm{GL}_n(k)$ is a differentiable real or complex representation, then $\pi = \pi_A \colon G \to \mathrm{GL}_n(k)$ with $A = \pi'(0) \in M_n(k)$.*

Proof. As before, we obtain the ordinary differential equation

$$\pi'(t) = \pi(t) \cdot \pi'(0)$$

with the initial condition $\pi(0) = E \in M_n(k)$, and it has $\pi = \pi_A$ with $A = \pi'(0)$ as its unique solution. □

Example 8. We consider

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{R})$$

and calculate

$$A^n = \begin{cases} (-1)^m E & \text{if } n = 2m \text{ is even} \\ (-1)^m A & \text{if } n = 2m+1 \text{ is odd,} \end{cases}$$

[1] The book writes $L_n(k)$ instead of $M_n(k)$.

which shows that

$$\begin{aligned}
e^{tA} &= \sum_{n \geq 0,\text{ even}} \frac{1}{n!}(tA)^n + \sum_{n \geq 0,\text{ odd}} \frac{1}{n!}(tA)^n \\
&= \sum_{m \geq 0} \frac{(-1)^m t^{2m}}{(2m)!} E + \sum_{m \geq 0} \frac{(-1)^m t^{2m+1}}{(2m+1)!} A \\
&= \cos t \cdot E + \sin t \cdot A \\
&= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.
\end{aligned}$$

So we conclude that the map $\pi \colon G = (\mathbb{R}, +) \to \mathrm{GL}_2(\mathbb{R})$ defined by

$$\pi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is a 2-dimensional real representation of G.

In general, if (V, π) is a k-linear representation of a group G, then we call

$$\ker(\pi) = \{g \in G \mid \pi(g) = \mathrm{id}_V\} \subset G$$

the kernel of π. It is a normal subgroup of G. The representation (V, π) will obviously not be of any help to study the elements in $\ker(\pi) \subset G$. We recall that $\ker(\pi) = \{e\}$ if and only if $\pi \colon G \to \mathrm{GL}(V)$ is injective.

Definition 9. A k-linear representation (V, π) of a group G is faithful if the group homomorphism $\pi \colon G \to \mathrm{GL}(V)$ is injective.

Example 10. (1) The representation $\pi_a \colon (\mathbb{R}, +) \to \mathrm{GL}_1(\mathbb{R})$ defined by $\pi_a(t) = e^{at}$ is faithful if and only if $a \neq 0$.

(2) The representation $\pi \colon (\mathbb{R}, +) \to \mathrm{GL}_2(\mathbb{R})$ defined by

$$\pi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is not faithful, since $\ker(\pi) = 2\pi\mathbb{Z} \subset \mathbb{R}$.

Example 11. We next let $G = S_n$ be the (finite) symmetric group on n letters. It is defined to be the set of all bijections

$$\{1, 2, \dots, n\} \xrightarrow{\sigma} \{1, 2, \dots, n\}$$

equipped with the group structure

$$S_n \times S_n \xrightarrow{\circ} S_n$$

that to (σ, τ) assigns the composite bijection $\sigma \circ \tau$.

If k is any field, then we define the n-dimensional matrix representation

$$S_n \xrightarrow{P} \mathrm{GL}_n(k)$$

to be the map that to $\sigma \in S_n$ assigns the permutation matrix[2]

$$P(\sigma) = \begin{pmatrix} \boldsymbol{e}_{\sigma(1)} & \boldsymbol{e}_{\sigma(2)} & \dots & \boldsymbol{e}_{\sigma(n)} \end{pmatrix} \in \mathrm{GL}_n(k).$$

[2] One much check that $P(\sigma \circ \tau) = P(\sigma) \cdot P(\tau)$ and that $P(e) = E$, which is not difficult, but we will give a high-tech proof later in Example 13.

Clearly, the kernel of P is trivial, so $P\colon S_n \to \mathrm{GL}_n(k)$ is a faithful representation. We recall that the determinant defines a group homomorphism

$$\mathrm{GL}_n(k) \xrightarrow{\det} \mathrm{GL}_1(k),$$

and therefore, the composite map

$$S_n \xrightarrow{P} \mathrm{GL}_n(k) \xrightarrow{\det} \mathrm{GL}_1(k)$$

is a 1-dimensional matrix representation of S_n. It is called the sign representation, since the sign of σ, by definition, is given by

$$\mathrm{sgn}(\sigma) = \det(P(\sigma)) \in \{\pm 1\} \subset \mathrm{GL}_1(k).$$

If $2 \neq 0$ in k, then the kernel $\ker(S_n) = A_n \subset S_n$ is the alternating group on n letters. In particular, the sign representation is not faithful, except in trivial cases.

The regular representation

If X is any set and k is a field, then we view the set of all maps $f\colon X \to k$ as a k-vector space $k[X]$ with vector sum and scalar multiplication defined by

$$(f+g)(x) = f(x) + g(x)$$
$$(a \cdot f)(x) = a \cdot f(x).$$

The k-linear representation $(k[G], L)$ of a group G, where $L\colon G \to k[G]$ is given by

$$L(g)(f)(x) = f(g^{-1}x),$$

is called the left regular representation, and the k-linear representation $(k[G], R)$ of G, where $R\colon G \to \mathrm{GL}(k[G])$ is given by

$$R(g)(f)(x) = f(xg),$$

is called the right regular representation. We check that $(k[G], L)$ and $(k[G], R)$ are representations of G. First, we clearly have

$$L(e) = \mathrm{id}_{k[G]} = R(e),$$

and second, the calculations

$$\begin{aligned} L(gh)(f)(x) &= f((gh)^{-1}x) = f(h^{-1}g^{-1}x) = L(h)(f)(g^{-1}x) \\ &= L(g)(L(h)(f))(x) = (L(g) \circ L(h))(f)(x) \\ R(gh)(f)(x) &= f(xgh) = R(h)(f)(xg) \\ &= R(g)(R(h)(f))(x) = (R(g) \circ R(h))(f)(x) \end{aligned}$$

show that $L(gh) = L(g) \circ L(h)$ and $R(gh) = R(g) \circ R(h)$ as required. The left regular representations give rise to a representation on a subspace $V \subset k[X]$, provided that V is G-invariant in the sense that $L(g)(V) \subset V$ for all $g \in G$. Similarly, for the right regular representation.

Example 12. If $G = (\mathbb{R}, +)$, then we have

$$L(t)(f)(x) = f(-t+x) = f(x-t),$$

so the following subspaces are G-invariant:

$$\begin{aligned} V &= \{f \in k[G] \mid f \text{ is a polynomial function}\} \subset k[G], \\ W &= \mathrm{span}(\cos, \sin) \subset \mathbb{R}[G]. \end{aligned}$$

In the case of $W \subset \mathbb{R}[G]$, we have

$$L(t)(\cos)(x) = \cos(-t+x) = \cos t \cos x + \sin t \sin x$$
$$L(t)(\sin)(x) = \sin(-t+x) = -\sin t \cos x + \cos t \sin x,$$

so we recover the representation $\pi\colon G \to \mathrm{GL}_2(\mathbb{R})$ given by

$$\pi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Permutation representations

Let X be any set, and let $S(X)$ be the group of all bijections $\sigma\colon X \to X$ with the composition group structure defined by $(\sigma \circ \tau)(x) = \sigma(\tau(x))$. For example, the group $S_n = S(\{1, 2, \dots, n\})$ is the symmetric group on n letters. If k is any field, then we may define a k-linear representation $(k[X], \pi)$ of $S(X)$ by [3]

$$\pi(\sigma)(f)(x) = f(\sigma^{-1}(x)).$$

A left action by a group G on a set X is defined to be a group homomorphism $\rho\colon G \to S(X)$. Thus, given a left action by G on X, the composite map

$$G \xrightarrow{\rho} S(X) \xrightarrow{\pi} \mathrm{GL}(k[X])$$

defines a k-linear representation $(k[X], \pi \circ \rho)$ of the group G. We say that a k-linear representation of this form is a permutation representation.

Example 13. The identity map $\rho\colon S_n \to S(\{1, 2, \dots, n\})$ is a left action, where

$$\rho(\sigma)(i) = \sigma(i).$$

So we obtain the permutation representation

$$S_n \xrightarrow{\pi = \pi \circ \rho} \mathrm{GL}(k[\{1, 2, \dots, n\}]).$$

Let us calculate the corresponding matrix representation with respect to the basis

$$(\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \dots, \boldsymbol{e}_n^*)$$

of $k[\{1, 2, \dots, n\}]$, where $\boldsymbol{e}_i^*\colon \{1, 2, \dots, n\} \to k$ is the map defined by

$$\boldsymbol{e}_i^*(j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

By the definition of π, we have

$$\begin{aligned} \pi(\sigma)(\boldsymbol{e}_i^*)(j) &= \boldsymbol{e}_i^*(\sigma^{-1}(j)) \\ &= \begin{cases} 1 & \text{if } i = \sigma^{-1}(j) \\ 0 & \text{if } i \neq \sigma^{-1}(j) \end{cases} \\ &= \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{if } \sigma(i) \neq j \end{cases} \\ &= \boldsymbol{e}_{\sigma(i)}^*(j), \end{aligned}$$

which shows that

$$\pi(\sigma)(\boldsymbol{e}_i^*) = \boldsymbol{e}_{\sigma(i)}^*.$$

[3] The book writes $\sigma_*(f)$ instead of $\pi(\sigma)(f)$.

Hence, we conclude that the matrix that represents the k-linear map

$$k[\{1,2,\dots,n\}] \xrightarrow{\pi(\sigma)} k[\{1,2,\dots,n\}]$$

with respect to the basis $(\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \dots, \boldsymbol{e}_n^*)$ is the permutation matrix

$$P(\sigma) = \begin{pmatrix} \boldsymbol{e}_{\sigma(1)} & \boldsymbol{e}_{\sigma(2)} & \dots & \boldsymbol{e}_{\sigma(n)} \end{pmatrix} \in \mathrm{GL}_n(k).$$

So we recover the matrix representation

$$S_n \xrightarrow{P} \mathrm{GL}_n(k)$$

from Example 11. In particular, we may conclude that the identities

$$P(\sigma \circ \tau) = P(\sigma) \cdot P(\tau)$$

and $P(e) = E$ hold.