INVARIANT SUBSPACES

In this lecture, we define semisimple representations and we prove three theorems that we will use repeatedly to show that various representations are semisimple. We apply these theorems to three important examples, all of which are representations of the group $G = \mathrm{GL}(V)$ with V a finite dimensional k-vector space.

Definition 1. Let (V, π) be a k-linear representation of a group G. A subspace $U \subset V$ is said to be π-invariant if for all $g \in G$ and $\boldsymbol{u} \in U$, $\pi(g)(\boldsymbol{u}) \in U$.

We note that the subspaces $U = \{0\} \subset V$ and $U = V \subset V$ always are π-invariant.

Example 2. Let $G = (\mathbb{R}, +)$ be the additive group of real numbers, and let $(\mathbb{R}[G], L)$ be the left regular representation of G on the real vector space $\mathbb{R}[G]$ of all functions $f\colon \mathbb{R} \to \mathbb{R}$, which, we recall, is defined by

$$L(t)(f)(x) = f(-t + x).$$

We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = \sum_{0 \leq n \leq d} a_n x^n$$

with $a_0, a_1, \dots, a_d \in \mathbb{R}$ is a polynomial function of degree $\leq d$, and we claim that the subspace $U_d \subset \mathbb{R}[G]$ of polynomial functions of degree $\leq d$ is L-invariant. Indeed, for $t \in G$ and $f\colon \mathbb{R} \to \mathbb{R}$ as above, we calculate that

$$\begin{aligned} L(t)(f)(x) = f(-t + x) &= \sum_{0 \leq n \leq d} a_n (-t + x)^n \\ &= \sum_{0 \leq n \leq d} a_n \Big(\sum_{0 \leq i \leq n} (-t)^{n-i} x^i \Big) = \sum_{0 \leq i \leq d} \Big(\sum_{i \leq n \leq d} a_n (-t)^{n-i} \Big) x^i, \end{aligned}$$

which shows that $L(t)(f) \in U_d$, as required.

Remark 3. Suppose that (V, π) is a k-linear representation of a group G with $\dim_k(V) < \infty$, and let $U \subset V$ be a subspace. We first choose a basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ of U, and then extend it to a basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m, \boldsymbol{e}_{m+1}, \dots, \boldsymbol{e}_{m+n})$ of V. In this situation, the subspace $U \subset V$ is π-invariant if and only if the matrix that represents the k-linear map $\pi(g)\colon V \to V$ with respect to this basis is of the form

$$\left(\begin{array}{c|c} A(g) & B(g) \\ \hline O & D(g) \end{array} \right)$$

with $A(g) \in M_m(k)$, $B(g) \in M_{m,n}(k)$, and $D(g) \in M_n(k)$.

We recall from algebra that if V is a k-vector space and $U \subset V$ is a subspace, then the quotient vector space V/U is defined to be the

$$V/U = \{\boldsymbol{v} + U \subset V \mid \boldsymbol{v} \in V\}$$

equipped with the vector sum $(\boldsymbol{v}+U)+(\boldsymbol{v}'+U)=(\boldsymbol{v}+\boldsymbol{v}')+U$ and the scalar multiplication $a\cdot(\boldsymbol{v}+U)=(a\cdot\boldsymbol{v})+U$. Moreover, a k-linear map $f\colon V\to V$ with the property that $f(U)\subset U$ gives rise to a k-linear map

$$V/U \xrightarrow{f/U} V/U$$

defined by $(f/U)(\boldsymbol{v}+U)=f(\boldsymbol{v})+U$. If $f_1,f_2\colon V\to V$ are two such maps, then we have $(f_1\circ f_2)/U=(f_1/U)\circ(f_2/U)$. In other words, if $\mathrm{GL}(V,U)\subset\mathrm{GL}(V)$ is the subgroup of k-linear automorphisms $f\colon V\to V$ such that $f(U)\subset U$, then the map

$$\mathrm{GL}(V,U) \xrightarrow{-/U} \mathrm{GL}(V/U)$$

that to $f\colon V\to V$ assigns $f/U\colon V/U\to V/U$ is a group homomorphism.[1]

Definition 4. Let (V,π) is a k-linear representation of a group G, and let $U\subset V$ is a π-invariant subspace.

(1) The representation (U,π_U), where $\pi_U\colon G\to\mathrm{GL}(U)$ is defined by

$$\pi_U(g)(\boldsymbol{u})=\pi(g)(\boldsymbol{u})$$

for $\boldsymbol{u}\in U$, is called the subrepresentation of (V,π) on U.

(2) The representation $(V/U,\pi_{V/U})$, where $\pi_{V/U}\colon G\to\mathrm{GL}(V/U)$ is defined by

$$\pi_{V/U}(g)(\boldsymbol{v}+U)=\pi(g)(\boldsymbol{v})+U$$

for $\boldsymbol{v}+U\in V/U$, is called the quotient representation of (V,π) on V/U.

It is common to abuse language and simply say that π_U is a subrepresentation of π and that $\pi_{V/U}$ is a quotient representation of π.

Remark 5. Let (V,π) be a k-linear representation of a group G, and let $U\subset V$ be a π-invariant subspace. Suppose that $\dim_k(V)<\infty$. If we choose a basis $(\boldsymbol{e}_1,\dots\boldsymbol{e}_m)$ of U and extend it to a basis $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_m,\boldsymbol{e}_{m+1},\dots,\boldsymbol{e}_{m+n})$ of V, then the family $(\boldsymbol{e}_{m+1}+U,\dots,\boldsymbol{e}_{m+n}+U)$ is a basis of V/U, and moreover, the matrices that represent the maps $\pi_U(g)\colon U\to U$ and $\pi_{V/U}(g)\colon V/U\to V/U$ with respect to these bases are $A(g)$ and $D(g)$, if $\pi(g)\colon V\to V$ is represented by the matrix

$$\left(\begin{array}{c|c} A(g) & B(g) \\ \hline O & D(g) \end{array}\right)$$

with respect to the basis $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_m,\boldsymbol{e}_{m+1},\dots,\boldsymbol{e}_{m+n})$.

Definition 6. A k-linear representation (V,π) of a group G is irredicible (or simple) if $V\neq\{0\}$ and if the only π-invariant subspaces of V are $\{0\}\subset V$ and $V\subset V$.[2]

We note the formal similarity of the definition of an irreducible representations to the definition of a prime number.

Example 7. (1) Every 1-dimensional representation if irreducible. In particular, the trivial representation of G on k given by the constant map $\pi\colon G\to\mathrm{GL}(k)$ to every $g\in G$ assigns $\mathrm{id}_k\in\mathrm{GL}(k)$ is irreducible.

(2) The identity representation $\pi=\mathrm{id}_{\mathrm{GL}(V)}\colon G=\mathrm{GL}(V)\to\mathrm{GL}(V)$ is irreducible.

[1] It is also common to write $\bar{f}$ instead of f/U.
[2] The assumption $V\neq\{0\}$ is missing in the book.

(3) The 2-dimensional representation $\pi\colon G = (\mathbb{R}, +) \to \mathrm{GL}_2(\mathbb{R})$ given by

$$\pi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is irreducible. Indeed, the map $\pi(t)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by counterclockwise rotation through t radians around the origin, so it leaves no line through the origin invariant, unless $t \in \ker(\pi) = 2\pi\mathbb{Z}$.

(4) Let $G = (\mathbb{R}, +)$, and let (U_d, π_{U_d}) be the subrepresentation of the left regular representation $(\mathbb{R}[G], L)$ from Example 2. If $d \geq 1$, then π_{U_d} is not irreducible, since any $U_e \subset U_d$ with $0 \leq e < d$ is π_{U_d}-invariant and $\{0\} \subsetneqq U_e \subsetneqq U_d$.

(5) Let $G = S_n$ be the symmetric group on n letters, and let $\pi\colon G \to \mathrm{GL}_n(k)$ be the standard permutation representation on $V = k^n$. The subspaces

$$V_0 = \{\boldsymbol{x} \in V \mid \textstyle\sum_{1 \leq i \leq n} x_i = 0\} \subset V$$
$$V_1 = k \cdot (1, 1, \dots, 1) \subset V$$

are π-invariant subspaces of dimension $n-1$ and 1, respectively. Moreover, if $\mathrm{char}(k)$ does not divide n, then $V_1 \not\subset V_0$, so in this case, the intertwining map

$$V_0 \oplus V_1 \xrightarrow{\sim} V$$

induced by the canonical inclusions is an isomorphism. We claim that both the subrepresentations $\pi_0 = \pi|_{V_0}$ and $\pi_1 = \pi|_{V_1}$ are irreducible. This is clear for π_1, since $\dim_k(V_1) = 1$. To prove that also π_0 is irreducible, we let $\{0\} \neq U \subset V_0$ be a π_0-invariant subspace and prove that $U = V_0$. We choose a nonzero vector

$$\boldsymbol{x} = \sum_{1 \leq i \leq n} \boldsymbol{e}_i x_i \in U.$$

Since $\boldsymbol{x} \notin V_1$, the coordinates x_i are not all equal, and since $U \subset V_0$ is π_0-invariant, we may assume that $x_1 \neq x_2$. But then

$$\pi_0((12))(\boldsymbol{x}) - \boldsymbol{x} = (\boldsymbol{e}_1 - \boldsymbol{e}_2)(x_2 - x_1) \in U,$$

so $\boldsymbol{e}_1 - \boldsymbol{e}_2 \in U$. Again, since $U \subset V_0$ is π_0-invariant, it follows that $\boldsymbol{e}_i - \boldsymbol{e}_j \in U$, for all $1 \leq i < j \leq n$. But this shows that

$$V_0 = \mathrm{span}_k(\boldsymbol{e}_i - \boldsymbol{e}_j \mid 1 \leq i < j \leq n) \subset U,$$

so we conclude that $U = V_0$. Hence, π_0 is irreducible as claimed.

Definition 8. A k-linear representation (V, π) of a group G is completely reducible if for every π-invariant subspace $U \subset V$, there exists a π-invariant subspace $W \subset V$ such that the map induced by the canonical inclusions

$$U \oplus W \xrightarrow{\sim} V$$

is an isomorphism.

If $U, W \subset V$ are as in the definition, then we say that $W \subset V$ is a π-invariant complement of $U \subset V$. We note, in this situation, that the composition

$$W \xrightarrow{i} V \xrightarrow{p} V/U$$

of the canonical inclusion and the canonical projection is a k-linear isomorphism, which intertwines between π_W and $\pi_{V/U}$. We also remark that if a π-invariant complement $W \subset V$ of $U \subset V$ exists, then it is typically *not* unique.

Remark 9. If (V, π) is a k-linear representation of a group G with $\dim_k(V) < \infty$, then a π-invariant subspace $U \subset V$ admits a π-invariant complement if and only if we can find bases $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ of U and $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m, \boldsymbol{e}_{m+1}, \dots, \boldsymbol{e}_{m+n})$ of V such that for all $g \in G$, the matrix that represents $\pi(g) \colon V \to V$ with respect to the latter basis has the form

$$\left(\begin{array}{c|c} A(g) & O \\ \hline O & D(g) \end{array}\right).$$

Example 10. We consider two representations $(\mathbb{R}^2, \pi_A)$ of the form

$$G = (\mathbb{R}, +) \xrightarrow{\pi_A} \mathrm{GL}_2(\mathbb{R})$$

where $\pi_A(t) = e^{tA}$ with $A \in M_2(\mathbb{R})$.

We first let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since $A^2 = O$, we have

$$\pi_A(t) = e^{tA} = E + tA = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

It follows that the subspace

$$U = \mathrm{span}_{\mathbb{R}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \subset \mathbb{R}^2,$$

is π_A-invariant but has no π_A-invariant complement. Therefore, the representation π_A is not completely irreducible.

We next let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

so that $A^2 = A$. It follows that

$$\pi_A(t) = e^{tA} = E + (e^t - 1)A = \begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}.$$

In this case, the same subspace

$$U = \mathrm{span}_{\mathbb{R}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \subset \mathbb{R}^2$$

is π_A-invariant, but it now has the π_A-invariant complement

$$W = \mathrm{span}_{\mathbb{R}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \subset \mathbb{R}^2.$$

Moreever, the subspaces $U, W \subset V = \mathbb{R}^2$ are the only 1-dimensional π_A-invariant subspaces, so we conclude that π_A is completely reducible. (Compare Theorem 13 below.)

We now prove three theorems that we will use repeatedly. The theorems are listed as Theorem 1, 2, and 3 in Chapter 1 of the book.

Theorem 11. *Let (V, π) be a k-linear representation of a group G, and let $U \subset V$ be a π-invariant subspace. If π is completely reducible, then so is π_U.*

Proof. Let $U_1 \subset U$ be a π_U-invariant subspace. We must show that $U_1 \subset U$ admits a π_U-invariant complement $W_1 \subset U$. Now, since $U_1 \subset V$ is π-invariant, there exists, by the assumption that π is completely reducible, a π-invariant subspace $W \subset V$ such that the map induced by the canonical inclusions

$$U_1 \oplus W \longrightarrow V$$

is an isomorphism. But then

$$U = (U_1 + W) \cap U = (U_1 \cap U) + (W \cap U) = U_1 + (W \cap U),$$

so $W_1 = W \cap U \subset U$ is a π_U-invariant subspace, and the map

$$U_1 \oplus W_1 \longrightarrow U$$

induced by the canonical inclusions is an isomorphism. This shows that $W_1 \subset U$ is a π_U-complement of $U_1 \subset U$ as desired. □

Theorem 12. *Let (V, π) be a completely reducible k-linear representation of a group G with $\dim_k(V) < \infty$. There exists π-invariant subspaces $V_1, \dots, V_m \subset V$ such that the map induced by the canonical inclusions*

$$V_1 \oplus \cdots \oplus V_m \longrightarrow V$$

is an isomorphism and such that $\pi_{V_1}, \dots, \pi_{V_m}$ are irreducible.

Proof. We argue by induction on $n = \dim_k(V)$. If $n = 0$, then the statement is trivial, so we assume, inductively, that the statement has been proved for $n < r$ and prove it for $n = r$. We claim that there exists a π-invariant subspace $V_1 \subset V$ such that π_{V_1} is irreducible. Granting the claim, there exists, by the assumption that π is completely reducible, a π-invariant complement $W \subset V$ of $V_1 \subset V$, and since $\dim_k(V_1) \geq 1$, we have

$$\dim_k(W) = \dim_k(V) - \dim_k(V_1) < r.$$

So by the inductive hypothesis, there exist π_W-invariant subspaces $V_2, \dots, V_m \subset W$ such that the map induced by the canonical inclusions

$$V_2 \oplus \cdots \oplus V_m \longrightarrow W$$

is an isomorphism and such that $\pi_{V_2}, \dots, \pi_{V_m}$ are irreducible. It follows that the map induced by the canonical inclusions

$$V_1 \oplus V_2 \oplus \cdots \oplus V_m \longrightarrow V$$

is an isomorphism and the subrepresentations $\pi_{V_1}, \pi_{V_2}, \dots, \pi_{V_m}$ all are irreducible, which proves the induction step. It remains to prove the claim. The set S of nonzero π-invariant subspaces $U \subset V$ is partially ordered under inclusion. It is nonempty, since $V \in S$, and it has a minimal element, since $\dim_k(V) = r < \infty$. Let $V_1 \in S$ be such a smallest element.[3] If $\{0\} \neq U \subset V_1$ is a π_{V_1}-invariant subspace, then we necessarily have $U = V_1$, since otherwise $U \in S$ is smaller than $V_1 \in S$. This shows that π_{V_1} is irreducible, which proves the claim. □

[3] In general, a minimal element $V_1 \in S$ is not unique.

Theorem 13. *Let (V, π) be a k-linear representation of a group G, and suppose that there exist π-invariant subspaces $V_1, \dots, V_m \subset V$ such that*

$$V = V_1 + \cdots + V_m$$

and such that $\pi_{V_1}, \dots \pi_{V_m}$ are irreducible. If $U \subset V$ is a π-invariant subspace, then there exist $\{i_1, \dots, i_p\} \subset \{1, \dots, m\}$ such that the map

$$U \oplus V_{i_1} \oplus \cdots \oplus V_{i_p} \longrightarrow V$$

induced by the canonical inclusions is an isomorphism. In particular, π is completely reducible.

Proof. We let S be the set of subsets $\{i_1, \dots, i_p\} \subset \{1, \dots, m\}$ with the property that the map induced by the canonical inclusions

$$U \oplus V_{i_1} \oplus \cdots \oplus V_{i_p} \longrightarrow V$$

is injective. The set S is partially ordered under inclusion. It is nonempty, since $\emptyset \in S$, and it is finite, since there are only finitely many subsets of $\{1, \dots, m\}$, and therefore, it has a maximal element. So we let $\{i_1, \dots, i_p\} \in S$ be a maximal element and prove that map induced by the canonical inclusions

$$U \oplus V_{i_1} \oplus \cdots \oplus V_{i_p} \longrightarrow V$$

is an isomorphism. By the definition of S, we know that the map is injective, so we only need to show that the map is surjective, or equivalently, that

$$V = U + V_{i_1} + \cdots + V_{i_p}.$$

Moreover, since $V = V_1 + \cdots + V_m$, it suffices to show that

$$V_i \subset U + V_{i_1} + \cdots + V_{i_p}$$

for all $1 \leq i \leq m$. If $i \in \{i_1, \dots, i_p\}$, then there is nothing to prove, so suppose that $i \notin \{i_1, \dots, i_p\}$. We consider the maps

$$U \oplus V_{i_1} \oplus \cdots \oplus V_{i_p} \oplus V_i \longrightarrow (U + V_{i_1} + \cdots + V_{i_p}) \oplus V_i \longrightarrow V$$

induced by the canonical inclusions. Since $\{i_1, \dots, i_p\} \in S$, the left-hand map is an isomorphism, and since $\{i_1, \dots, i_p\} \in S$ is maximal, the composite map is *not* injective, so we conclude that the right-hand map is not injective. Therefore, its kernel, which is equal to

$$(U + V_{i_1} + \cdots + V_{i_p}) \cap V_i \subset V_i$$

is nonzero. But π_{V_i} is irreducible, so this implies that

$$(U + V_{i_1} + \cdots + V_{i_p}) \cap V_i = V_i,$$

so $V_i \subset U + V_{i_1} + \cdots + V_{i_p}$ as desired. □

Remark 14. A representation (V, π) is defined to be semisimple, if there exists a finite number of π-invariant subspaces $V_1, \dots, V_m \subset V$ such that the map

$$V_1 \oplus \cdots \oplus V_m \longrightarrow V$$

induced by the canonical inclusions is an isomorphism and such that each of the subrepresentations π_{V_i} is irreducible. Thus, Theorems 12 and 13 shows that a finite dimensional representation is semisimple if and only if it is completely reducible.

Corollary 15. *Let (V, π) be a k-linear representation of a group G, and let $U \subset V$ be a π-invariant subspace. If $\dim_k(V) < \infty$ and if π is completely reducible, then also the quotient representation $\pi_{V/U}$ is completely reducible.*

Proof. By Theorem 12, there exists π-invariant subspaces $V_1, \dots, V_m \subset V$ such that the map induced by the canonical inclusions

$$V_1 \oplus \cdots \oplus V_m \longrightarrow V$$

is an isomorphism and such that each π_{V_i} is irreducible. We let $\overline{V_i} \subset V/U$ be the image of the composition

$$V_i \longrightarrow V \longrightarrow V/U$$

of the canonical inclusion and the canonical projection and note that $\overline{V}_i$ is zero if and only if $V_i \subset U$. So let $S = \{i_1, \dots, i_p\} \subset \{1, \dots, m\}$ be the subset consisting those $i \in \{1, \dots, m\}$ for which $V_i \not\subset U$. The subspace $V_i \cap U \subset V_i$ is π_{V_i}-invariant, so if $i \in S$, then $V_i \cap U = \{0\}$, because π_{V_i} is irreducible. Therefore, if $i \in S$, then the canonical map $V_i \to \overline{V_i}$ is an isomorphism. This shows that the $\pi_{V/U}$-invariant subspaces $\overline{V}_{i_1}, \dots, \overline{V}_{i_p} \subset V/U$ satisfy the hypothesis of Theorem 13, we conclude that $\pi_{V/U}$ is completely reducible, as stated. □

We consider three examples, in all of which $G = \mathrm{GL}(V)$ with V a k-vector space of finite dimension n. We first consider the k-vector space $\mathrm{End}_k(V)$[4] of all k-linear maps $f\colon V \to V$ with vector sum and scalar multiplication defined by

$$\begin{aligned}(f_1 + f_2)(\boldsymbol{v}) &= f_1(\boldsymbol{v}) + f_2(\boldsymbol{v}) \\ (a \cdot f)(\boldsymbol{v}) &= a \cdot f(\boldsymbol{v}).\end{aligned}$$

We consider the representation of G on $\mathrm{End}_k(V)$ by left multiplication:

Proposition 16. *Let V be a k-vector space of finite dimension n, and define*

$$G = \mathrm{GL}(V) \xrightarrow{\ \lambda\ } \mathrm{GL}(\mathrm{End}_k(V))$$

by $\lambda(g)(f) = g \circ f$. The representation $(\mathrm{End}_k(V), \lambda)$ is completely reducible.

Proof. We choose a basis $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ of V and define

$$L_j = \{f \in \mathrm{End}_k(V) \mid f(\boldsymbol{v}_i) = \boldsymbol{0} \text{ for } i \neq j\} \subset \mathrm{End}_k(V).$$

It is a λ-invariant subspace. Indeed, if $g \in G$ and $f \in L_j$, then

$$\lambda(g)(f)(\boldsymbol{v}_i) = g(f(\boldsymbol{v}_i)) = \boldsymbol{0}$$

for $i \neq j$, because g is k-linear, so $\lambda(g)(f) \in L_j$. Moreover, the map

$$L_j \xrightarrow{\ h_j\ } V$$

defined by $h_j(f) = f(\boldsymbol{v}_j)$ is an isomorphism. It is also intertwining between λ and the identity representation of G on V. Indeed,

$$h_j(\lambda(g)(f)) = h_j(g \circ f) = (g \circ f)(\boldsymbol{v}_j) = g(f(\boldsymbol{v}_j)) = \mathrm{id}(g)(f(\boldsymbol{v}_j)) = \mathrm{id}(g)(h_j(f)).$$

[4] The book writes $L(V)$ instead of $\mathrm{End}_k(V)$.

Thus, $h_j\colon (L_j, \lambda_{L_j}) \to (V, \mathrm{id})$ is an isomorphism, and since (V, id) is irreducible, so is (L_j, λ_{L_j}). Finally, the map induced by the canonical inclusions

$$L_1 \oplus \cdots \oplus L_n \longrightarrow \mathrm{End}_k(V)$$

is an isomorphism, since every $f \in \mathrm{End}_k(V)$ can be written uniquely as

$$f = f_1 + \cdots + f_n$$

with $f_j \in \mathrm{End}_k(V)$ defined by

$$f_j(\boldsymbol{v}_i) = \begin{cases} f(\boldsymbol{v}_j) & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence, Theorem 13 shows that $(\mathrm{End}_k(V), \lambda)$ is completely reducible, as stated. □

We next consider the adjoint representation of G on $\mathrm{End}_k(V)$. It is an example of the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ of a reductive group on its Lie algebra.

Proposition 17. *Let V be a k-vector space of finite dimension n, and define*

$$G = \mathrm{GL}(V) \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathrm{End}_k(V))$$

by $\mathrm{Ad}(g)(f) = g \circ f \circ g^{-1}$. The adjoint representation $(\mathrm{End}_k(V), \mathrm{Ad})$ is completely reducible, provided that $\mathrm{char}(k)$ does not divide n.

Proof. We let $\mathfrak{t} \subset \mathrm{End}_k(V)$ be the 1-dimensional subspace spanned by id_V, and let $\mathfrak{sl}_n \subset \mathrm{End}_k(V)$ be the subspace consisting of the k-linear maps $f\colon V \to V$ with $\mathrm{tr}(f) = 0$. Both subspaces are Ad-invariant. In the case of $\mathfrak{sl}_n$, we use the fact from linear algebra that $\mathrm{tr}(g \circ f \circ g^{-1}) = \mathrm{tr}(f)$. By our assumption that $\mathrm{char}(k)$ does not divide n, we have $\mathrm{tr}(\mathrm{id}_V) = n \neq 0 \in k$, so $\mathfrak{t} \cap \mathfrak{sl}_n = \{0\}$, and hence, the map

$$\mathfrak{t} \oplus \mathfrak{sl}_n \longrightarrow \mathrm{End}_k(V)$$

induced by the canonical inclusions is an isomorphism. It turns out that $\mathrm{Ad}_{\mathfrak{t}}$ and $\mathrm{Ad}_{\mathfrak{sl}_n}$ both are irreducible. This is trivial in the case of the $\mathfrak{t}$, but the proof for $\mathfrak{sl}_n$ is not so simple. We prove this for $n = 2$ in the appendix. So Theorem 13 shows that π is completely reducible. □

Finally, we consider a representation of $G = \mathrm{GL}(V)$ on the k-vector space

$$B(V) = \{f\colon V \times V \to k \mid f \text{ is } k\text{-bilinear}\} \simeq \mathrm{Hom}_k(V \otimes_k V, k)$$

of k-bilinear forms on V.

Proposition 18. *Let V be a k-vector space of finite dimension n, and define*

$$G = \mathrm{GL}(V) \xrightarrow{\pi} \mathrm{GL}(B(V))$$

by $\pi(g)(f)(\boldsymbol{x}, \boldsymbol{y}) = f(g^{-1}(\boldsymbol{x}), g^{-1}(\boldsymbol{y}))$. The representation $(B(V), \pi)$ is completely reducible, provided that $\mathrm{char}(k) \neq 2$.

Proof (Incomplete). Let $B^{\pm}(V) \subset B(V)$ be the subspaces of symmetric forms and skew-symmetric forms, respectively. We recall that $f \in B^{+}(V)$ if and only if

$$f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y}, \boldsymbol{x})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in V$, and that $f \in B^-(V)$ if and only if

$$f(\boldsymbol{x}, \boldsymbol{y}) = -f(\boldsymbol{y}, \boldsymbol{x})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in V$. Clearly, $B^{\pm}(V) \subset B(V)$ are both π-invariant, and if $\operatorname{char}(k) \neq 2$, then the map induced by the canonical inclusions

$$B^+(V) \oplus B^-(V) \longrightarrow B(V)$$

is an isomorphism. One can prove that $\pi_{B^{\pm}(V)}$ both are irreducible, but we will not do so here. So Theorem 13 shows that π is completely reducible. $\square$

Appendix: The adjoint representation

We include a proof of the following theorem, which we used above.

Theorem 19. *If* $\operatorname{char}(k) \neq 2$*, then the adjoint representation*

$$\mathrm{GL}_2(k) \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathfrak{sl}_2(k))$$

is irreducible.

Proof. We must show that if $U \subset \mathfrak{sl}_2(k)$ is an Ad-invariant subspace, then either $U = \{0\}$ or $U = \mathfrak{sl}_2(k)$. So we assume that $U \neq \{0\}$ and proceed to prove that $U = \mathfrak{sl}_2(k)$. We fix the basis (H, X, Y) of $\mathfrak{sl}_2(k)$, where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We claim that $H \in U$ if and only if $X \in U$ if and only if $Y \in U$. First, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = Y, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X,$$

so if $X \in U$, then $Y \in U$ and vice versa. Second, we use the fact that $X^2 = 0$ so that $1 + X \in \mathrm{GL}_2(k)$ with inverse $1 - X$. Hence, the calculation

$$HX = X, \qquad XH = -X, \qquad XHX = 0$$

shows that

$$(1 + X)H(1 - X) = H - HX + XH - XHX = H - 2X.$$

Therefore, if $H \in U$, then $2X \in U$, and hence $X \in U$, since we are assuming that $2 \neq 0$ in k. Similarly, the calculation

$$YX = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad XY = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad XYX = X$$

shows that

$$(1 + X)Y(1 - X) = Y - YX + XY - XYX = Y + H - X.$$

Therefore, since we have already seen that $Y \in U$ if and only if $X \in U$, we conclude that if $Y \in U$, then $H = (Y + H - X) - Y + X \in U$. This proves the claim.

It remains to prove that at least one of H, X, and Y is in U. Since U is nonzero, there exists $0 \neq A \in \mathfrak{sl}_2(k)$. We write

$$A = aH + bX + cY = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

with $(a,b,c) \neq (0,0,0)$. For all $t \in k^*$, we have

$$g(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(k)$$

with inverse $g(t)^{-1} = g(t^{-1})$. Since U is assumed Ad-invariant, the calculation

$$\mathrm{Ad}(g(t))(A) = g(t)Ag(t)^{-1} = \begin{pmatrix} a & tb \\ t^{-1}c & -a \end{pmatrix} = aH + tbX + t^{-1}cY$$

shows that $aH + tbX + t^{-1}cY \in U$ for all $t \in k^*$. We wish to conclude that each of aH, bX, and cY is in U. So we wish to show that the system of linear equations

$$\begin{aligned} aH + rbX + r^{-1}cY &= aH \\ aH + sbX + s^{-1}cY &= bX \\ aH + tbX + t^{-1}cY &= cY \end{aligned}$$

has a solution with $r,s,t \in k^*$. The calculation

$$\det \begin{pmatrix} 1 & r & r^{-1} \\ 1 & s & s^{-1} \\ 1 & t & t^{-1} \end{pmatrix} = -(rst)^{-1}(r-s)(r-t)(s-t)$$

shows that a solution exists, provided that k^* has order at least three. Hence, if this is the case, then aH, bX, and cY are all in U, and since $(a,b,c) \neq (0,0,0)$, it follows that at least one of H, X, and Y is in U, so we are done.

The only missing case is $k = \mathbb{F}_3$, where $k^* = \{\pm 1\}$ only has order 2. In this case, the argument above shows that

$$\mathrm{span}_k(aH + bX + cY, aH - bX - cY) \subset U,$$

so $aH \in U$ and $bX + cY \in U$. If $a \neq 0$, then $H \in U$. Also, if $a = b = 0$, then $c \neq 0$, so $Y \in U$, and similarly, if $a = c = 0$, then $b \neq 0$, so $X \in U$. Hence, it only remains to prove that both the subspaces

$$\begin{aligned} V &= \mathrm{span}_k(\mathrm{Ad}(g)(X+Y) \mid g \in \mathrm{GL}_2(k)) \subset U \\ W &= \mathrm{span}_k(\mathrm{Ad}(g)(X-Y) \mid g \in \mathrm{GL}_2(k)) \subset U \end{aligned}$$

are equal to U. The calculation

$$\mathrm{Ad}(\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix})(X+Y) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}(X+Y)\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} = H$$

shows that $H \in V$, so that $V = U$, and the calculation

$$\mathrm{Ad}(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})(X-Y) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}(X-Y)\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = -H - X - Y$$

shows that $H + X + Y \in W$, so $W = U$, since, by the argument above, $H \in W$. □

Remark 20. We note that if $\mathrm{char}(k) = 2$, then the adjoint representation

$$\mathrm{GL}_2(k) \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathfrak{sl}_2(k))$$

is not irreducible. Indeed, since $H = 1$, the 1-dimensional subspace

$$\mathrm{span}_k(H) \subset \mathfrak{sl}_2(k)$$

is Ad-invariant.