SMOOTH MANIFOLDS

We recall that a topological group is defined to be a group $G = (G, \mu, \iota)$ together with a topology on the set G such that the maps $\mu\colon G \times G \to G$ and $\iota\colon G \to G$ are continuous. Similarly, a Lie group is defined to be a group $G = (G, \mu, \iota)$ together with a structure of smooth manifold on the set G such that the maps $\mu\colon G \times G \to G$ and $\iota\colon G \to G$ are smooth. We first discuss smooth manifolds.

Smooth manifolds belong to geometry rather than topology. Geometric objects are pairs $(X, \mathcal{O}_X)$ of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X, where for $U \subset X$ open, the set $\Gamma(U, \mathcal{O}_X)$ of sections of $\mathcal{O}_X$ over U should be thought of as the set "geometric functions" on U. The geometric functions that we allow will depend on the geometric situation that we consider. For instance, we could consider "smooth functions," "analytic functions," or "algebraic functions," but note that we have not yet assigned any precise mathematical meaning to these terms. Moreover, in some situations, the elements of $\Gamma(U, \mathcal{O}_X)$ may not be functions in the usual sense. A map of geometric objects $f\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a pair $(f, f^\sharp)$ of a continuous map $f\colon Y \to X$ and a map of sheaves of rings $f^\sharp\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$. Let us now define sheaves properly.

Let X be a topological space, and let X_{Zar} be the category, whose objects are the open subsets $U \subset X$, and whose morphisms are

$$\mathrm{Hom}_{X_{\mathrm{Zar}}}(U, V) = \begin{cases} \{\mathrm{incl}_U^V\} & \text{if } U \subset V \\ \emptyset & \text{if } U \not\subset V. \end{cases}$$

So if $U \subset V$, then there is a unique morphism $\mathrm{incl}_U^V\colon U \to V$, and if $U \not\subset V$, then there are no morphisms from U to V. A presheaf of sets on X is a defined to be a functor $\mathcal{F}\colon X_{\mathrm{Zar}}^{\mathrm{op}} \to \mathrm{Set}$. To specify a functor $\mathcal{F}\colon X_{\mathrm{Zar}}^{\mathrm{op}} \to \mathrm{Set}$, we must specify for every open subset $U \subset X$, a set $\mathcal{F}(U)$, and for every inclusion $U \subset V$ of open subsets of X, a map $\mathcal{F}(\mathrm{incl}_U^V)\colon \mathcal{F}(V) \to \mathcal{F}(U)$. We may think of $F(U)$ as the set of "functions defined on U" and of $\mathcal{F}(\mathrm{incl}_U^V)$ as the map that to a "function defined on U" assigns the restriction of this function to a "function defined on V." To emphasize this interpretation, we also write $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ and call it the set of sections of $\mathcal{F}$ over U, and we write $\mathrm{Res}_U^V = \mathcal{F}(\mathrm{incl}_U^V)$ and call it the restriction from V to U. A presheaf $\mathcal{F}\colon X_{\mathrm{Zar}}^{\mathrm{op}} \to \mathrm{Set}$ is defined to be a sheaf if it satisfies the following sheaf condition: For every covering $(U_i \to U)_{i \in I}$ of an open subset $U \subset X$ by open subsets $U_i \subset U$, the diagram

$$\mathcal{F}(U) \xrightarrow{h} \prod_{i \in I} \mathcal{F}(U_i) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{(i,j) \in I \times I} \mathcal{F}(U_i \cap U_j)$$

is an equalizer. Here h is the unique map such that for all $i \in I$,

$$\mathrm{pr}_i \circ h = \mathrm{Res}_{U_i}^U,$$

and a and b are the unique maps such that for all $(i,j) \in I \times I$,

$$\mathrm{pr}_{(i,j)} \circ a = \mathrm{Res}^{U_i}_{U_i \cap U_j} \circ \mathrm{pr}_i$$
$$\mathrm{pr}_{(i,j)} \circ b = \mathrm{Res}^{U_j}_{U_i \cap U_j} \circ \mathrm{pr}_j \,.$$

That the diagram is an equalizer means that for all $(\varphi_i)_{i \in I} \subset \prod_{i \in I} \mathcal{F}(U_i)$ such that

$$a((\varphi_i)_{i \in I}) = b((\varphi_i)_{i \in I}),$$

there exists a unique $\varphi \in \mathcal{F}(U)$ such that

$$(\varphi_i)_{i \in I} = (\mathrm{Res}^U_{U_i}(\varphi))_{i \in I}.$$

Informally, the sheaf condition expresses that if we are given "functions" φ_i on U_i for all $i \in I$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ for all $(i,j) \in I \times I$, then there exists a unique "function" φ on U such that $\varphi_i = \varphi|_{U_i}$ for all $i \in I$.

Example 1. Let X be a topological space, and let $k = \mathbb{R}$ or $k = \mathbb{C}$. The presheaf $\mathcal{O}^{\mathrm{cont}}_X \colon X^{\mathrm{op}}_{\mathrm{Zar}} \to \mathrm{Set}$, where $\Gamma(U, \mathcal{O}^{\mathrm{cont}}_X)$ is defined to be the set of continuous functions $\varphi \colon U \to k$, and where $\mathrm{Res}^V_U \colon \Gamma(V, \mathcal{O}^{\mathrm{cont}}_X) \to \Gamma(U, \mathcal{O}^{\mathrm{cont}}_X)$ is defined to be the map $\mathrm{Res}^V_U(\varphi) = \varphi \circ \mathrm{incl}^V_U$, is a sheaf, because "being continuous" is a local property.

We define the category of presheaves of sets on X to be the category

$$\mathrm{PreShv}(X) = \mathrm{Fun}(X^{\mathrm{op}}_{\mathrm{Zar}}, \mathrm{Set}),$$

whose objects are functors and whose morphisms are natural transformations, and we define the category of sheaves on X to be the full subcategory

$$\mathrm{Shv}(X) \subset \mathrm{PreShv}(X),$$

whose objects are the sheaves on X. One can prove that there is an adjunction

$$\mathrm{PreShv}(X) \underset{\iota_X}{\overset{\mathrm{ass}_X}{\rightleftarrows}} \mathrm{Shv}(X)$$

where the right adjoint functor ι_X is the canonical inclusion of the subcategory of sheaves in the category of presheaves, and where the left adjoint functor ass_X takes a presheaf to its associated sheaf. This functor is called "sheafification."

Example 2. Let X be a topological space, and let $F \in \mathrm{PreShv}(X)$ be the presheaf of constant functions, $\mathcal{F}(U) = \{\varphi \colon U \to k \mid \varphi \text{ constant}\}$. It is not a sheaf, since "being constant" is not a local property. The associated sheaf $\mathrm{ass}_X(F) \in \mathrm{Shv}(X)$ is the sheaf of locally constant functions, $\mathrm{ass}_X(F)(U) = \{\varphi \colon U \to \mathbb{R} \mid \varphi \text{ locally constant}\}$.

It is a fundamental result of Grothendieck[1] that "sheafification" preserves finite limits. (The inclusion functor ι_X preserves all limits, as does every right adjoint functor.) In particular, it preserves finite products, which implies that it takes "presheaves of rings" to "sheaves of rings." Indeed, we define a presheaves of rings and sheaves of rings to be ring objects in $\mathrm{PreShv}(X)$ and $\mathrm{Shv}(X)$, respectively. A ring object in a category $\mathcal{C}$ with finite products is defined to be a sixtuple $(R, +, \cdot, -, 0, 1)$ of an object $R \in \mathcal{C}$, two morphisms $+, \cdot \colon R \times R \to R$, one morphism $- \colon R \to R$, and two morphisms $0, 1 \colon e \to R$ that satisfy the usual ring axioms. Here the empty product $e = R^0 \in \mathcal{C}$ is a terminal object.

[1] This result and many results are consequences of Grothendieck's theorem that, in the category of sets, filtered colimits and finite limits commute.

Let $f\colon Y \to X$ be a continous map. If $U \subset X$ is open, then $f^{-1}(U) \subset Y$ is open, so we obtain a functor $f^{-1}\colon X_{\mathrm{Zar}} \to Y_{\mathrm{Zar}}$. The functor

$$\mathrm{PreShv}(Y) \xrightarrow{f_p} \mathrm{PreShv}(X)$$

is defined by $f_p(G) = G \circ f^{-1}$ has a left adjoint functor

$$\mathrm{PreShv}(X) \xrightarrow{f^p} \mathrm{PreShv}(Y)$$

given by left Kan extension along $f^{-1}\colon X^{\mathrm{op}}_{\mathrm{Zar}} \to Y^{\mathrm{op}}_{\mathrm{Zar}}$. More concretely, we have

$$f^p(\mathcal{F})(V) = \mathrm{colim}_{f(V)\subset U}\, \mathcal{F}(U),$$

where the colimit is indexed by the opposite of the "slice category"

$$(X_{\mathrm{Zar}})_{/f^{-1}} \times_{Y_{\mathrm{Zar}}} \{V\}$$

with objects open subsets $U \subset X$ such that $V \subset f^{-1}(U)$ and with morphisms inclusions among such open subsets. It is a filtered category, so f^p preserves finite limits by Grothendieck's theorem. The functor f_p preserves sheaves in the sense that there is a unique functor f_* making the diagram

$$\begin{array}{ccc} \mathrm{Shv}(Y) & \xrightarrow{f_*} & \mathrm{Shv}(X) \\ \downarrow{\scriptstyle \iota_X} & & \downarrow{\scriptstyle \iota_Y} \\ \mathrm{PreShv}(Y) & \xrightarrow{f_p} & \mathrm{PreShv}(X) \end{array}$$

commute, but the functor f^p does not. However, the functor

$$\mathrm{Shv}(X) \xrightarrow{f^*} \mathrm{Shv}(Y)$$

defined by $f^* = \mathrm{ass}_Y \circ f^p \circ \iota_X$ is left adjoint of f_*. We call f^* the inverse image functor and we call f_* the direct image functor. So we have an adjunction

$$\mathrm{Shv}(X) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Shv}(Y)$$

and the functor f^* preserves finite limits. In particular, it preserves ring objects.

Example 3. (1) Let $j\colon U \to X$ be the inclusion of an open subset. It is an open map in the sense that if $V \subset U$ is open, then so is $V = j(V) \subset X$. This implies $j^p\colon \mathrm{PreShv}(X) \to \mathrm{PreShv}(U)$ preserves sheaves and that $j^*\colon \mathrm{Shv}(X) \to \mathrm{Shv}(U)$ is given by $j^*(\mathcal{F})(V) = \mathcal{F}(j(V))$. Therefore, we also write $\mathcal{F}|_U = j^*(\mathcal{F})$.

(2) Let $i_x\colon \{x\} \to X$ be the inclusion of a point and note that $\mathrm{Shv}(\{x\}) \simeq \mathrm{Set}$. Indeed, a presheaf $\mathcal{G}\colon \{x\}_{\mathrm{Zar}} \to \mathrm{Set}$ is a sheaf if and only if $\mathcal{G}(\emptyset)$ is a one-element set, so, up to unique isomorphism, a sheaf $\mathcal{G} \in \mathrm{Shv}(\{x\})$ is determined by the set $\mathcal{G}(\{x\})$. We say that $\mathcal{F}_x = i_x^*(\mathcal{F})(\{x\})$ is the stalk of $\mathcal{F} \in \mathrm{Shv}(X)$ at $x \in X$. Concretely, we have $\mathcal{F}_x = \mathrm{colim}_{x\in U}\, \mathcal{F}(U)$, where the colimit is indexed by the opposite of the category of open neighborhoods $x \in U \subset X$ under inclusion. One can prove that a morphism $h\colon \mathcal{F} \to \mathcal{F}'$ in $\mathrm{Shv}(X)$ is an isomorphism if and only if the induced map of stalks $h_x\colon \mathcal{F}_x \to \mathcal{F}'_x$ is an isomorphism for all $x \in X$.[2]

[2] We refer to this statement by saying that the Zariski topos $\mathrm{Shv}(X)$ has "enough points."

The sheaf $\mathcal{O}_X^{\text{cont}}$ continuous k-valued functions on X is a sheaf of commutative rings, and therefore, its stalk $\mathcal{O}_{X,x}^{\text{cont}}$ at $x \in X$ is a commutative ring.

Lemma 4. *For every $x \in X$, $\mathcal{O}_{X,x}^{\text{cont}}$ is a local ring.*

Proof. The elements $h \in \mathcal{O}_{X,x}^{\text{cont}}$ are germs of continuous k-valued functions at $x \in X$, that is, equivalence classes of pairs (U, φ) of an open neighborhood $x \in U \subset X$ and a continuous function $\varphi \colon U \to k$, where two such pairs (U_1, φ_1) and (U_2, φ_2) are equivalent, if there exists $x \in V \subset U_1 \cap U_2$ open such that $\varphi_1|_V = \varphi_2|_V$. The map $i_x^\sharp \colon \mathcal{O}_{X,x}^{\text{cont}} \to k$ that to the class of (U, φ) assigns $\varphi(x)$ is a surjective ring homomorphism to a field, so its kernel $\mathfrak{m}_x \subset \mathcal{O}_{X,x}^{\text{cont}}$ is a maximal ideal. Now, if $h \in \mathcal{O}_{X,x}^{\text{cont}}$ and $h \notin \mathfrak{m}_x$, then we can represent h by a pair (U, φ) such that $\varphi(u) \neq 0$ for all $u \in U$. This shows that h is invertible with h^{-1} given by the class of the pair (U, ψ), where $\psi(u) = \varphi(u)^{-1}$. This proves the lemma. □

Let $f \colon Y \to X$ be a continuous map. We define the morphism

$$\mathcal{O}_X^{\text{cont}} \xrightarrow{f^\sharp} f_* \mathcal{O}_Y^{\text{cont}}$$

of sheaves of rings on X as follows. If $U \subset X$ is open with $V = f^{-1}(U) \subset Y$, then

$$\Gamma(U, \mathcal{O}_X^{\text{cont}}) \xrightarrow{f_U^\sharp} \Gamma(U, f_* \mathcal{O}_Y^{\text{cont}}) = \Gamma(V, \mathcal{O}_Y^{\text{cont}})$$

is the ring homomorphism that to $\varphi \colon U \to k$ assigns $\varphi \circ f|_V \colon V \to k$. By adjunction, it determines and is determined by a morphism

$$f^* \mathcal{O}_X^{\text{cont}} \xrightarrow{\tilde{f}^\sharp} \mathcal{O}_Y^{\text{cont}}.$$

of sheaves of rings on Y. We will abuse notation and write also $f^\sharp$ instead of $\tilde{f}^\sharp$ for this map. The induced map of stalks at $y \in Y$ is a ring homomorphism

$$\mathcal{O}_{X,x}^{\text{cont}} = i_x^* \mathcal{O}_X^{\text{cont}} \simeq (f \circ i_y)^* \mathcal{O}_X^{\text{cont}} \simeq i_y^* f^* \mathcal{O}_X^{\text{cont}} \xrightarrow{f_y^\sharp} i_y^* \mathcal{O}_Y = \mathcal{O}_{Y,y}^{\text{cont}},$$

where the indicated isomorphisms are the unique natural isomorphisms between different choices of left adjoint functors of the functor $i_{x*} = (f \circ i_y)_*$.

Lemma 5. *The ring homomorphism $f_y^\sharp \colon \mathcal{O}_{X,x}^{\text{cont}} \to \mathcal{O}_{Y,y}^{\text{cont}}$ is a local homomorphism.*

Proof. That $f_y^\sharp$ is a local homomorphism means that it is a ring homomorphism and that $(f_y^\sharp)^{-1}(\mathfrak{m}_y) = \mathfrak{m}_x$, or equivalently, that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{O}_{X,x}^{\text{cont}} & \xrightarrow{i_x^\sharp} & k \\ \downarrow{\scriptstyle f_y^\sharp} & & \| \\ \mathcal{O}_{Y,y}^{\text{cont}} & \xrightarrow{i_y^\sharp} & k \end{array}$$

Now, if $h \in \mathcal{O}_{X,x}^{\text{cont}}$ is represented by the pair (U, φ), where $x \in U \subset X$ is an open neighborhood and $\varphi \colon U \to k$ is a continuous map, then $y \in V = f^{-1}(U) \subset Y$ is an open neighborhood, and the pair $(V, \varphi \circ f|_V)$ represents $f_y^\sharp(h) \in \mathcal{O}_{Y,y}$. So

$$i_y^\sharp(f_y^\sharp(h)) = (\varphi \circ f|_V)(y) = \varphi(f(y)) = \varphi(x) = i_x^\sharp(h),$$

as desired. □

We will consider other kinds of "functions," but we always want them to retain the properties that we proved in Lemmas 4 and 5 for continuous functions. We encode these properties in the following definition.

Definition 6. (1) A locally ringed space is a pair $(X, \mathcal{O}_X)$ of a topological space X and a sheaf of rings $\mathcal{O}_X$ such that for all $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.

(2) A morphism of locally ringed spaces is a pair $(f, f^\sharp)\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ of a continuous map $f\colon Y \to X$ and a morphism $f^\sharp\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ of sheaves of rings on X such that for all $y \in Y$, the induced map of stalks $f^\sharp_y\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,f(y)}$ is a local ring homomorphism.

If $(X, \mathcal{O}_X)$ is a locally ringed space, if $U \subset X$ is open, and if $\varphi \in \Gamma(U, \mathcal{O}_X)$, then we define its value $\varphi(x)$ at $x \in U$ to be the image of φ by the composite map

$$\Gamma(U, \mathcal{O}_X) \xrightarrow{i_U} \mathcal{O}_{X,x} \xrightarrow{i^\sharp_x} k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x.$$

We note that the value $\varphi(x) \in k(x)$ is an element of a field $k(x)$ that may vary with $x \in U$. It may also happen that $\varphi \neq 0$ even though $\varphi(x) = 0$ for all $x \in U$.

We now define the "geometric functions" relevant for smooth manifolds, namely, the smooth functions. However, our discussion below applies mutatis mutandis to holomorphic functions and complex manifolds and to analytic functions and real analytic manifolds. Let $U \subset \mathbb{R}^n$ be an open subset. A function $\varphi\colon U \to \mathbb{R}$ is defined to be smooth if the partial derivatives $\partial^k\varphi/\partial x_{i_1} \dots \partial x_{i_k}\colon U \to \mathbb{R}$ exist and are continuous for all $k \geq 0$ and $1 \leq i_1, \dots, i_k \leq n$. The sheaf of standard smooth functions on U is defined to be the subsheaf $\mathcal{O}_U^{\mathrm{sm}} \subset \mathcal{O}_U^{\mathrm{cont}}$ given by

$$\Gamma(V, \mathcal{O}_U^{\mathrm{sm}}) = \{\varphi\colon V \to \mathbb{R} \mid \varphi \text{ smooth}\} \subset \Gamma(V, \mathcal{O}_U^{\mathrm{cont}})$$

for all $V \subset U$ open. We say that a locally ringed space $(X, \mathcal{O}_X)$ is an affine smooth manifold, if there exists an isomorphism of locally ringed spaces

$$(X, \mathcal{O}_X) \xrightarrow{(f, f^\sharp)} (U, \mathcal{O}_U^{\mathrm{sm}})$$

with $U \subset \mathbb{R}^n$ open. The number n is uniquely determined by $(X, \mathcal{O}_X)$ and is called the dimension of the affine smooth manifold.

Definition 7. A smooth manifold[3] is a locally ringed space $(X, \mathcal{O}_X)$ for which there exists an open covering $(U_i \to X)_{i \in I}$ such that for all $i \in I$, $(U_i, \mathcal{O}_X|_{U_i})$ is an affine smooth manifold. A morphism $f\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ between smooth manifolds is a morphism of locally ringed spaces.

Remark 8. (1) If $(X, \mathcal{O}_X)$ is a smooth manifold, then $\mathcal{O}_X$ is canonically isomorphic to a subsheaf of $\mathcal{O}_X^{\mathrm{cont}}$. Indeed, by definition, this is true locally, so by the sheaf condition, it is also true globally. Moreover, if $(f, f^\sharp)\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism between smooth manifolds, then the diagram

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{f^\sharp} & f_*\mathcal{O}_Y \\ \downarrow & & \downarrow \\ \mathcal{O}_X^{\mathrm{cont}} & \xrightarrow{f^\sharp} & f_*\mathcal{O}_Y^{\mathrm{cont}} \end{array}$$

[3] In the literature, the requirement that X be Hausdorff is often included in the definition of a smooth manifold, but we will not do so. Note that "being Hausdorff" is not a local property.

commutes, and therefore, the top horizontal map is uniquely determined by the bottom horizontal map. So we may view "being smooth" as the property of the continuous map $f\colon Y \to X$ that a map $f^\sharp\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ making the diagram commute exist. We also note that an isomorphism between smooth manifolds is traditionally called a diffeomorphism.

(2) We define the dimension of a smooth manifold $(X, \mathcal{O}_X)$ to be the map

$$X \xrightarrow{\dim} \mathbb{Z}_{\geq 0}$$

that to $x \in X$ assigns $n = \dim(x)$, if there exists $x \in U \subset X$ open with $(U, \mathcal{O}_X|_U)$ an affine smooth manifold of dimension n. It is well-defined and locally constant, and if it is constant with value n, then we say that $(X, \mathcal{O}_X)$ has pure dimension n or that $(X, \mathcal{O}_X)$ is a smooth n-manifold. We define a chart of $(X, \mathcal{O}_X)$ around $x \in X$ to be a pair (U, h) of an open neighborhood $x \in U \subset X$ and a diffeomorphism

$$(U, \mathcal{O}_X|_U) \xrightarrow{h} (V, \mathcal{O}_V^{\mathrm{sm}})$$

with $V \subset \mathbb{R}^{\dim(x)}$ an open subset.

Proposition 9. *The category of smooth manifolds and their morphisms admits finite products. More precisely, if $f\colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_Y)$ and $g\colon (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ are morphisms between smooth manifolds, then, up to unique isomorphism, there is a unique sheaf of rings $\mathcal{O}_{X\times Y}$ on $X \times Y$ such that $(X \times Y, \mathcal{O}_{X\times Y})$ is a smooth manifold and such that, in the diagram*

$$\begin{array}{ccccc} & & (Z, \mathcal{O}_Z) & & \\ & \swarrow^{f} & \downarrow{\scriptstyle (f,g)} & {}^{g}\searrow & \\ (X, \mathcal{O}_X) & \xleftarrow{p} & (X \times Y, \mathcal{O}_{X\times Y}) & \xrightarrow{q} & (Y, \mathcal{O}_Y), \end{array}$$

the projections p and q and the unique map (f, g) that makes the diagram commute are morphisms of smooth manifolds.

Proof. Up to isomorphism, there is a unique sheaf $\mathcal{O}_{X\times Y}$ on $X \times Y$ such that given $(x, y) \in X \times Y$ and charts $f\colon (U, \mathcal{O}_X|_U) \to (A, \mathcal{O}_A^{\mathrm{sm}})$ and $g\colon (V, \mathcal{O}_Y|_V) \to (B, \mathcal{O}_B^{\mathrm{sm}})$ around $x \in X$ and $y \in Y$, respectively, the map

$$(U \times V, \mathcal{O}_{X\times Y}|_{U\times V}) \xrightarrow{f\times g} (A \times B, \mathcal{O}_{A\times B}^{\mathrm{sm}})$$

is a chart around $(x, y) \in X \times Y$.[4] Since the subsets of the form $U \times V \subset X \times Y$, where $U \subset X$ and $V \subset Y$ are open, form a basis for the product topology, this shows that $(X \times Y, \mathcal{O}_{X\times Y})$ is a smooth manifold. That the maps p, q, and (f, g) are smooth can be checked locally in charts, where it is clear. □

One can construct new smooth manifolds is by gluing existing smooth manifolds together. To state the result, we introduce some terminology. In general, we define

[4] There is a canonical map $\mathcal{O}_X \otimes_k \mathcal{O}_Y \to \mathcal{O}_{X\times Y}$ of sheaves of k-algebras on $X \times Y$, but it is not an isomorphism. Rather the target is a suitable completion of the source.

a morphism $(s,t)\colon R \to Y \times Y$ in a category $\mathcal{C}$ that admits finite products to be an equivalence relation if for all $Z \in \mathcal{C}$, the induced map of sets

$$\mathrm{Hom}_{\mathcal{C}}(Z,R) \xrightarrow{(s,t)} \mathrm{Hom}_{\mathcal{C}}(Z,Y) \times \mathrm{Hom}_{\mathcal{C}}(Z,Y)$$

exhibits $\mathrm{Hom}_{\mathcal{C}}(Z,R)$ as an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(Z,Y)$ in the usual sense. In particular, the morphism (s,t) is a monomorphism.

A morphism $f\colon (Y,\mathcal{O}_Y) \to (X,\mathcal{O}_X)$ of smooth manifolds is étale if there exists an open covering $(V_i \to Y)_{i\in I}$ such that each $f|_{V_i}\colon (V_i,\mathcal{O}_Y|_{V_i}) \to (f(V_i),\mathcal{O}_X|_{f(V_i)})$ is a diffeomorphism. It is an open immersion if, in addition, the map $f\colon Y \to X$ is injective. The image $f(Y) \subset X$ of an étale morphism is an open subset.

Proposition 10. *Given an equivalence relation of smooth manifolds*

$$(R,\mathcal{O}_R) \xrightarrow{(s,t)} (Y,\mathcal{O}_Y) \times (Y,\mathcal{O}_Y)$$

such that $Y = \coprod_{i\in I} Y_i$ *and* $R = \coprod_{(i,j)\in I\times I} U_{i,j}$ *and such that* s *and* t *restrict to open immersions* $s|_{U_{i,j}}\colon U_{i,j} \to Y_i$ *and* $t|_{U_{i,j}}\colon U_{i,j} \to Y_j$*, the coequalizer*

$$(R,\mathcal{O}_R) \underset{t}{\overset{s}{\rightrightarrows}} (Y,\mathcal{O}_Y) \xrightarrow{f} (X,\mathcal{O}_X)$$

exists. Moreover, the morphism f *is étale.*

Proof. Let $X = Y/R$ with the quotient topology, and let $f\colon Y \to X$ be the canonical projection. It is the coequalizer of $s,t\colon R \to Y$ in the category of topological spaces and continuous maps. We claim that for all $i \in I$, the map $f|_{Y_i}\colon Y_i \to f(Y_i)$ is a homeomorphism. First, it is a bijection, since the maps $s|_{U_{i,i}}\colon U_{i,i} \to Y_i$ and $t_{U_{i,i}}\colon U_{i,i} \to Y_i$ necessarily are equal. For they are both open immersions and the diagonal map $\Delta\colon Y_i \to Y_i \times Y_i$ factors through $(s,t)|_{U_{i,i}}\colon U_{i,i} \to Y_i \times Y_i$, since (s,t) is an equivalence relation. Second, it is an open map. Indeed, if $V \subset Y_i$ is an open subset, then so is the subset

$$f^{-1}(f(V)) = \textstyle\coprod_{j\in I}(t\circ s^{-1})(V\cap U_{i,j}) \subset \coprod_{j\in I} Y_j = Y.$$

This shows that $f|_{Y_i}\colon Y_i \to f(Y_i) \subset X$ is a homeomorphism.

Finally, the sheaf of rings $\mathcal{O}_X$ given by the equalizer

$$\mathcal{O}_X \xrightarrow{f^\sharp} f_*\mathcal{O}_Y \underset{f_*t^\sharp}{\overset{f_*s^\sharp}{\rightrightarrows}} h_*\mathcal{O}_R,$$

where $h = f\circ s = f\circ t$, makes $(X,\mathcal{O}_X)$ a smooth manifold and makes the diagram in the statement a coequalizer in the category of smooth manifolds and morphisms of smooth manifolds. □

Remark 11. The morphisms $s,t\colon (R,\mathcal{O}_R) \to (Y,\mathcal{O}_Y)$ in Proposition 10 are étale, but they are a very particular kind of étale morphisms. We would like the result to hold more generally for every étale equivalence relation, that is, for every equivalence relation $(s,t)\colon (R,\mathcal{O}_R) \to (Y\times Y,\mathcal{O}_{Y\times Y})$ such that s and t are étale, but this is not true.[5] To remedy this, one builds the larger category of "smooth spaces" in which the result holds for every étale equivalence relation.

[5] A counterexample is $(s,t)\colon \mathbb{Z}\times S^1 \to S^1 \times S^1$, where $s(n,z) = z$ and $t(n,z) = w^n z$, where w is some fixed irrational roation of the circle S^1.

Example 12. (1) Let $\mathbb{A}^1_k = (\mathbb{R}, \mathcal{O}^{\mathrm{sm}}_{\mathbb{R}})$ be the affine line, let $\mathbb{A}^1_k \smallsetminus \{0\} \subset \mathbb{A}^1_k$ be the open complement of $\{0\} \subset \mathbb{A}^1_k$, and let (s,t) be the equivalence relation with

$$R = R_{11} \sqcup R_{12} \sqcup R_{22} = \mathbb{A}^1_k \sqcup (\mathbb{A}^1_k \smallsetminus \{0\}) \sqcup \mathbb{A}^1_k \underset{t}{\overset{s}{\rightrightarrows}} Y = Y_1 \sqcup Y_2 = \mathbb{A}^1_k \sqcup \mathbb{A}^1_k$$

where the maps $s,t\colon R_{12} \to Y_1$ are defined to be the canonical inclusion and the map $t \mapsto t^{-1}$, respectively. The coequalizer $(X, \mathcal{O}_X)$ is the projective line $\mathbb{P}^1_k$.
(2) We consider the equivalence relation defined as in (1), except that we now define both $s,t\colon R_{12} \to Y_1$ to be the canonical inclusion. The coequalizer $(X, \mathcal{O}_X)$ is an affine line with a double point at the origin. The space X is not Hausdorff.

We will use Proposition 10 to construct the tangent bundle of a smooth manifold. It is a functor that to a smooth manifold $(X, \mathcal{O}_X)$ assigns a morphism

$$T(X, \mathcal{O}_X) = (TX, \mathcal{O}_{TX}) \xrightarrow{p_X} (X, \mathcal{O}_X)$$

of smooth manifolds together with a structure of real vector space on the fiber

$$T(X, \mathcal{O}_X)_x = p_X^{-1}(x) \subset T(X, \mathcal{O}_X)$$

for all $x \in X$, and that to a morphism $f\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ of smooth manifolds assigns a commutative diagram of morphisms of smooth manifolds

$$\begin{array}{ccc} T(Y, \mathcal{O}_Y) & \xrightarrow{df} & T(X, \mathcal{O}_X) \\ \downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle p_X} \\ (Y, \mathcal{O}_Y) & \xrightarrow{f} & (X, \mathcal{O}_X) \end{array}$$

such that for all $y \in Y$ with image $x = f(y) \in X$, the induced map of fibers

$$T(Y, \mathcal{O}_Y)_y \xrightarrow{df_y} T(X, \mathcal{O}_X)_x$$

is linear. The "chain rule" is the statement that this assignment is a functor.

First, if $U \subset \mathbb{R}^m$ is an open subset, then we define

$$T(U, \mathcal{O}^{\mathrm{sm}}_U) = (U \times \mathbb{R}^m, \mathcal{O}^{\mathrm{sm}}_{U \times \mathbb{R}^m}) \xrightarrow{p_U} (U, \mathcal{O}^{\mathrm{sm}}_U)$$

to the projection on the first factor. We define the structure of real vector space on the fiber $T(U, \mathcal{O}^{\mathrm{sm}}_U)_x$ by $(x, \boldsymbol{v}) + (x, \boldsymbol{w}) = (x, \boldsymbol{v} + \boldsymbol{w})$ and $(x, \boldsymbol{v}) \cdot a = (x, \boldsymbol{v} \cdot a)$, where $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^m$ and $a \in \mathbb{R}$. If $f\colon (V, \mathcal{O}^{\mathrm{sm}}_V) \to (U, \mathcal{O}^{\mathrm{sm}}_U)$ is a morphism of smooth manifolds with $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ open, then we define

$$T(V, \mathcal{O}^{\mathrm{sm}}_V) \xrightarrow{df} T(U, \mathcal{O}^{\mathrm{sm}}_U)$$

to be the morphism of smooth manifolds defined by

$$df(y, \boldsymbol{v}) = (f(y), D_{\boldsymbol{v}} f(y)),$$

where $y \in V$ and $\boldsymbol{v} \in \mathbb{R}^n$, and where

$$D_{\boldsymbol{v}} f(y) = \lim_{h \to 0} (f(y + \boldsymbol{v}h) - f(y))/h$$

is the directional derivative. If $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ and $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ are the standard bases for $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and if we write $f(y) = \sum_{i=1}^m \boldsymbol{e}_i f_i(y)$, then

$$D_{\boldsymbol{e}_j} f(y) = \sum_{i=1}^m \boldsymbol{e}_i \cdot (\partial f_i / \partial y_j)(y).$$

It follows that the diagram

$$\begin{array}{ccc} T(V, \mathcal{O}_V^{\mathrm{sm}}) & \xrightarrow{df} & T(U, \mathcal{O}_U^{\mathrm{sm}}) \\ \downarrow{\scriptstyle p_V} & & \downarrow{\scriptstyle p_U} \\ (V, \mathcal{O}_V^{\mathrm{sm}}) & \xrightarrow{f} & (U, \mathcal{O}_U^{\mathrm{sm}}) \end{array}$$

commutes, and that for all $y \in V$ with image $x = f(y) \in U$, the induced map

$$T(V, \mathcal{O}_V^{\mathrm{sm}})_y \xrightarrow{df_y} T(U, \mathcal{O}_U^{\mathrm{sm}})_x$$

is linear. Moreover, the chain rule from calculus shows that

$$d(f \circ g) = df \circ dg$$

for all composable morphisms of smooth manifolds

$$(W, \mathcal{O}_W^{\mathrm{sm}}) \xrightarrow{g} (V, \mathcal{O}_V^{\mathrm{sm}}) \xrightarrow{f} (U, \mathcal{O}_U^{\mathrm{sm}})$$

with $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$, and $W \subset \mathbb{R}^p$.

Second, given any smooth manifold $(X, \mathcal{O}_X)$, we let $(Y_i, h_i \colon Y_i \to V_i)_{i \in I}$ be a family of charts with $V_i \subset \mathbb{R}^{n_i}$. The canonical map

$$(Y, \mathcal{O}_Y) = \textstyle\coprod_{i \in I} (Y_i, \mathcal{O}_X|_{Y_i}) \xrightarrow{f} (X, \mathcal{O}_X)$$

is étale, the canonical inclusion

$$(R, \mathcal{O}_R) = (Y, \mathcal{O}_Y) \times_{(X, \mathcal{O}_X)} (Y, \mathcal{O}_Y) \xrightarrow{(s,t)} (Y, \mathcal{O}_Y) \times (Y, \mathcal{O}_Y)$$

is an equivalence relation, and the diagram

$$(R, \mathcal{O}_R) \underset{t}{\overset{s}{\rightrightarrows}} (Y, \mathcal{O}_Y) \xrightarrow{f} (X, \mathcal{O}_X)$$

is a coequalizer. We have $Y = \coprod_{i \in I} Y_i$ and $R = \coprod_{(i,j) \times I \times I} U_{i,j}$ with $U_{i,j} = Y_i \cap Y_j$, so the existence of the coequalizer also is a consequence of Proposition 10. We now define $p_X \colon T(X, \mathcal{O}_X) \to (X, \mathcal{O}_X)$ to be the induced morphism of coequalizers

$$\begin{array}{ccccc} T(R, \mathcal{O}_R) & \underset{dt}{\overset{ds}{\rightrightarrows}} & T(Y, \mathcal{O}_Y) & \xrightarrow{df} & T(X, \mathcal{O}_X) \\ \downarrow{\scriptstyle p_R} & & \downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle p_X} \\ (R, \mathcal{O}_R) & \underset{t}{\overset{s}{\rightrightarrows}} & (Y, \mathcal{O}_Y) & \xrightarrow{f} & (X, \mathcal{O}_X), \end{array}$$

and we give the fiber $T(X, \mathcal{O}_X)_x$ the unique structure of real vector space such that for any $y \in Y$ with $f(y) = x$, the induced map of fibers

$$T(Y, \mathcal{O}_Y)_y \xrightarrow{df_y} T(X, \mathcal{O}_X)_x$$

is a linear isomorphism. To see that $p_X \colon T(X, \mathcal{O}_X) \to (X, \mathcal{O}_X)$ is well-defined, up to canonical isomorphism, one has to prove two things. First, one much show that the equivalence relation (ds, dt) satisfies the hypothesis of Proposition 10, which is not difficult. Second, if $p'_X \colon T(X, \mathcal{O}_X)' \to (X, \mathcal{O}_X)$ is obtained as above but beginning with a different choice of family of charts $(Y_i', h_i' \colon Y_i' \to V_i')_{i \in I'}$, then one must produce a canonical diffeomorphism g making the diagram

$$\begin{array}{ccc} T(X, \mathcal{O}_X) & \xrightarrow{g} & T(X, \mathcal{O}_X)' \\ \big\downarrow{\scriptstyle p_X} & & \big\downarrow{\scriptstyle p_X'} \\ (X, \mathcal{O}_X) & = & (X, \mathcal{O}_X) \end{array}$$

commute. This is more delicate, since we have not characterized the tangent bundle by some universal property, and therefore, there is not a unique choice of "canonical" diffeomorphism.[6] We will not go further into this here.

Definition 13. A tangent vector field on a smooth manifold $(X, \mathcal{O}_X)$ is a morphism of smooth manifolds $\boldsymbol{v} \colon (X, \mathcal{O}_X) \to T(X, \mathcal{O}_X)$ such that $p_X \circ \boldsymbol{v} = \mathrm{id}_X$.

We note that the value of the map $\boldsymbol{v}$ at $x \in X$ is a vector $\boldsymbol{v}(x) \in T(X, \mathcal{O}_X)_x$ in a vector space that varies with x. We give the set $\mathrm{Vect}(X, \mathcal{O}_X)$ of tangent vector fields on $(X, \mathcal{O}_X)$ the structure of a left $\Gamma(X, \mathcal{O}_X)$-module, where

$$(\boldsymbol{v} + \boldsymbol{w})(x) = \boldsymbol{v}(x) + \boldsymbol{w}(x)$$
$$(\varphi \cdot \boldsymbol{v})(x) = \varphi(x) \cdot \boldsymbol{v}(x)$$

for $\boldsymbol{v}, \boldsymbol{w} \in \mathrm{Vect}(X, \mathcal{O}_X)$ and $\varphi \in \Gamma(X, \mathcal{O}_X)$.

Let $(X, \mathcal{O}_X)$ be a smooth manifold, and let $\boldsymbol{v} \in \mathrm{Vect}(X, \mathcal{O}_X)$ be a tangent vector field. The directional derivative along $\boldsymbol{v}$ is a k-linear map of sheaves

$$\mathcal{O}_X \xrightarrow{D_{\boldsymbol{v}}} \mathcal{O}_X,$$

which we now define. We must define, for all $U \subset X$ open, a k-linear map

$$\Gamma(U, \mathcal{O}_X) \xrightarrow{D_{\boldsymbol{v},U}} \Gamma(U, \mathcal{O}_X)$$

such that for all $U \subset V \subset X$ open, the diagram

$$\begin{array}{ccc} \Gamma(V, \mathcal{O}_X) & \xrightarrow{D_{\boldsymbol{v},V}} & \Gamma(V, \mathcal{O}_X) \\ \big\downarrow{\scriptstyle \mathrm{Res}_U^V} & & \big\downarrow{\scriptstyle \mathrm{Res}_U^V} \\ \Gamma(U, \mathcal{O}_X) & \xrightarrow{D_{\boldsymbol{v},U}} & \Gamma(U, \mathcal{O}_X) \end{array}$$

commutes. We first note that the smooth tangent vector field $\boldsymbol{v}$ on $(X, \mathcal{O}_X)$ restricts to a smooth tangent vector field $\boldsymbol{v}|_U$ on $(U, \mathcal{O}_X|_U)$ for all $U \subset X$ open. Indeed, if $j \colon U \to X$ is the open immersion of U in X, then the diagram

$$\begin{array}{ccc} T(U, \mathcal{O}_X|_U) & \xrightarrow{dj} & T(X, \mathcal{O}_X) \\ \big\downarrow{\scriptstyle p_U} & & \big\downarrow{\scriptstyle p_X} \\ (U, \mathcal{O}_X|_U) & \xrightarrow{j} & (X, \mathcal{O}_X) \end{array}$$

is cartesian, and therefore, we may define $\boldsymbol{v}|_U \colon (U, \mathcal{O}_X|_U) \to T(U, \mathcal{O}_X|_U)$ to be the unique morphism such that $dj \circ \boldsymbol{v}|_U = \boldsymbol{v} \circ j$ and $p_U \circ \boldsymbol{v}|_U = \mathrm{id}_U$. Next, we may view

[6] It would of course be much better to give a global definition of the tangent bundle similar to the definition $p_X \colon T(X, \mathcal{O}_X) = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_X}(\Omega^1_{X/k})) \to (X, \mathcal{O}_X)$ in algebraic geometry.

$\varphi \in \Gamma(U, \mathcal{O}_X)$ as a morphism of smooth manifolds $\varphi\colon (U, \mathcal{O}_X|_U) \to (\mathbb{R}, \mathcal{O}_{\mathbb{R}}^{\mathrm{sm}})$, so we have the commutative diagram

$$\begin{array}{ccc} T(U, \mathcal{O}_X|_U) & \xrightarrow{d\varphi} & T(\mathbb{R}, \mathcal{O}_{\mathbb{R}}) \\ \downarrow{\scriptstyle p_U} & & \downarrow{\scriptstyle p_{\mathbb{R}}} \\ U & \xrightarrow{\varphi} & \mathbb{R}. \end{array}$$

We also have a tangent vector field $\boldsymbol{w}$ on $(\mathbb{R}, \mathcal{O}_{\mathbb{R}}^{\mathrm{sm}})$ defined by $\boldsymbol{w}(t) = (t, \boldsymbol{e}_1)$, and we now define $D_{\boldsymbol{v},U}(\varphi) \in \Gamma(U, \mathcal{O}_X|_U)$ to be the unique element such that

$$d\varphi \circ \boldsymbol{v}|_U = \boldsymbol{w} \cdot D_{v,U}(\varphi).$$

It is clear from the definition that the map $D_{\boldsymbol{v},U}$ is k-linear and that if $U \subset V \subset X$ are open subsets, then $D_{\boldsymbol{v},U} \circ \mathrm{Res}_U^V = \mathrm{Res}_U^V \circ D_{\boldsymbol{v},V}$. Therefore, we have defined a k-linear map of sheaves $D_{\boldsymbol{v}}\colon \mathcal{O}_X \to \mathcal{O}_X$ as desired.

In general, given a morphism $f\colon (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ of locally ringed spaces and a right $\mathcal{O}_X$-module $\mathcal{F}$, an $f^*\mathcal{O}_S$-linear morphism of sheaves

$$\mathcal{O}_X \xrightarrow{\delta} \mathcal{F}$$

is an $f^*\mathcal{O}_S$-linear derivation if for all $U \subset X$ open and $\varphi, \psi \in \Gamma(U, \mathcal{O}_X)$,

$$\delta_U(\varphi \cdot \psi) = \delta_U(\varphi) \cdot \psi + \delta_U(\psi) \circ \varphi.$$

We write $\mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{F})$ for the set of $f^*\mathcal{O}_S$-linear derivations $\delta\colon \mathcal{O}_X \to \mathcal{F}$. It has a structure of abelian group given by the pointwise sum of derivations. Moreover, if $h\colon \mathcal{F} \to \mathcal{F}$ is an $\mathcal{O}_X$-linear morphism and if $\delta\colon \mathcal{O}_X \to \mathcal{F}$ is an $f^*\mathcal{O}_S$-linear derivation, then $h \circ \delta\colon \mathcal{O}_X \to \mathcal{F}$ again is an $f^*\mathcal{O}_S$-linear derivation. So $(h, \delta) \mapsto h \circ \delta$ defines a structure of left $\mathrm{End}_{\mathcal{O}_X}(\mathcal{F})$-module on the abelian group $\mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{F})$.

Lemma 14. *If $(X, \mathcal{O}_X)$ is a smooth manifold, then for all $\boldsymbol{v} \in \mathrm{Vect}(X, \mathcal{O}_X)$, the directional derivative $D_{\boldsymbol{v}}\colon \mathcal{O}_X \to \mathcal{O}_X$ is a k-linear derivation.*

Proof. Given $\boldsymbol{v} \in \mathrm{Vect}(X, \mathcal{O}_X)$, an open subset $U \subset X$, and a point $x \in U$, we give a formula for $D_{\boldsymbol{v},U}(\varphi)(x)$ for $\varphi \in \Gamma(U, \mathcal{O}_X|_U)$. There exists a smooth curve $\gamma\colon (I, \mathcal{O}_I^{\mathrm{sm}}) \to (U, \mathcal{O}_X|_U)$ defined on an open interval $0 \in I \subset \mathbb{R}$ such that $\gamma(0) = x$ and such that, in the diagram

$$\begin{array}{ccccc} T(I, \mathcal{O}_I^{\mathrm{sm}}) & \xrightarrow{d\gamma} & T(U, \mathcal{O}_X|_U) & \xrightarrow{d\varphi} & T(\mathbb{R}, \mathcal{O}_{\mathbb{R}}) \\ \downarrow{\scriptstyle p_I} & & \downarrow{\scriptstyle p_U} & & \downarrow{\scriptstyle p_{\mathbb{R}}} \\ (I, \mathcal{O}_I) & \xrightarrow{\gamma} & (U, \mathcal{O}_X|_U) & \xrightarrow{\varphi} & (\mathbb{R}, \mathcal{O}_{\mathbb{R}}^{\mathrm{sm}}), \end{array}$$

we have $(d\gamma \circ \boldsymbol{w}|_I)(0) = \boldsymbol{v}|_U(x) = (\boldsymbol{v}|_U \circ \gamma)(0)$. Therefore,

$$(d\varphi \circ \boldsymbol{v}|_U)(x) = (d\varphi \circ \boldsymbol{v}|_U \circ \gamma)(0) = (d\varphi \circ d\gamma \circ \boldsymbol{w}|_I)(0) = (d(\varphi \circ \gamma) \circ \boldsymbol{w}|_I)(0),$$

from which we obtain the formula

$$D_{\boldsymbol{v},U}(\varphi)(x) = (\varphi \circ \gamma)'(0).$$

Hence, for all $\varphi, \psi \in \Gamma(U, \mathcal{O}_X)$, we have

$$\begin{aligned} D_{\boldsymbol{v},U}(\varphi \cdot \psi)(x) &= ((\varphi \cdot \psi) \circ \gamma)'(0) = ((\varphi \circ \gamma) \cdot (\psi \circ \gamma))'(0) \\ &= (\varphi \circ \gamma)'(0) \cdot (\psi \circ \gamma)(0) + (\psi \circ \gamma)'(0) \cdot (\varphi \circ \gamma)(0) \\ &= D_{\boldsymbol{v},U}(\varphi)(x) \cdot \psi(x) + D_{\boldsymbol{v},U}(\psi)(x) \cdot \varphi(x), \end{aligned}$$

and since $x \in U$ was arbitrary, we conclude that

$$D_{\boldsymbol{v},U}(\varphi \cdot \psi) = D_{\boldsymbol{v},U}(\varphi) \cdot \psi + D_{\boldsymbol{v},U}(\psi) \cdot \varphi$$

as desired. □

We now obtain the promised global description of the left $\Gamma(X, \mathcal{O}_X)$-module of tangent vector fields.

Proposition 15. *Let* $(X, \mathcal{O}_X)$. *The directional derivative*

$$\operatorname{Vect}(X, \mathcal{O}_X) \xrightarrow{D} \operatorname{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$$

is an isomorphism of left $\Gamma(X, \mathcal{O}_X)$*-modules.*

Proof. For all open subsets $U \subset V \subset X$, we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Vect}(V, \mathcal{O}_X|_V) & \xrightarrow{D_V} & \operatorname{Der}_k(\mathcal{O}_X|_V, \mathcal{O}_X|_V) \\ \downarrow{\scriptstyle \operatorname{Res}^V_U} & & \downarrow{\scriptstyle \operatorname{Res}^V_U} \\ \operatorname{Vect}(U, \mathcal{O}_X|_U) & \xrightarrow{D_U} & \operatorname{Der}_k(\mathcal{O}_X|_U, \mathcal{O}_X|_U), \end{array}$$

so the family $(D_U)_{U \subset X}$ is a morphism of presheaves of left $\mathcal{O}_X$-modules

$$\mathcal{V}ect(X, \mathcal{O}_X) \xrightarrow{D} \mathcal{D}er_k(\mathcal{O}_X, \mathcal{O}_X).$$

Both of these presheaves are in fact sheaves, because they are defined in terms of by local conditions. We will prove that this morphism of sheaves is an isomorphim. Since the map in the statement is obtained from this morphism of sheaves by applying the global sections functor $\Gamma(X, -)$, this will prove the proposition.

Since the statement that the map of sheaves in question is an isomorphism is local on X, we may assume that $(X, \mathcal{O}_X)$ is equal to $(U, \mathcal{O}_U^{\mathrm{sm}})$ with $U \subset \mathbb{R}^n$ open. We may further assume that $U \subset \mathbb{R}^n$ is convex, since every open subset of $\mathbb{R}^n$ admits a covering by convex open subsets. So it suffices to prove that for $U \subset \mathbb{R}^n$ convex open, the directional derivative

$$\operatorname{Vect}(U, \mathcal{O}_U^{\mathrm{sm}}) \xrightarrow{D} \operatorname{Der}_k(\mathcal{O}_U^{\mathrm{sm}}, \mathcal{O}_U^{\mathrm{sm}})$$

is an isomorphism of left $\Gamma(U, \mathcal{O}_U^{\mathrm{sm}})$-modules. The left-hand $\Gamma(U, \mathcal{O}_U^{\mathrm{sm}})$-module is free of rank n, and a basis is given by the family $(\boldsymbol{w}_1, \dots, \boldsymbol{w}_n)$ of vector fields defined by $\boldsymbol{w}_i(x) = (x, \boldsymbol{e}_i)$, where $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ is the standard basis of $\mathbb{R}^n$. By the definition of the directional derivative, we have

$$D_{\boldsymbol{w}_i}(\varphi) = \partial\varphi/\partial x_i,$$

so we must prove that the family $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ of derivations is a basis of the left $\Gamma(U, \mathcal{O}_U^{\mathrm{sm}})$-module $\mathrm{Der}_k(\mathcal{O}_U^{\mathrm{sm}}, \mathcal{O}_U^{\mathrm{sm}})$. It is linearly independent, since

$$\partial x_i/\partial x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

and to show that it also generates the left $\Gamma(U, \mathcal{O}_U^{\mathrm{sm}})$-module $\mathrm{Der}_k(\mathcal{O}_U^{\mathrm{sm}}, \mathcal{O}_U^{\mathrm{sm}})$, we prove that for all $\delta \in \mathrm{Der}_k(\mathcal{O}_U^{\mathrm{sm}}, \mathcal{O}_U^{\mathrm{sm}})$, the following identity holds,

$$\delta = \textstyle\sum_{i=1}^n \delta(x_i) \cdot \partial/\partial x_i.$$

It suffices to show that for all $\delta \in \mathrm{Der}_k(\mathcal{O}_U^{\mathrm{sm}}, \mathcal{O}_U^{\mathrm{sm}})$, $\varphi \in \Gamma(U, \mathcal{O}_U^{\mathrm{sm}})$, and $a \in U$,

$$\delta(\varphi)(a) = \textstyle\sum_{i=1}^n \delta(x_i)(a) \cdot (\partial\varphi/\partial x_i)(a).$$

Indeed, the sheaf $\mathcal{F} = \mathcal{O}_U^{\mathrm{sm}}$ has the special property that a section $\psi \in \Gamma(U, \mathcal{O}_U^{\mathrm{sm}})$ is zero if and only if all its values $\psi(a) \in \mathcal{F}(a) = \mathcal{F}_a \otimes_{\mathcal{O}_{U,a}^{\mathrm{sm}}} k(a)$ are zero. Now, since we assumed that the open subset $U \subset \mathbb{R}^n$ is convex, Corollary 21 below shows that there exist unique $\varphi_{i,j} \in \Gamma(U, \mathcal{O}_U^{\mathrm{sm}})$ such that

$$\varphi(x) = \varphi(a) + \textstyle\sum_{i=1}^n (x_i - a_i)(\partial\varphi/\partial x_i)(a) + \sum_{i,j=1}^n (x_i - a_i)(x_j - a_j)\varphi_{i,j}(x),$$

and since δ is a k-linear derivation, the desired identity ensues. □

Example 16. If $(X, \mathcal{O}_X)$ is a smooth manifold, and and if $h\colon U \to V$ is a chart with $V \subset \mathbb{R}^n$ open, then the family of derivations $(\delta_1, \dots, \delta_n)$, where

$$\delta_i(\varphi)(x) = (\partial(\varphi \circ h^{-1})/\partial x_i)(h(x)),$$

is a basis of the left $\Gamma(U, \mathcal{O}_X)$-module $\mathrm{Der}_k(\mathcal{O}_X|_U, \mathcal{O}_X|_U)$. Hence, there is a unique basis $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ of the left $\Gamma(U, \mathcal{O}_X)$-module $\mathrm{Vect}(U, \mathcal{O}_X|_U)$ such that $D_{\boldsymbol{v}_i} = \delta_i$.

According to Proposition 15, tangent vector fields may analogously be defined to be k-linear derivations $\delta\colon \mathcal{O}_X \to \mathcal{O}_X$. This definition has the advantage of being truly global. We define the "Lie bracket"

$$\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) \otimes_k \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) \xrightarrow{[-,-]} \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$$

to be the map that to $\delta_1 \otimes \delta_2$ assigns the k-linear morphism

$$[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1.$$

To verify that $[\delta_1, \delta_2] \in \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$, we let $\varphi, \psi \in \Gamma(U, \mathcal{O}_X|_U)$ and calculate

$$\begin{aligned}
[\delta_1, \delta_2](\varphi \cdot \psi) &= \delta_1(\delta_2(\varphi \cdot \psi)) - \delta_2(\delta_1(\varphi \cdot \psi)) \\
&= \delta_1(\delta_2(\varphi) \cdot \psi + \varphi \cdot \delta_2(\psi)) - \delta_2(\delta_1(\varphi) \cdot \psi + \varphi \cdot \delta_1(\psi)) \\
&= \delta_1(\delta_2(\varphi)) \cdot \psi + \delta_2(\varphi) \cdot \delta_1(\psi) + \delta_1(\varphi) \cdot \delta_2(\psi) + \varphi \cdot \delta_1(\delta_2(\psi)) \\
&\quad - \delta_2(\delta_1(\varphi)) \cdot \psi - \delta_1(\varphi) \cdot \delta_2(\psi) - \delta_2(\varphi) \cdot \delta_1(\psi) - \varphi \cdot \delta_2(\delta_1(\psi)) \\
&= [\delta_1, \delta_2](\varphi) \cdot \psi + \varphi \cdot [\delta_1, \delta_2](\psi).
\end{aligned}$$

It is clear that the map $[-,-]$ is k-linear in both arguments so that we obtain the stated map. A similar and equally straightforward calculation shows that given three k-linear derivations δ_1, δ_2, and δ_3, the "Jacobi identity"

$$[[\delta_1, \delta_2], \delta_3] + [[\delta_2, \delta_3], \delta_1] + [[\delta_3, \delta_1], \delta_2] = 0$$

holds. This makes $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ a Lie algebra over k.[7]

We proved earlier that the category of smooth manifolds and morphisms of smooth manifolds has finite products. It does not have all fiber products, but the implicit function theorem shows that it does have some fiber products. Given a cartesian square of smooth manifolds and morphism of smooth manifolds

$$\begin{array}{ccc} (Y', \mathcal{O}_{Y'}) & \xrightarrow{g'} & (Y, \mathcal{O}_Y) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ (X', \mathcal{O}_{X'}) & \xrightarrow{g} & (X, \mathcal{O}_X), \end{array}$$

we say that f' is the base-change of f along g. If such a square exists for given f and g, then we say that the base-change of f along g exists.

A morphism of smooth manifolds $f\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a submersion[8] (resp. an immersion) if for all $y \in Y$ with image $x = f(y) \in X$, the differential

$$T(Y, \mathcal{O}_Y)_y \xrightarrow{df_y} T(X, \mathcal{O}_X)_x$$

is surjective (resp. injective). We note that, in this case, it follows from linear algebra that $\dim(y) \geq \dim(x)$ (resp. $\dim(y) \leq \dim(x)$).

Theorem 17 (Implicit function theorem)**.** *In the category of smooth manifolds and morphisms of smooth manifolds, the base-change of a submersion along any morphism exists and is a submersion.*

Proof. This is based on the inverse function theorem. It states that a morphism of smooth manifolds, which is both an immersion and a submersion, is étale. The proof has a number of steps. First, if $(Y, \mathcal{O}_Y) = (X \times Z, \mathcal{O}_{X \times Z})$ and f is the projection on the first factor, then the base-change along any g exists with $Y' = X' \times Z$, with f' the projection on the first factor, and with $g' = g \times \mathrm{id}_Z$. Second, the inverse function theorem shows if f is any submersion, then for all $y \in Y$, we find open neighborhoods $y \in V \subset Y$, $x = f(y) \in U \subset X$, and $0 \in W \subset \mathbb{R}^p$ together with a diffeomorphism h making the diagram

$$\begin{array}{ccc} (V, \mathcal{O}_Y|_V) & \xrightarrow{h} & (U \times W, \mathcal{O}_{X \times W}|_{U \times W}) \\ \downarrow{\scriptstyle f|_U} & & \downarrow{\scriptstyle p} \\ (U, \mathcal{O}_X|_U) & = & (U, \mathcal{O}_X|_U), \end{array}$$

where p is the canonical projection, commute. Hence, it follows from the first step that the base-change of $f|_U$ along any morphism g exists and is a submersion. Finally, we use Proposition 10 to glue together the local solutions obtained in the second step to a global solution. To do so, we also use the fact that the base-change of an open immersion along any morphism exists and is an open immersion and the fact that base-change along an open immersion preserves both coequivalizers and submersions. ∎

[7] This Lie algebra is infinite dimensional, unless X is finite. We will define the Lie algebra of a Lie group to be a subalgebra of this Lie algebra.

[8] In algebraic geometry, the analogue of submersions are called smooth morphisms. It is for this reason, that I say "morphism of smooth manifolds" instead of "smooth map."

Remark 18. Let $f\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a morphism of smooth manifolds. We say that $y \in Y$ is a regular point of f if df_y is surjective and that $x \in X$ is a regular value of f if every $y \in Y$ with $f(y) = x$ is a regular point. Therefore, given a morphism $g\colon (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ for which there exists $g(X') \subset U \subset X$ open such that every $x \in U$ is a regular point of f, then the base-change of f along g exists and is equal to the base-change of $f|_{f^{-1}(U)}$ along g.

Example 19. Let $Y = M_n(\mathbb{R})$, and let $X \subset M_n(\mathbb{R})$ be the subset of symmetric matrices. So Y and X are both real vector spaces of dimension n^2 and $(n+1)n/2$, respectively, which we view as smooth manifolds of the same dimensions. The map $f\colon (Y, \mathcal{O}_Y^{\mathrm{sm}}) \to (X, \mathcal{O}_X^{\mathrm{sm}})$ defined by $f(A) = A^*A$ is smooth, and we claim that

$$T(Y, \mathcal{O}_Y^{\mathrm{sm}})_A \xrightarrow{df_A} T(X, \mathcal{O}_X^{\mathrm{sm}})_{f(A)}$$

is surjective for all $A \in Y$ with $f(A) = E \in X$. To see this, use the identity maps of Y and X as charts and calculate

$$\begin{aligned} df_A(B) &= \lim\nolimits_{h\to 0}(f(A+hB) - f(A))/h \\ &= \lim\nolimits_{h\to 0}((A+hB)^*(A+hB) - A^*A)/h \\ &= \lim\nolimits_{h\to 0}(A^*A + hA^*B + hB^*A + h^2B^*B - A^*A)/h \\ &= A^*B + B^*A. \end{aligned}$$

Now, if $f(A) = A^*A = E$, then given $C = C^* \in X$, we set $B = \frac{1}{2}AC$ and calculate

$$df_A(B) = A^*B + B^*A = \tfrac{1}{2}A^*AC + \tfrac{1}{2}C^*A^*A = \tfrac{1}{2}(C + C^*) = C.$$

So the implicit function theorem shows that the base-change

$$\begin{array}{ccc} (O(n), \mathcal{O}_{O(n)}) & \xrightarrow{g'} & (Y, \mathcal{O}_Y^{\mathrm{sm}}) \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ (\{E\}, \mathcal{O}_{\{E\}}) & \xrightarrow{g} & (X, \mathcal{O}_X^{\mathrm{sm}}) \end{array}$$

exists; see Remark 18. Hence, the subspace $O(n) \subset M_n(\mathbb{R})$ of orthogonal matrices has a structure of smooth manifold of dimension $n^2 - (n+1)n/2 = n(n-1)/2$.

Appendix: Hadamard's lemma

We have used the following result, commonly referred to as Hadamard's lemma.

Lemma 20. *Let $U \subset \mathbb{R}^n$ be an open subset that is star-convex with respect to $a \in U$, and let $\varphi\colon U \to \mathbb{R}$ is a smooth function. Then there exists unique smooth functions $\varphi_i\colon U \to \mathbb{R}$ such that for all $x \in U$,*

$$\varphi(x) = \varphi(a) + \textstyle\sum_{i=1}^n (x_i - a_i)\varphi_i(x).$$

Moreover, for all $1 \leq i \leq n$, $\varphi_i(a) = (\partial\varphi/\partial x_i)(a)$.

Proof. We define $h\colon [0,1] \to \mathbb{R}$ by $h(t) = \varphi(a + (x-a)t)$, which is possible by the assumption that U' be star-convex with respect to a, and calculate that

$$\begin{aligned}\varphi(x) - \varphi(a) &= h(1) - h(0) = \textstyle\int_0^1 (dh/dt)(t)dt \\ &= \textstyle\int_0^1 \sum_{i=1}^n (\partial\varphi/\partial x_i)(a + (x-a)t)(x_i - a_i)dt \\ &= \textstyle\sum_{i=1}^n (x_i - a_i) \int_0^1 (\partial\varphi/\partial x_i)(a + (x-a)t)dt.\end{aligned}$$

So the lemma holds with $\varphi_i(x) = \int_0^1 (\partial\varphi/\partial x_i)(a + (x-a)t)dt$. □

Corollary 21. *Let $U \subset \mathbb{R}^n$ be an open subset that is star-convex with respect to $a \in U$, and let $\varphi\colon U \to \mathbb{R}$ is a smooth function. Then there exists unique smooth functions $\varphi_{i,j}\colon U \to \mathbb{R}$ such that for all $x \in U$,*

$$\varphi(x) = \varphi(a) + \textstyle\sum_{i=1}^n (x_i - a_i)(\partial\varphi/\partial x_i)(a) + \sum_{i,j=1}^n (x_i - a_i)(x_j - a_j)\varphi_{i,j}(x).$$

Proof. We first write $\varphi(x)$ as in the statement of Lemma 20 and the apply the lemma again to write each of the functions $\varphi_i\colon U \to \mathbb{R}$ as

$$\varphi_i(x) = \varphi_i(a) + \textstyle\sum_{j=1}^n (x_j - a_j)\varphi_{i,j}(x) = (\partial\varphi/\partial x_i)(a) + \sum_{j=1}^n (x_j - a_j)\varphi_{i,j}(x)$$

with $\varphi_{i,j}\colon U \to \mathbb{R}$ smooth. □