INDUCTION AND RESTRICTION

This time, we will apply the general theory that we developed last time to the particular map of left G-sets given by the unique map

$$G/H \xrightarrow{\ p\ } G/G = \{G\},$$

where $H \subset G$ is a subgroup.[1] We have defined functors

$$\begin{array}{ccc}
BG & \xleftarrow{\quad f \quad} & BH \\
\| & & \| \\
[G \backslash (G/G)] & \xleftarrow{\ f\ } & [H \backslash (H/H)] \\
{\scriptstyle p}\nwarrow & & \swarrow{\scriptstyle i} \\
& [G \backslash (G/H)] &
\end{array}$$

with i an equilvalence of categories, and adjoint pairs of functors

$$\begin{array}{ccc}
\mathrm{Rep}_k(G) & \underset{\mathrm{Ind}_H^G}{\overset{\mathrm{Res}_H^G}{\rightleftarrows}} & \mathrm{Rep}_k(H) \\
\| & & \| \\
\mathrm{QCoh}([G \backslash (G/G)]) & \underset{f_*}{\overset{f^*}{\rightleftarrows}} & \mathrm{QCoh}([H \backslash (H/H)]) \\
{\scriptstyle p^*}\searrow\nwarrow{\scriptstyle p_*} & & {\scriptstyle i^*}\nearrow\swarrow{\scriptstyle i_*} \\
& \mathrm{QCoh}([G \backslash (G/H)]). &
\end{array}$$

We call $\mathrm{Res}_H^G = f^*$ the restriction from G to H and its right adjoint $\mathrm{Ind}_H^G = f_*$ the induction from H to G. (We also defined a left adjoint $f_!$ of f^*, which we call compact induction from H to G. It is sometimes written ind_H^G in all lower-case.) Since composition of functors is (strictly) associative, we have

$$f^* = (p \circ i)^* = i^* \circ p^*,$$

but it is *not* true that

$$f_* = (p \circ i)_* = p_* \circ i_*.$$

What is true, however, is that the two composite natural transformations

$$f_* \longrightarrow p_* p^* f_* \longrightarrow p_* i_* i^* p^* f_* = p_* i_* f^* f_* \longrightarrow p_* i_*$$

$$p_* i_* \longrightarrow f_* f^* p_* i_* = f_* i^* p^* p_* i_* \longrightarrow f_* i^* i_* \longrightarrow f_*$$

defined using the counits and units of the three adjunctions are each other's inverses. In this way, the two adjoints f_* and $p_* i_*$ of f^* are uniquely naturally isomorphic. This is a general fact:

[1] We do *not* assume that $H \subset G$ is normal.

Proposition 1. *Let* $(f^*, f_*, \epsilon, \eta)$ *and* $(f^*, \bar{f}_*, \bar{\epsilon}, \bar{\eta})$ *be two adjunctions with the same left adjoint functor* f^**. In this situation, the composite natural transformation*

$$f_* \xrightarrow{\bar{\eta} \circ f_*} \bar{f}_* \circ f^* \circ f_* \xrightarrow{\bar{f}_* \circ \epsilon} \bar{f}_*$$

is the unique natural transformation $\sigma \colon f_* \to \bar{f}_*$ *that makes the diagrams*

$$\begin{array}{ccc} f^* \circ f_* & \xrightarrow{f^* \circ \sigma} & f^* \circ \bar{f}_* \\ & {\scriptstyle \epsilon} \searrow \quad \swarrow {\scriptstyle \bar{\epsilon}} & \\ & \mathrm{id} & \end{array} \qquad \begin{array}{ccc} f_* \circ f^* & \xrightarrow{\sigma \circ f^*} & \bar{f}_* \circ f^* \\ & {\scriptstyle \eta} \nwarrow \quad \nearrow {\scriptstyle \bar{\eta}} & \\ & \mathrm{id} & \end{array}$$

commute. In particular, it is a natural isomorphism with inverse

$$\bar{f}_* \xrightarrow{\eta \circ \bar{f}_*} f_* \circ f^* \circ \bar{f}_* \xrightarrow{f_* \circ \bar{\epsilon}} f_*.$$

Proof. This is not so easy to show. See for example Saunders MacLane, Categories for the Working Mathematician, Chapter IV, Section 7, Theorem 2. □

Here is an application:

Corollary 2. *The adjunction*

$$\mathrm{QCoh}([G \backslash (G/H)]) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Rep}_k(H)$$

is an adjoint equivalence.

Proof. In the adjunction $(i^*, i_*, \epsilon, \eta)$, the functors i^* and i_* are given by restriction and right Kan extension along the canonical inclusion

$$BH = [H \backslash (H/H)] \xrightarrow{i} [G \backslash (G/H)],$$

and we wish to prove that ϵ and η are natural isomorphisms. We have proved last time that i is an equilvance of categories. So if h be a quasi-inverse of i, then h^* is a quasi-inverse of i^*, and we can choose natural isomorphisms $\bar{\epsilon} \colon i^* \circ h^* \to \mathrm{id}$ and $\bar{\eta} \colon \mathrm{id} \to h^* \circ i^*$ such that $(i^*, h^*, \bar{\epsilon}, \bar{\eta})$ is an adjunction. By Proposition 1, the natural transformation $\sigma \colon i_* \to h^*$ defined as the composition

$$i_* \xrightarrow{\bar{\eta} \circ i_*} i_* i^* h^* \xrightarrow{\epsilon \circ h^*} h^*$$

is an isomorphism and is unique with the property that the diagrams

$$\begin{array}{ccc} i^* \circ i_* & \xrightarrow{i^* \circ \sigma} & i^* \circ h^* \\ & {\scriptstyle \epsilon} \searrow \quad \swarrow {\scriptstyle \bar{\epsilon}} & \\ & \mathrm{id} & \end{array} \qquad \begin{array}{ccc} i_* \circ i^* & \xrightarrow{\sigma \circ i^*} & h^* \circ i^* \\ & {\scriptstyle \eta} \nwarrow \quad \nearrow {\scriptstyle \bar{\eta}} & \\ & \mathrm{id} & \end{array}$$

commute. In particular, we conclude that ϵ and η are natural isomorphisms. □

Proposition 1 also implies that to "calculate" the induction functor

$$\mathrm{Rep}_k(H) \xrightarrow{\mathrm{Ind}_H^G} \mathrm{Rep}_k(G),$$

it suffices to produce an adjunction $(\mathrm{Res}_H^G, \mathrm{Ind}_H^G, \epsilon, \eta)$ with $\mathrm{Res}_H^G = f^*$. For in this situation, the proposition will give a unique natural isomorphism $\sigma\colon \mathrm{Ind}_H^G \to f_*$ to any other right adjoint functor f_* of f^*, say, to the right Kan extension along the functor $f\colon BH \to BG$.

Now, given a k-linear representation of H,

$$BH \xrightarrow{\tau} \mathrm{Vect}_k$$

with $W = \tau(0)$, we define the induced k-linear representation

$$BG \xrightarrow{\pi = \mathrm{Ind}_H^G(\tau)} \mathrm{Vect}_k$$

as follows. The k-vector space $\pi(0) = V = \mathrm{Map}_H(G, W)$ is given by the set of all maps $f\colon G \to W$ such that for all $h \in H$ and $x \in G$,

$$f(h \cdot x) = h \cdot f(x) = \tau(h)(f(x)),$$

with vector sum and scalar multiplication by $a \in k$ defined pointwise by

$$(f + f')(x) = f(x) + f'(x)$$
$$(f \cdot a)(x) = f(x) \cdot a,$$

and for $g \in G$, the k-linear map $\pi(g)\colon V \to V$ is given by

$$\pi(g)(f)(x) = f(xg).$$

We define the counit $\epsilon_\tau\colon (\mathrm{Res}_H^G \circ \mathrm{Ind}_H^G)(\tau) \to \tau$ to be the k-linear map

$$\mathrm{Map}_H(G, W) \longrightarrow W$$

that to $f\colon G \to W$ assigns $\epsilon_\tau(f) = f(e)$, and the calculation

$$\epsilon_\tau(h \cdot f) = (h \cdot f)(e) = f(e \cdot h) = f(h \cdot e) = h \cdot f(e) = h \cdot \epsilon_\tau(f)$$

shows intertwines between the two k-linear representations of H in question. Finally, if we define the unit $\eta_\pi\colon \pi \to (\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G)(\pi)$ as follows. Let $\pi\colon BG \to \mathrm{Vect}_k$ be a k-linear representation of G, and let $V = \pi(0)$. We define

$$V \xrightarrow{\eta_\pi} \mathrm{Map}_H(G, V)$$

by $\eta_\pi(\boldsymbol{v})(x) = \pi(x)(\boldsymbol{v})$. The calculation

$$\eta_\pi(\boldsymbol{v})(h \cdot x) = \pi(h \cdot x)(\boldsymbol{v}) = (\pi(h) \circ \pi(x))(\boldsymbol{v}) = \pi(h)(\eta_\pi(\boldsymbol{v}))$$

shows that $\eta_\pi \in \mathrm{Map}_H(G, V)$, so the map is well-defined. And the calculation

$$\eta_\pi(g \cdot \boldsymbol{v})(x) = \pi(x)(\pi(g)(\boldsymbol{v})) = \pi(x \cdot g)(\boldsymbol{v}) = \eta_\pi(\boldsymbol{v})(x \cdot g) = (g \cdot \eta_\pi(\boldsymbol{v}))(x)$$

shows that it intertwines between the two representations in question. Thus, we obtain the following special case of Frobenius reciprodity II, which we proved in Theorem 8 of Lecture 9.

Theorem 3 (Frobenius reciprocity II)**.** *In the situation above, the maps*

$$\mathrm{Hom}(\mathrm{Res}_H^G(\pi), \tau) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Hom}(\pi, \mathrm{Ind}_H^G(\tau))$$

defined by $\alpha(h) = \mathrm{Ind}_H^G(h) \circ \eta_\pi$ *and* $\beta(k) = \epsilon_\tau \circ \mathrm{Res}_H^G(k)$ *are each other's inverses.*

Example 4. Let $G = \Sigma_4$ be the group of permutations of the set $\{1,2,3,4\}$, and let $H \subset G$ be the subgroup of permutations σ such that $\sigma(4) = 4$. We identify H with the group Σ_3 of permutations $\{1,2,3\}$ via the group isomorphism $\rho\colon H \to \Sigma_3$ defined by $\rho(\sigma) = \sigma|_{\{1,2,3\}}$. We let $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5$ be the irreducible complex representations of G defined in Lecture 7, and let τ_1, τ_2, τ_3 be the irreducible complex representations of H defined in Lecture 1. So π_1 and τ_1 are the 1-dimensional trivial representations, π_2 and τ_2 are the 1-dimensional sign representations, π_3 and τ_3 are the standard representations of dimension 3 and 2, respectively, $\pi_4 \simeq \pi_2 \otimes \pi_3$ is 3-dimensional, and π_5 is 2-dimensional. We wish to understand

$$\pi = \operatorname{Ind}_H^G(\tau_1),$$

which has $\dim_{\mathbb{C}}(\pi) = [G : H] \cdot \dim_{\mathbb{C}}(\tau_1) = 4$. We have the canonical isomorphism

$$\textstyle\bigoplus_{1 \leq i \leq 5} \operatorname{Hom}(\pi_i, \pi) \otimes \pi_i \longrightarrow \pi$$

that to $f_i \otimes \boldsymbol{x}_i$ assigns $f_i(\boldsymbol{x}_i)$, and by Frobenius reciprocity,

$$\operatorname{Hom}(\pi_i, \pi) = \operatorname{Hom}(\pi_i, \operatorname{Ind}_H^G(\tau_1)) \simeq \operatorname{Hom}(\operatorname{Res}_H^G(\pi_i), \tau_1).$$

We see immediately form the definitions that

$$\begin{aligned} \operatorname{Res}_H^G(\pi_1) &\simeq \tau_1 \\ \operatorname{Res}_H^G(\pi_2) &\simeq \tau_2 \\ \operatorname{Res}_H^G(\pi_3) &\simeq \tau_1 \oplus \tau_3, \end{aligned}$$

so by Schur's lemma, we conclude that the canonical map

$$\operatorname{Hom}(\pi_1, \pi) \otimes \pi_1 \oplus \operatorname{Hom}(\pi_3, \pi) \otimes \pi_3 \longrightarrow \pi$$

is an isomorphism. Hence, less canonically, we have an isomorphism

$$\operatorname{Ind}_H^G(\tau_1) \simeq \pi_1 \oplus \pi_3.$$

Let us finish the calculation of $\operatorname{Res}_H^G(\pi_i)$. Using that $\pi_4 = \pi_2 \otimes \pi_3$, we get

$$\begin{aligned} \operatorname{Res}_H^G(\pi_4) = \operatorname{Res}_H^G(\pi_2 \otimes \pi_3) &\simeq \operatorname{Res}_H^G(\pi_2) \otimes \operatorname{Res}_H^G(\pi_3) \\ &\simeq \tau_2 \otimes (\tau_1 \oplus \tau_3) \simeq \tau_2 \oplus \tau_3, \end{aligned}$$

where the second identification uses the "symmetric monoidal" structure on Res_H^G. Finally, we consider the diagram of groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \overset{q}{\longrightarrow} & H & \longrightarrow & 1, \\ & & & & \uparrow{\scriptstyle f} & \diagup\!\!\diagup & & & \\ & & & & H & & & & \end{array}$$

where $N = \{e, (12)(34), (13)(24), (14)(23)\}$, and where q maps $g \in G$ to the unique element $q(g) \in H \cap gN$. In Lecture 7, we defined $\pi_5 = q^*(\tau_3)$, so we find that

$$\operatorname{Res}_H^G(\pi_5) = (f^* \circ q^*)(\tau_3) = (q \circ f)^*(\tau_3) = \tau_3.$$

Remark 5. As Example 4 shows, if π is irreducible, then $\operatorname{Res}_H^G(\pi)$ may well not be so. (Physicists call this "symmetry breaking.") The example also shows that if τ is irreducible, then $\operatorname{Ind}_H^G(\tau)$ may also not be irreducible.

Suppose $H \subset G$ is a subgroup of finite index $[G : H] = n$. In this case, the map

$$G/H \xrightarrow{p} G/G$$

is proper, so by Theorem 9 from Lecture 9, the norm map $\mathrm{Nm}_p \colon p_! \to p_*$ is a natural isomorphism. This means that, under this assumption, the functor Ind_H^G is also left adjoint to Res_H^G. Let us spell out the adjunction

$$(\mathrm{Ind}_H^G, \mathrm{Res}_H^G, \epsilon', \eta').$$

We choose a family $(g_1, \ldots, g_n)$ of representatives of the right cosets $Hg \in H\backslash G$. If (V, π) is a k-linear representation of G, then we define the counit

$$(\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G)(\pi) \xrightarrow{\epsilon'_\pi} \pi$$

to be the k-linear map $\epsilon'_\pi \colon \mathrm{Map}_H(G, V) \to V$ given by $\epsilon'_\pi(f) = \sum_{1 \leq i \leq n} f(g_i)$, and if (W, τ) is a k-linear representation of H, then we define the unit

$$\tau \xrightarrow{\eta'_\tau} (\mathrm{Res}_H^G \circ \mathrm{Ind}_H^G)(\tau)$$

to be the k-linear map $\eta'_\tau \colon W \to \mathrm{Map}_H(G, W)$ given by

$$\eta'_\tau(\boldsymbol{w})(x) = \begin{cases} \tau(x)(\boldsymbol{w}) & \text{if } x \in H, \\ 0 & \text{if } x \notin H. \end{cases}$$

Therefore, by invoking Proposition 1, Frobenius preciprodicity I, which we proved in Theorem 7 of Lecture 9, specializes to the following result.

Theorem 6. *Let G be a group, and let $H \subset G$ be a subgroup of finite index. Given k-linear representations π and τ of G and H, respectively, the maps*

$$\mathrm{Hom}(\mathrm{Ind}_H^G(\tau), \pi) \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} \mathrm{Hom}(\tau, \mathrm{Res}_H^G(\pi))$$

defined by $\alpha'(h) = \mathrm{Res}_H^G(h) \circ \eta'_\tau$ and $\beta'(k) = \epsilon'_\pi \circ \mathrm{Ind}_H^G(k)$ are each other's inverses.

Remark 7. The restriction $\mathrm{Res}_H^G = f^*$ always has the left adjoint $\mathrm{ind}_H^K = f_!$, but the norm map $\mathrm{Nm}_f \colon f_! \to f_*$ is a natural isomorphism only if $[G : H] < \infty$.

Let $H, K \subset G$ be two subgroups, and let σ and τ be k-linear representations of H and K, respectively. Frobenius reciprocity gives us the canonical isomorphism

$$\mathrm{Hom}(\mathrm{Ind}_H^G(\sigma), \mathrm{Ind}_K^G(\tau)) \xrightarrow{\beta} \mathrm{Hom}((\mathrm{Res}_K^G \circ \mathrm{Ind}_H^G)(\sigma), \tau),$$

so we would like to understand the functor $\mathrm{Res}_K^G \circ \mathrm{Ind}_H^G$, and this is exactly what the base-change theorem allows us to do. We first determine the set

$$\mathrm{Map}_G(G/H, G/K)$$

of G-equivariant maps $f \colon G/H \to G/K$. Given such a map, we have $f(H) = aK$, for some $a \in G$, and hence, by the G-equivariance of f, we have

$$f(gH) = gaK$$

for all $g \in G$. In particular, we have $haK = aK$ for all $h \in H$, or equivalently,

$$a^{-1}Ha \subset K.$$

Conversely, given $a \in G$ such that $a^{-1}Ha \subset K$, the map $f_a \colon G/H \to G/K$ defined by $f_a(gH) = gaK$ is G-equivariant. Moreover, we observe that $f_a = f_b$ if and only if $aK = bK$, or equivalently, if and only if

$$a^{-1}b \in K.$$

If $a^{-1}Ha = K$, then $f_a = r_a$ is the G-equivariant map

$$G/H \xrightarrow{r_a} G/a^{-1}Ka$$

given by right multiplication by a. Indeed,

$$f_a(gH) = gaa^{-1}Ha = gHa = r_a(gH).$$

In general, if $a^{-1}Ha \subset K$, then f_a factors in two ways

$$\begin{array}{ccc} G/H & \xrightarrow{p_H^{aKa^{-1}}} & G/aKa^{-1} \\ {\scriptstyle r_a}\downarrow & \searrow^{f_a} & \downarrow{\scriptstyle r_a} \\ G/a^{-1}Ha & \xrightarrow{p_{a^{-1}Ha}^{K}} & G/K \end{array}$$

as the composition of r_a and the canonical projections.

We now assume that G is finite and consider the cartesian square of left G-sets

$$\begin{array}{ccc} X & \xrightarrow{p_1} & G/H \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_H^G} \\ G/K & \xrightarrow{p_K^G} & G/G, \end{array}$$

where $H, K \subset G$ are subgroups and $X = G/H \times G/K$. The base-change theorem, Theorem 11 in Lecture 9, gives a canonical natural isomorphism

$$(p_K^G)^* \circ (p_H^G)_* \longrightarrow p_{2*} \circ p_1^*,$$

so we wish to understand the left G-set X. The map $s \colon G/K \to X$ defined by $s(aK) = (H, aK)$ is not G-equivariant, unless $H = G$, but it induces a surjection

$$G/K \xrightarrow{\bar{s}} G \backslash X = \pi_0([G \backslash X])$$

that maps aK to the G-orbit $\bar{s}(aK) = G \cdot (H, aK)$ through $s(aK) = (H, aK)$, and moreover, (H, aK) and (H, bK) are in the same G-orbit if and only if $ab^{-1} \in H$. This shows that we have a bijection

$$H \backslash G/K \longrightarrow G \backslash X$$

that to HaK assigns the G-orbit $G \cdot (H, aK)$. Moreover, the isotropy subgroup at (H, aK) for the left action by G on X is equal to

$$G_{(H,aK)} = H \cap aKa^{-1},$$

since $(H, aK) = (gH, gaK)$ if and only if $g \in H$ and $g \in aKa^{-1}$. We now choose a map $a \colon \{1, 2, \dots, m\} \to G$, whose composition with the canonical projection

$$\{1, 2, \dots, m\} \xrightarrow{a} G \xrightarrow{q} H \backslash G/K$$

is a bijection. We write $a_s = a(s)$ and say that $(a_1, a_2, \dots, a_m)$ is a family of double coset representatives. With this choice, we obtain a G-equivariant bijection

$$\textstyle\coprod_{1 \leq s \leq m} G/(H \cap a_s K a_s^{-1}) \xrightarrow{\ u\ } X$$

that to $g(H \cap a_s K a_s^{-1})$ assigns $(gH, ga_s K)$. Moreover, we have

$$p_1 \circ u = \textstyle\sum_{1 \leq s \leq m} p^H_{H \cap a_s K a_s^{-1}}$$

$$p_2 \circ u = \textstyle\sum_{1 \leq s \leq m} r_{a_s} \circ p^{a_s K a_s^{-1}}_{H \cap a_s K a_s^{-1}},$$

where "Σ" is notation for the map from the disjoint union that on the sth summand is given by the indicated map. Finally, we note that the diagram

$$\begin{array}{ccc} B(aKa^{-1}) & \xrightarrow{\ i_{aKa^{-1}}\ } & [G \backslash (G/aKa^{-1})] \\ \downarrow{\scriptstyle c_a} & & \downarrow{\scriptstyle r_a} \\ BK & \xrightarrow{\ i_K\ } & [G \backslash (G/K)], \end{array}$$

where $c_a \colon aKa^{-1} \to K$ maps aka^{-1} to k, commutes, up to the natural isomorphism

$$i_K \circ c_a \longrightarrow r_a \circ i_{aKa^{-1}}$$

defined by the isomorphism

$$\begin{array}{ccc} (i_K \circ c_a)(0) & \longrightarrow & (r_a \circ i_{aKa^{-1}})(0) \\ \| & & \| \\ K & \xrightarrow{\ (a,K)\ } & aK \end{array}$$

in the category $[G \backslash (G/K)]$.

With all these choices made, the base-change theorem gives rise to the following result known as the double coset formula.

Theorem 8. *In the situation above, there is a natural isomorphism*

$$\textstyle\bigoplus_{1 \leq s \leq m} c_{a_s *} \circ \mathrm{Ind}^{a_s K a_s^{-1}}_{H \cap a_s K a_s^{-1}} \circ \mathrm{Res}^H_{H \cap a_s K a_s^{-1}} \longrightarrow \mathrm{Res}^G_K \circ \mathrm{Ind}^G_H$$

that depends on the various choices made.

Proof. By the base-change theorem, the diagram

$$\begin{array}{ccc} \mathrm{QCoh}([G \backslash X]) & \xleftarrow{\ p_1^*\ } & \mathrm{QCoh}([G \backslash (G/H)]) \\ {\scriptstyle p_{2*}}\downarrow & & \downarrow{\scriptstyle (p^G_H)_*} \\ \mathrm{QCoh}([G \backslash (G/K)]) & \xleftarrow{\ (p^G_K)^*\ } & \mathrm{QCoh}([G \backslash (G/G)]) \end{array}$$

commutes, up to canonical natural isomorphism. Moreover, using the (non-canonical) G-equivariant bijection

$$\textstyle\coprod_{1 \leq s \leq m} G/(H \cap a_s K a_s^{-1}) \xrightarrow{\ u\ } X = G/H \times G/K,$$

this translates into a diagram

$$\begin{array}{ccc}
\prod_{1\le s\le m}\operatorname{Rep}_k(H\cap a_sKa_s^{-1}) & \xleftarrow{(\operatorname{Res}^H_{H\cap a_sKa_s^{-1}})} & \operatorname{Rep}_k(H) \\
\downarrow \prod \operatorname{Ind}^{a_sKa_s^{-1}}_{H\cap a_sKa_s^{-1}} & & \\
\prod_{1\le s\le m}\operatorname{Rep}_k(a_sKa_s^{-1}) & & \\
\downarrow \prod\, c_{a_s *} & & \downarrow \operatorname{Ind}^G_H \\
\prod_{1\le s\le m}\operatorname{Rep}_k(K) & & \\
\downarrow \oplus & & \\
\operatorname{Rep}_k(K) & \xleftarrow{\operatorname{Res}^G_K} & \operatorname{Rep}_k(G),
\end{array}$$

which commutes, up to a natural isomorphism that depends on the (many) choices made. The translation uses the fact, which we stated as Proposition 1, that adjoints of functors, if they exist, are unique, up to unique natural isomorphism. □

We will use these results to prove a theorem called the intertwining number theorem. So we let G be a finite group, and let $H, K \subset G$ be subgroup. Let (V, σ) and (W, τ) be k-linear representations of H and K, respectively. By Frobenius reciprodicity I+II and the double coset formula, we obtain isomorphisms

$$\begin{aligned}
&\operatorname{Hom}(\operatorname{Ind}^G_H(\sigma), \operatorname{Ind}^G_K(\tau)) \simeq \operatorname{Hom}((\operatorname{Res}^G_K \circ \operatorname{Ind}^G_H)(\sigma), \tau) \\
&\simeq \textstyle\bigoplus_{1\le s\le m} \operatorname{Hom}((c_{a_s *} \circ \operatorname{Ind}^{a_sKa_s^{-1}}_{H\cap a_sKa_s^{-1}} \circ \operatorname{Res}^H_{H\cap a_sKa_s^{-1}})(\sigma), \tau) \\
&\simeq \textstyle\bigoplus_{1\le s\le m} \operatorname{Hom}(\operatorname{Res}^H_{H\cap a_sKa_s^{-1}}(\sigma), (\operatorname{Res}^{a_sKa_s^{-1}}_{H\cap a_sKa_s^{-1}} \circ c^*_{a_s})(\tau)).
\end{aligned}$$

We note that for $a \in G$, the k-vector space

$$\operatorname{Hom}(\operatorname{Res}^H_{H\cap aKa^{-1}}(\sigma), (\operatorname{Res}^{aKa^{-1}}_{H\cap aKa^{-1}} \circ c^*_a)(\tau))$$

consists of the k-linear maps $f \colon V \to W$ such that

$$f(\sigma(h)(\boldsymbol{v})) = \tau(a^{-1}ha)(f(\boldsymbol{v}))$$

for all $h \in G$ and $\boldsymbol{v} \in V$, or equivalently, such that

$$f \circ \sigma(h) = \tau(k) \circ f$$

for all $(h, k) \in H \times K$ with $ha = ak$. Let us write $d(\sigma, \tau; s)$ for the dimension of this k-vector space for $a = a_s$. To see that it only depends on σ, τ, and s, and not on the choice of $a_s \in Ha_sK \in H\backslash G/K$, we rewrite the calculation of

$$\operatorname{Hom}(\operatorname{Ind}^G_H(\sigma), \operatorname{Ind}^G_K(\tau))$$

in a way that does not involve any choices. If we let

$$X_s \xrightarrow{i_s} X = G/H \times G/K$$

be the inclusion of the sth orbit, then the calculation becomes

$$\begin{aligned}
&\operatorname{Hom}((p^G_H)_*(\sigma), (p^G_K)_*(\tau)) \simeq \operatorname{Hom}(((p^G_K)^* \circ (p^G_H)_*)(\sigma), \tau) \\
&\simeq \operatorname{Hom}(p_{2*}\, p_1^*(\sigma), \tau) \simeq \oplus_{1\le s\le m} \operatorname{Hom}(p_{2*}\, i_{s*}\, i_s^*\, p_1^*(\sigma), \tau) \\
&\simeq \oplus_{1\le s\le m} \operatorname{Hom}(p_{2!}\, i_{s!}\, i_s^*\, p_1^*(\sigma), \tau) \simeq \oplus_{1\le s\le m} \operatorname{Hom}(i_s^*\, p_1^*(\sigma), i_s^*\, p_2^*(\tau)),
\end{aligned}$$

which, in turn, gives the formula

$$d(\sigma,\tau;s) = \dim_k \operatorname{Hom}(i_s^*\,p_1^*(\sigma), i_s^*\,p_2^*(\tau)).$$

So this number manifestly only depends on σ, τ, and s. Finally, by taking dimensions everywhere, we obtain the following theorem due to Mackey.

Theorem 9 (Intertwining number theorem)**.** *In the situation above,*

$$\dim_k \operatorname{Hom}(\operatorname{Ind}_H^G(\sigma), \operatorname{Ind}_K^G(\tau)) = \textstyle\sum_{1\le s\le m} d(\sigma,\tau;s).$$