REPRESENTATIONS OF THE SYMMETRIC GROUPS

Let X be a finite set with n elements, and let $G = \mathrm{Aut}(X)$ be its group of automorphisms. We proceed to construct representatives for all isomorphism classes of irreducible complex representations of G. Since the set of isomorphism classes of irreducible complex representations is bijective to the set $C(G)$ of conjugacy classes of elements in G, we first introduce some language to understand this set.

We recall that every permutation $g \in G$ can be written as a product

$$g = g_1 \cdots g_m$$

of disjoint cycles and that this product decomposition is unique, up to a reordering of the factors. The cycle type of $g \in G$ is the sequence $(\lambda_1, \dots, \lambda_m)$ of lengths of the cycles $g_1, \dots, g_m$, listed in non-increasing order, and it is a basic fact that two permutations $g, h \in G$ are conjugate if and only if they have the same cycle type.

Definition 1. Let n be a non-negative integer. A partition of n is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that $\sum_{i \geq 1} \lambda_i = n$.

Let $\mathrm{Part}(n)$ be the set of partitions of n. The map that to a permutation $g \in G$ assigns its cycle type $\lambda(g) \in \mathrm{Part}(n)$ induces a bijection

$$C(G) \longrightarrow \mathrm{Part}(n).$$

Example 2. Let $n = 7$, and let $g \in G$ be the permutation given by

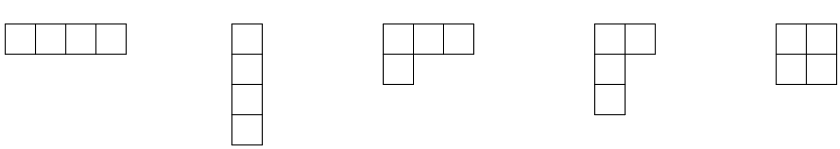
i	1	2	3	4	5	6	7
$g(i)$	5	1	6	3	2	4	7

We have $g = (152)(643)(7)$, so g has cycle type $\lambda(g) = (3, 3, 1)$.

Let us write $\mathbb{N}$ for the set of positive integers.

Definition 3. A Young diagram is a finite subset $S \subset \mathbb{N} \times \mathbb{N}$ with the property that for all $(i, j) \in \mathbb{N} \times \mathbb{N}$, if either $(i+1, j) \in S$ or $(i, j+1) \in S$ or both, then $(i, j) \in S$. The cardinality of the set S is called the size of the Young diagram.

Example 4. We picture a Young diagram as a collection of boxes arranged as the entries in a matrix. For instance, there are five Young diagrams of size $n = 4$:



We will see that these correspond to the five isomorphism classes of irreducible complex representations $G = \Sigma_4$.

Given a Young diagram S of size n, we define its row partition $\lambda(S)$ by

$$\lambda(S)_i = \mathrm{card}(\{j \in \mathbb{N} \mid (i,j) \in S\}),$$

and we define its column partition $\mu(S)$ by

$$\mu(S)_j = \mathrm{card}(\{i \in \mathbb{N} \mid (i,j) \in S\}).$$

We write $\mathrm{Young}(n)$ for the set of Young diagrams of size n.

Proposition 5. *The maps* $\lambda, \mu \colon \mathrm{Young}(n) \to \mathrm{Part}(n)$ *that to a Young diagram assign its row partition* $\lambda(S)$ *and column partition* $\mu(S)$ *are bijections.*

Proof. This is clear from the definitions. $\square$

Example 6. In Example 2, the row partition (resp. column partition) of the first (resp. third) Young diagram is equal to the column partition (resp. row partition) of the second (resp. fourth) Young diagram. The row and column partitions of the fifth Young diagram are equal.

Definition 7. Let X be a finite set. A Young tableau on X is an injective map

$$X \xrightarrow{\;u\;} \mathbb{N} \times \mathbb{N}$$

whose image $S = u(X)$ is a Young diagram.

Given a Young tableau, the map $u \colon X \to S = u(X)$ is a bijection. So to specify a Young tableau $u \colon X \to \mathbb{N} \times \mathbb{N}$ with a given Young diagram S as its image amounts to assigning an element of X to each "box" in S.

Example 8. The figures

8	12	4	9	1
11	3	10		
5	6	13		
7	2			

illustrate a Young tableau on $X = \{1, 2, \dots, 13\}$ and its underlying Young diagram.

Let $\mathrm{Tabl}(X)$ be the set of Young tableaux on X. The group homomorphism

$$G^{\mathrm{op}} = \mathrm{Aut}(X)^{\mathrm{op}} \xrightarrow{\;\rho\;} \mathrm{Aut}(\mathrm{Tabl}(X))$$

given by $\rho(g)(u) = u \circ g$ defines a right action by the group G on $\mathrm{Tabl}(X)$. It is free action. Indeed, if $u \circ g = u$, then $g = e$, since u is injective.

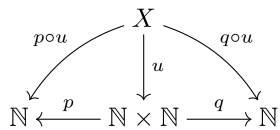
Proposition 9. *The map to a Young tableau assigns its image induces a bijection*

$$\mathrm{Tabl}(X)/G \longrightarrow \mathrm{Young}(n)$$

from the set of orbits of the right action by G *on* $\mathrm{Tabl}(X)$ *onto the set of Young diagrams of size* $n = \mathrm{card}(X)$.

Proof. Indeed, the map is surjective, by the definition of a Young tableau, and it is injective, since two Young tableaux $u, v \colon X \to \mathbb{N} \times \mathbb{N}$ have the same image if and only if there exists a bijection $g \colon X \to X$ such that $v = u \circ g$. $\square$

Given a Young tableau u, we consider its composition with the projections



given by $p(i, j) = i$ and $q(i, j) = j$, respectively.

Definition 10. The row stabilizer $H \subset G$ and the column stabilizer $K \subset G$ of a Young tableau $u\colon X \to \mathbb{N} \times \mathbb{N}$ are the subgroups

$$H = \{g \in G \mid p \circ u \circ g = p \circ u\} \subset G,$$
$$K = \{g \in G \mid q \circ u \circ g = q \circ u\} \subset G.$$

More informally, the row stabilizer consists of the permutations, which permute the elements within the rows of a Young tableau, but which do not permute elements that belong to separate rows. Similarly for the column stabilizer.

Lemma 11. *Let $u\colon X \to \mathbb{N} \times \mathbb{N}$ be a Young tableau. If $H, K \subset G = \mathrm{Aut}(X)$ are its row and column stabilizers, then $H \cap K = \{e\}$.*

Proof. Indeed, if $p \circ u \circ g = p \circ u$ and $q \circ u \circ g = q \circ u$, then $u \circ g = u$, and, as we have already noticed, this implies that $g = e$, since u is injective. □

We give the set $\mathrm{Part}(n)$ of partitions of n the lexicographic order, where $\lambda > \mu$ if there exists an $m \geq 1$ such that $\lambda_m > \mu_m$ and $\lambda_i = \mu_i$ for $1 \leq i < m$. It is a total order in the sense that if $\lambda \neq \mu$, then either $\lambda > \mu$ or $\mu > \lambda$.

Lemma 12. *Let $u, v\colon X \to \mathbb{N} \times \mathbb{N}$ be Young tableaux, let H be the row stabilizer of u, and let K be the column stabilizer of v. Let $S = u(X)$ and $T = v(X)$ be the underlying Young diagrams, and suppose that $\lambda(S) \geq \lambda(T)$. If, in addition, every row in u and every column in v have at most one element in common, then $S = T$ and there exists $h \in H$ and $k \in K$ such that $u \circ h = v \circ k$.*

Proof. We prove the statement by induction on $n = \mathrm{card}(X)$, the case $n = 1$ being trivial. So we let $n = m$ and assume that the statement has been proved for $n < m$. Let $X_i = (p \circ u)^{-1}(i) \subset X$ be the set of elements in the ith row of u, and let $Y_j = (q \circ v)^{-1}(j) \subset X$ for the set of elements in the jth column of v. By assumption, the intersection $X_i \cap Y_j$ has at most one element for all (i, j). Also by assumption, $\lambda(S) \geq \lambda(T)$, so in particular that $\lambda(S)_1 \geq \lambda(T)_1$. But since there are $\lambda(S)_1$ elements in X_1, and since at most one of them belongs to each of the columns $Y_1, \dots, Y_{\lambda(T)_1}$, we also have $\lambda(S)_1 \leq \lambda(T)_1$, so $\lambda(S)_1 = \lambda(T)_1$. We can now choose $h \in H$ such that for all $x \in X_1$,

$$(q \circ u \circ h)(x) = (q \circ v)(x),$$

and we can further choose $k \in K$ such that for $x \in X_1$,

$$(p \circ u)(x) = (p \circ v \circ k)(x).$$

It follows that for all $x \in X_1$, we have

$$(p \circ u \circ h)(x) = (p \circ u)(x) = (p \circ v \circ k)(x),$$
$$(q \circ u \circ h)(x) = (q \circ v)(x) = (q \circ v \circ k)(x),$$

which, in turn, implies that for all $x \in X_1$, we have

$$(u \circ h)(x) = (v \circ k)(x).$$

We can now bring ourselves in a position to invoke the inductive hypothesis. Indeed, we let $X' = X \smallsetminus X_1$, and define $u', v' \colon X' \to \mathbb{N} \times \mathbb{N}$ by

$$\begin{aligned}(p \circ u')(x) &= (p \circ u \circ h)(x) - 1,\\ (q \circ u')(x) &= (q \circ u \circ h)(x),\\ (p \circ v')(x) &= (p \circ v \circ k)(x) - 1,\\ (q \circ v')(x) &= (q \circ v \circ k)(x).\end{aligned}$$

It is clear that $S' = u'(X')$ and $T' = v'(X')$ again are Young diagrams, so that u' and v' are Young tableaux; that $\lambda(S') \geq \lambda(T')$; and that every row in u' and every column in v' at most have one element of X' in common. Let $G' = \operatorname{Aut}(X')$, and let $H', K' \subset G'$ be the row stabilizer of u' and the column stabilizer of v'. Since $\operatorname{card}(X') < \operatorname{card}(X)$, we conclude from the inductive hypothesis that $S' = T'$ and that there exist $h' \in H'$ and $k' \in K'$ such that $u' \circ h' = v' \circ k'$. We conclude that $S = T$. Moreover, since the group homomorphism $\rho \colon G' \to G$ defined by

$$\rho(g')(x) = \begin{cases} x & \text{if } x \in X_1,\\ g'(x) & \text{if } x \in X',\end{cases}$$

maps H' and K' into H and K, respectively, we further conclude that

$$u \circ h \circ \rho(h') = v \circ k \circ \rho(k').$$

This completes the proof. □

We can now reap the benefits of the work that we did in the last two lectures together with the lemmas above and classify all irreducible complex representations of $G = \operatorname{Aut}(X)$, up to non-canonical isomorphism. Let S be a Young diagram, let $u \colon X \to \mathbb{N} \times \mathbb{N}$ be a Young tableau with $u(X) = S$, and let $H, K \subset G$ be its row and column stabilizers. We define

$$\begin{aligned}\pi_S^+ &= (\operatorname{Ind}_H^G \circ \operatorname{Res}_H^G)(\tau)\\ \pi_S^- &= (\operatorname{Ind}_K^G \circ \operatorname{Res}_K^G)(\sigma),\end{aligned}$$

where τ is the 1-dimensional trivial representation of G and σ is the 1-dimensional sign representation of G.

Theorem 13. *Let X be a finite set with n elements, and let $G = \operatorname{Aut}(X)$.*

(1) *If S is a Young diagram of size n, then, up to non-canonical isomorphism, there is a unique irreducible complex representation π_S of G, which occurs in the decompositions of both π_S^+ and π_S^-.*
(2) *If S and T are distinct Young diagrams of size n, then the representations π_S and π_T are non-isomorphic.*
(3) *If π is an irreducible complex representation of G, then $\pi \simeq \pi_S$, for some Young diagram S of size n.*

Proof. To prove (1), it suffices to show that

$$\dim_{\mathbb{C}} \operatorname{Hom}(\pi_S^+, \pi_S^-) = 1,$$

and to do so, we will use the results on induced representations that we proved in the last two lectures. We consider the cartesian diagram of left G-sets

$$\begin{array}{ccc} G/H \times G/K & \xrightarrow{p_1} & G/H \\ \downarrow{\scriptstyle p_2} & \overset{f}{\searrow} & \downarrow{\scriptstyle p_H^G} \\ G/K & \xrightarrow{p_K^G} & G/G \end{array}$$

where we have included $f = p_H^G \circ p_1 = p_K^G \circ p_2$. We have canonical isomorphisms

$$\begin{aligned} \operatorname{Hom}(\pi_S^+, \pi_S^-) &= \operatorname{Hom}((p_H^G)_*(p_H^G)^*\tau, (p_K^G)_*(p_K^G)^*\sigma) \\ &\simeq \operatorname{Hom}((p_K^G)^*(p_H^G)_*(p_H^G)^*\tau, (p_K^G)^*\sigma) \\ &\simeq \operatorname{Hom}((p_K^G)_!\,(p_K^G)^*(p_H^G)_*(p_H^G)^*\tau, \sigma) \\ &\simeq \operatorname{Hom}((p_K^G)_!\,(p_K^G)^*(p_H^G)_!\,(p_H^G)^*\tau, \sigma) \\ &\simeq \operatorname{Hom}((p_K^G)_!\,p_{2!}\,p_1^*(p_H^G)^*\tau, \sigma) \\ &\simeq \operatorname{Hom}(f_!\,f^*\tau, \sigma) \\ &\simeq \operatorname{Hom}(f^*\tau, f^*\sigma). \end{aligned}$$

Moreover, we defined a non-canonical isomorphism of left G-sets

$$\textstyle\coprod_{1 \leq s \leq m} G/H \cap a_s K a_s^{-1} \longrightarrow G/H \times G/K,$$

which depends on a choice of a family $(a_1, \dots, a_m)$ of representatives of the double cosets $H \backslash G/K$, among other things. So we conclude that

$$\operatorname{Hom}(\pi_S^+, \pi_S^-) \simeq \textstyle\prod_{1 \leq s \leq m} \operatorname{Hom}(\operatorname{Res}^G_{H \cap a_s K a_s^{-1}}(\tau), \operatorname{Res}^G_{H \cap a_s K a_s^{-1}}(\sigma)).$$

Since both $\operatorname{Res}^G_{H \cap aKa^{-1}}(\tau)$ and $\operatorname{Res}^G_{H \cap aKa^{-1}}(\sigma)$ are 1-dimensional representations of $H \cap aKa^{-1}$, we find that

$$\begin{aligned} &\dim_{\mathbb{C}} \operatorname{Hom}(\operatorname{Res}^G_{H \cap aKa^{-1}}(\tau), \operatorname{Res}^G_{H \cap aKa^{-1}}(\sigma)) \\ &\qquad = \begin{cases} 1 & \text{if } \operatorname{sgn}(g) = 1 \text{ for all } g \in H \cap aKa^{-1}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For the double coset $HaK = HK$, we have

$$\dim_{\mathbb{C}} \operatorname{Hom}(\operatorname{Res}^G_{H \cap K}(\tau), \operatorname{Res}^G_{H \cap K}(\sigma)) = 1,$$

since $H \cap K = \{e\}$ by Lemma 11. Hence, we must show that if $a \notin HK$, then there exists $g \in H \cap aKa^{-1}$ such that $\operatorname{sgn}(g) = -1$. To this end, we consider, in addition to the tableau $u \colon X \to \mathbb{N} \times \mathbb{N}$, the tableau $v = u \circ a^{-1} \colon X \to \mathbb{N} \times \mathbb{N}$, whose column stabilizer is aKa^{-1}. We claim that there exists a row in u and a column in v, which have at least two elements in common. Granting this claim, the transposition g that interchanges these two elements belongs to $H \cap aKa^{-1}$ and has $\operatorname{sgn}(g) = -1$, which proves (1). To prove claim, we assume that every row in u and every column in v have at most one element in common. In this case, Lemma 12 shows that there exists $h \in H$ and $aka^{-1} \in aKa^{-1}$ such that $u \circ h = v \circ aka^{-1}$. But then $a = h^{-1}k \in HK$, which is a contradicts that $a \notin HK$.

To prove (2), it suffices to show that

$$\dim_{\mathbb{C}} \operatorname{Hom}(\pi_S^+, \pi_T^-) = 0.$$

Arguing as in the proof of (1), we see that it further suffices to show that for all Young tableaux $u, v\colon X \to \mathbb{N}\times\mathbb{N}$ with $u(X) = S$ and $v(X) = T$, there exists a row in u and a column in v that have at least two elements in common. But this follows immediately from Lemma 12. Indeed, since the lexicographic order on $\mathrm{Part}(n)$ is a total order, we can assume without loss of generality that $\lambda(S) \geq \lambda(T)$.

Finally, we prove (3). We have constructed the family

$$(\pi_S)_{S\in\mathrm{Young}(n)}$$

of pairwise non-isomorphic irreducible complex representations of G. But the set of Young diagrams of size n and the set of conjugacy classes of elements in G are bijective, so we have found all irreducible complex representations of G, up to non-canonical isomorphism. □

The representation π_S is called the Specht representation associated with the Young diagram S. Its isomorphism class is independent of the choice of Young tableau u that we made in its definition.

Remark 14. We defined $\pi_S^+ = (\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G)(\tau)$ and $\pi_S^- = (\mathrm{Ind}_K^G \mathrm{Res}_K^G)(\sigma)$, but we could of course just as well have switched τ and σ in this definition.

If the subset $S \subset \mathbb{N}\times\mathbb{N}$ is a Young diagram of size n, then so is the subset

$$S' = \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid (j,i) \in S\},$$

which we call the conjugate Young diagram of S.

Lemma 15. *If S is a Young diagram of size n, and if S' is its conjugate Young diagram, then the associated Specht representations are related by*

$$\pi_{S'} \simeq \pi_S \otimes \sigma.$$

Proof. Let $u\colon X \to \mathbb{N}\times\mathbb{N}$ be a Young tableau with $u(X) = S$, and let H and K be its row stabilizer and column stabilizer. Let $u'\colon X \to \mathbb{N}\times\mathbb{N}$ be the unique map with $p \circ u' = q \circ u$ and $q \circ u' = p \circ u$. Then $u'(X) = S'$ and u' has row stabilizer K and column stabilizer H. Thus,

$$\begin{aligned}
\pi_{S'}^+ &= (\mathrm{Ind}_K^G \circ \mathrm{Res}_K^G)(\tau) \simeq (\mathrm{Ind}_K^G \circ \mathrm{Res}_K^G)(\sigma \otimes \sigma) \\
&\simeq (\mathrm{Ind}_K^G \circ \mathrm{Res}_K^G)(\sigma) \otimes \sigma \simeq \pi_S^- \otimes \sigma \\
\pi_{S'}^- &= (\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G)(\sigma) \simeq (\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G)(\tau \otimes \sigma) \\
&\simeq (\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G)(\tau) \otimes \sigma \simeq \pi_S^+ \otimes \sigma.
\end{aligned}$$

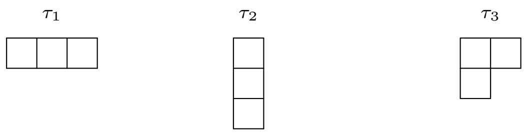
Here we have used that, in general, for $H \subset G$, one has

$$\begin{aligned}
\mathrm{Res}_H^G(\pi \otimes \rho) &\simeq \mathrm{Res}_H^G(\pi) \otimes \mathrm{Res}_H^G(\rho) \\
\mathrm{Ind}_H^G(\sigma \otimes \mathrm{Res}_H^G(\rho)) &\simeq \mathrm{Ind}_H^G(\sigma) \otimes \rho
\end{aligned}$$

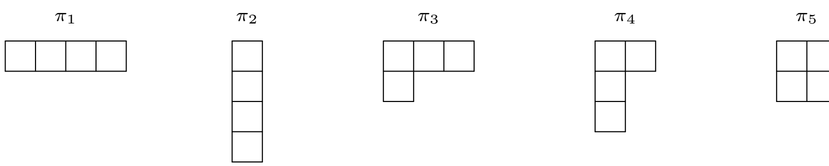
for all representations π and ρ of G and σ of H. The latter identity is called the projection formula. □

Example 16. For $H = \Sigma_3$, we have earlier found three irreducible finite dimensional complex representations of H, namely, the 1-dimensional trivial representation τ_1

and sign representation τ_2, and the 2-dimensional standard representation τ_3. These correspond to the following Specht representations:



Similarly, for $G = \Sigma_4$, we have earlier found five irreducible finite dimensional complex representations of G, namely, the 1-dimensional trivial representation π_1 and sign representation π_2, the 3-dimensional standard representation π_3 and its tensor product $\pi_4 = \pi_2 \otimes \pi_3$ with the sign representation, and the 2-dimensional representation π_5. These correspond to the following Specht representations:



Using Lemma 15, we see immediately from these listings that $\tau_2 \otimes \tau_3 \simeq \tau_3$ and that $\pi_2 \otimes \pi_5 \simeq \pi_5$. If we identify H with the subgroup of G consisting of all $g \in G$ with $g(4) = 4$, then one can also show that, in terms of Young diagrams, Res_H^G takes an irreducible G-representation π to the sum with multiplicity one of all irreducible H-representations τ corresponding to the Young diagrams obtained from the Young diagram for π by removing one box. So we have

$$\begin{aligned}
\operatorname{Res}_H^G(\pi_1) &\simeq \tau_1 \\
\operatorname{Res}_H^G(\pi_2) &\simeq \tau_2 \\
\operatorname{Res}_H^G(\pi_3) &\simeq \tau_1 \oplus \tau_3 \\
\operatorname{Res}_H^G(\pi_4) &\simeq \tau_2 \oplus \tau_3 \\
\operatorname{Res}_H^G(\pi_5) &\simeq \tau_3
\end{aligned}$$

Similarly, one can show that Ind_H^G takes an irreducible H-representation τ to the sum with multiplicity one of all irreducible G-representations π corresponding to the Young diagrams obtained from the Young diagram associated with τ by adding one box. So we find that

$$\begin{aligned}
\operatorname{Ind}_H^G(\tau_1) &\simeq \pi_1 \oplus \pi_3 \\
\operatorname{Ind}_H^G(\tau_2) &\simeq \pi_2 \oplus \pi_3 \\
\operatorname{Ind}_H^G(\tau_3) &\simeq \pi_3 \oplus \pi_4 \oplus \pi_5,
\end{aligned}$$

which is also what we have calculated directly before.

Finally, we mention that for Young diagrams S and T, Frobenius has given a formula for the value $\chi_{\pi_S}(g)$ of the character of the Specht representation π_S on an element g in the conjugacy class corresponding to T in terms of combinatorial data that can be read off from the Young diagrams S and T directly. The formula is called the Frobenius character formula.