BASIC OPERATIONS ON REPRESENTATIONS

The dual representation

We first discuss the dual vector space. To do so (and not make mistakes), we will let k be any skew-field. So we do not assume $a \cdot b$ and $b \cdot a$ are equal for $a, b \in k$. A skew-field $k = (k, +, \cdot)$ has an opposite skew-field $k^{\mathrm{op}} = (k, +, \star)$ with the same underlying set and the same addition, but with the new multiplication

$$a \star b = b \cdot a.$$

By a k-vector space, we will always mean a *right* k-vector space. So we agree that scalars multiply from the right and not from the left. We have to do so, if we want matrices (that represent linear maps) to multiply from the left, and I think that we all agree that we want that. Let us recall how this works.

So let $\varphi \colon W \to V$ be a linear map between right k-vector spaces, which we will assume to be finite dimensional, and let $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$ and $(\boldsymbol{w}_1, \dots, \boldsymbol{w}_n)$ be bases for V and W, respectively. Every $\boldsymbol{w} \in W$ and $\boldsymbol{v} \in V$ can be unique written as

$$\boldsymbol{v} = \boldsymbol{v}_1 x_1 + \boldsymbol{v}_2 x_2 + \cdots + \boldsymbol{v}_m x_m$$
$$\boldsymbol{w} = \boldsymbol{w}_1 y_1 + \boldsymbol{w}_2 y_2 + \cdots + \boldsymbol{w}_n y_n$$

with $\boldsymbol{x} = (x_j) \in M_{m,1}(k)$ and $\boldsymbol{y} = (y_i) \in M_{n,1}(k)$. Now, there is a unique matrix

$$A = (a_{ij}) \in M_{m,n}(k)$$

such that for all $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in W$, $\boldsymbol{v} = \varphi(\boldsymbol{w})$ if and only if $\boldsymbol{x} = A\boldsymbol{y}$, namely, the matrix whose entries a_{ij} are the unique solutions to the linear equations

$$\varphi(\boldsymbol{w}_j) = \boldsymbol{v}_1 a_{1j} + \boldsymbol{v}_2 a_{2j} + \cdots + \boldsymbol{v}_m a_{mj}$$

with $1 \leq j \leq n$. So the jth column in A is the coordinate vector of $\varphi(\boldsymbol{w}_j)$ with respect to the basis $(\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n)$. We say that A is the matrix that represents $\varphi \colon W \to V$ with respect to the bases $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$ and $(\boldsymbol{w}_1, \dots, \boldsymbol{w}_n)$.

If $U = (U, +, \cdot)$ is a *left* k-vector space, then we view it as a right k^{op}-vector space $U = (U, +, \star)$ with the same underlying set (of vectors) and the same vector sum, but with the new scalar multiplication $\star \colon U \times k^{\mathrm{op}} \to U$ given by

$$\boldsymbol{u} \star a = a \cdot \boldsymbol{u}.$$

We now discuss the dual vector space. So suppose that $V = (V, +, \cdot)$ is a *right* k-vector space. Its dual is the *left* k-vector space[1]

$$V^* = (\mathrm{Hom}_k(V, k), +, \cdot)$$

with vector sum and *left* scalar multiplication given by

$$(f + g)(\boldsymbol{v}) = f(\boldsymbol{v}) + g(\boldsymbol{v})$$
$$(a \cdot f)(\boldsymbol{v}) = a \cdot f(\boldsymbol{v}).$$

[1] The book writes V' instead of V^*.

Let us check that $a \cdot f \in V^*$. It is clear that $a \cdot f$ is additive, and the calculation

$$(a \cdot f)(\boldsymbol{v} \cdot b) = a \cdot f(\boldsymbol{v} \cdot b) = a \cdot (f(\boldsymbol{v}) \cdot b) = (a \cdot f(\boldsymbol{v})) \cdot b = (a \cdot f)(\boldsymbol{v}) \cdot b.$$

shows that it is also preserves right multiplication by b, as required. Note that this would not be true, if we instead let a multiply from the right, unless $a \cdot b = b \cdot a$. We agreed to consider this left k-vector space as the right k^{op}-vector space

$$V^* = (\mathrm{Hom}_k(V, k), +, \star),$$

with the right scalar multiplication by $a \in k^{\mathrm{op}}$ given by

$$(f \star a)(\boldsymbol{v}) = (a \cdot f)(\boldsymbol{v}) = a \cdot f(\boldsymbol{v}) = f(\boldsymbol{v}) \star a.$$

If $\dim_k(V) < \infty$, then a basis $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$ of the right k-vector space V gives rise to a basis $(\boldsymbol{v}_1^*, \dots, \boldsymbol{v}_m^*)$ of the dual right k^{op}-vector space V^* defined by

$$\boldsymbol{v}_i^*(\boldsymbol{v}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

which we call the dual basis.

A k-linear map $\varphi \colon W \to V$ between right k-vector spaces V and W determine a k^{op}-linear map $\varphi^* \colon V^* \to W^*$ between right k^{op}-vector spaces defined by

$$\varphi^*(f)(\boldsymbol{w}) = f(\varphi(\boldsymbol{w})).$$

Moreover, if V and W are finite dimensional, and if $A \in M_{m,n}(k)$ is the matrix that represents $\varphi \colon W \to V$ with respect to bases $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$ and $(\boldsymbol{w}_1, \dots, \boldsymbol{w}_n)$ of V and W, respectively, then the matrix that represents the map $\varphi^* \colon V^* \to W^*$ with respect to the dual bases $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m^*)$ and $(\boldsymbol{w}_1^*, \dots, \boldsymbol{w}_n^*)$ is the transpose matrix

$$A^t = (a_{ji}) \in M_{n,m}(k^{\mathrm{op}}).$$

If V is a right k-vector space, then its double dual $V^{**} = (V^*)^*$ is also a right k-vector space, so it is possible to compare them. There is a natural k-linear map

$$V \xrightarrow{\delta_V} V^{**}$$

defined by $\delta(\boldsymbol{v})(f) = f(\boldsymbol{v})$. That the map δ_V is natural[2] means that if $\varphi \colon W \to V$ is any k-linear map, then the diagram

$$\begin{array}{ccc} W & \xrightarrow{\delta_W} & W^{**} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi^{**}} \\ V & \xrightarrow{\delta_V} & V^{**} \end{array}$$

commutes. If $\dim_k(V) < \infty$, then δ_V is an isomorphism. Indeed, if $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$ is a basis of V, then $(\boldsymbol{v}_1^{**}, \dots, \boldsymbol{v}_m^{**})$ is a basis of V^{**}, and the calculation

$$\delta_V(\boldsymbol{v}_i)(\boldsymbol{v}_j^*) = \boldsymbol{v}_j^*(\boldsymbol{v}_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

shows that $\delta_V(\boldsymbol{v}_i) = \boldsymbol{v}_i^{**}$.

[2] We use the word "natural" to indicate natural transformations between functors, whereas we use the word "canonical" to indicate some particular or preferred choice. So "natural" has a precise mathematical meaning, whereas "canonical" does not.

Warning 1. By contrast, there is *no* preferred way to compare V and V^*. If V is a right k-vector space, then V^* is a right k^{op}-vector space. So to convert V^* into a right k-vector space $\sigma_*(V^*)$, we need a ring homomorphism

$$k \xrightarrow{\sigma} k^{\mathrm{op}}.$$

Such a ring homomorphism may not exist, and if it does, then it may not be unique. For instance, if $k = \mathbb{C}$, then we can choose σ to be the identity map, but we can also choose σ to be the map given by complex conjugation, which is different! Given $\sigma \colon k \to k^{\mathrm{op}}$, we must choose a map of right k-vector spaces

$$V \xrightarrow{b} \sigma_*(V^*).$$

The map b determines and is determined by the map

$$V \times V \xrightarrow{\langle -,- \rangle} k$$

defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = b(\boldsymbol{x})(\boldsymbol{y})$, and b is a well-defined and k-linear map if and only if the map $\langle -, - \rangle$ satisfies

(S1) For all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$, $\langle \boldsymbol{x}, \boldsymbol{y} + \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{x}, \boldsymbol{z} \rangle$.
(S2) For all $\boldsymbol{x}, \boldsymbol{y} \in V$ and $a \in k$, $\langle \boldsymbol{x}, \boldsymbol{y} \cdot a \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle \cdot a$.
(S3) For all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$, $\langle \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{z} \rangle + \langle \boldsymbol{y}, \boldsymbol{z} \rangle$.
(S4) For all $\boldsymbol{x}, \boldsymbol{y} \in V$ and $a \in k$, $\langle \boldsymbol{x} \cdot a, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle \star \sigma(a) = \sigma(a) \cdot \langle \boldsymbol{x}, \boldsymbol{y} \rangle$.

We say that $\langle -, - \rangle$ is a σ-sesquilinear form, and we say that it is non-singular if the map b is an isomorphism. Therefore, in order to compare V and V^*, we must both choose a ring homomorphism $\sigma \colon k \to k^{\mathrm{op}}$ and a σ-sesquilinear form $\langle -, - \rangle \colon V \times V \to k$. Obviously, we should never do so, if we can avoid it! Let us also mention that if $\sigma \circ \sigma = \mathrm{id}_k$, then the σ-sesquilinear form $\langle -, - \rangle$ is said to be σ-hermitian if, in addition, it satisfies:

(H) For all $\boldsymbol{x}, \boldsymbol{y} \in V$, $\langle \boldsymbol{y}, \boldsymbol{x} \rangle = \sigma(\langle \boldsymbol{x}, \boldsymbol{y} \rangle)$.

The requirement (H) is equivalent to the statement that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad b \quad} & \sigma_*(V^*) \\ \big\downarrow{\scriptstyle \delta_V} & & \big\uparrow{\scriptstyle \sigma_*(b^*)} \\ V^{**} & = & \sigma_*((\sigma_*(V^*))^*) \end{array}$$

commutes.

We now assume that k is a field. Since $a \cdot b = b \cdot a$ for all $a, b \in k$, the identity map is a ring homomorphism $\mathrm{id}_k \colon k \to k^{\mathrm{op}}$. If V is a right k-vector space, then we agree that we will *always* use the identity map $\sigma = \mathrm{id}_k \colon k \to k^{\mathrm{op}}$ to view the right k^{op}-vector space V^* as a right k-vector. In particular, if $k = \mathbb{C}$, then we will *not* use complex conjugation to view V^* as a right $\mathbb{C}$-vector space.

Definition 2. Let k be a field, and let (V, π) be a k-linear representation of a group G. The dual representation is the pair (V^*, π^*) of the dual k-vector space V^* and the group homomorphism

$$G \xrightarrow{\pi^*} \mathrm{GL}(V^*)$$

defined by $\pi^*(g) = \pi(g^{-1})^*$.

Let us check that π^* is indeed a group homomorphism. We have

$$\begin{aligned}\pi^*(g \cdot h) = \pi((g \cdot h)^{-1})^* &= \pi(h^{-1} \cdot g^{-1})^* = (\pi(h^{-1}) \circ \pi(g^{-1}))^* \\ &= \pi(g^{-1})^* \circ \pi(h^{-1})^* = \pi^*(g) \circ \pi^*(h)\end{aligned}$$

as required. Also, if V is finite dimensional, and if the matrix

$$A(g) \in M_n(k)$$

represents $\pi(g) \colon V \to V$ with respect to the basis $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$, then the matrix

$$A(g^{-1})^t = (A(g)^{-1})^t \in M_n(k)$$

represents $\pi^*(g) \colon V^* \to V^*$ with respect to the dual basis $(\boldsymbol{v}_1^*, \dots, \boldsymbol{v}_n^*)$.

Example 3. Let (V, π) be a finite dimensional real representation of a group G. We claim that if π is orthogonal, then $\pi^* \simeq \pi$. To see this, recall that π is said to be orthogonal if there exists an inner product $\langle -, - \rangle \colon V \times V \to \mathbb{R}$ such that

$$\langle \pi(g)(\boldsymbol{x}), \pi(g)(\boldsymbol{y}) \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$$

for all $g \in G$ and $\boldsymbol{x}, \boldsymbol{y} \in V$. Therefore, the matrix $Q(g) \in M_n(\mathbb{R})$ that represents $\pi(g) \colon V \to V$ with respect to a basis $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ that is orthonormal with respect to $\langle -, - \rangle$ is orthogonal, that is, it satisfies $Q(g) = (Q(g)^{-1})^t$. So the map

$$V \xrightarrow{b} V^*$$

defined by $b(\boldsymbol{x})(\boldsymbol{y}) = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ is intertwining between π and π^*. Since $\langle -, - \rangle$ is an inner product, the map b is also an isomorphism of vector spaces, so the claim follows. We note, as in Warning 1, that the isomorphism $\pi \simeq \pi^*$ is not canonical, let alone natural, but depends on the choice of inner product.

Example 4. If k is a field and if (V, π) is a k-linear representation of a group G, then the map $\delta_V \colon V \to V^{**}$ is intertwining between π and $\pi^{**} = (\pi^*)^*$. Indeed,

$$\pi^{**}(g) = \pi^*(g^{-1})^* = \pi(g)^{**},$$

and the diagram

$$\begin{array}{ccc} V & \xrightarrow{\delta_V} & V^{**} \\ \big\downarrow{\scriptstyle \pi(g)} & & \big\downarrow{\scriptstyle \pi(g)^{**}} \\ V & \xrightarrow{\delta_V} & V^{**} \end{array}$$

commutes by the naturality of δ.

Theorem 5. *Let k be a field, and let (V, π) be a finite dimensional k-linear representation of a group G.*

(1) *π is irreducible if and only π^* is so.*
(2) *π is completely reducible if and only if π^* is so.*

Proof. Indeed, the sequence

$$0 \longrightarrow U \xrightarrow{i} V \xrightarrow{p} W \longrightarrow 0$$

is exact (resp./ split exact) if and only if the sequence

$$0 \longrightarrow W^* \xrightarrow{p^*} V^* \xrightarrow{i^*} U^* \longrightarrow 0$$

is exact (resp. split exact). □

Remark 6. In elementary particle physics, an elementary particle is an irreducible representation π of the gauge group $\mathcal{G}$. The corresponding antiparticle is the dual (irreducible) representation π^*.

Sums of representations

We will next define (direct) sums of representations. There are two version, the exterior sum denoted "$\boxplus$" and the (interior) sum denoted "$\oplus$." First, let

$$V_1 \xrightarrow{i_1} V_1 \oplus V_2 \xleftarrow{i_2} V_2$$

be a direct sum of k-vector spaces V_1 and V_2. Given $\boldsymbol{v}_1 \in V_1$ and $\boldsymbol{v}_2 \in V_2$, we write

$$\boldsymbol{v}_1 \oplus \boldsymbol{v}_2 = i_1(\boldsymbol{v}_1) + i_2(\boldsymbol{v}_2) \in V_1 \oplus V_2.$$

If $f_1 \colon W_1 \to V_1$ and $f_2 \colon W_2 \to V_2$ are k-linear maps, then there is a unique k-linear map $f_1 \oplus f_2 \colon W_1 \oplus W_2 \to V_1 \oplus V_2$ that makes the diagram

$$\begin{array}{ccccc} W_1 & \xrightarrow{j_1} & W_1 \oplus W_2 & \xleftarrow{j_2} & W_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_1 \oplus f_2} & & \downarrow{\scriptstyle f_2} \\ V_1 & \xrightarrow{i_1} & V_1 \oplus V_2 & \xleftarrow{i_2} & V_2 \end{array}$$

commute. In terms of elements, we have

$$(f_1 \oplus f_2)(\boldsymbol{w}_1 \oplus \boldsymbol{w}_2) = f_1(\boldsymbol{w}_1) \oplus f_2(\boldsymbol{w}_2).$$

Moreover, if also $g_1 \colon V_1 \to U_1$ and $g_2 \colon V_2 \to U_2$ are k-linear maps, then

$$(g_1 \oplus g_2) \circ (f_1 \oplus f_2) = (g_1 \circ f_1) \oplus (g_2 \circ f_2).$$

In particular, we have a well-defined group homomorphism

$$\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \xrightarrow{\oplus} \mathrm{GL}(V_1 \oplus V_2)$$

that to (f_1, f_2) assigns $f_1 \oplus f_2$.

Definition 7. Let (V_1, π_1) and (V_2, π_2) be k-linear representations of two groups G_1 and G_2, respectively. The k-linear representation $(V_1 \oplus V_2, \pi_1 \boxplus \pi_2)$ of the product group $G_1 \times G_2$, where $\pi_1 \boxplus \pi_2$ is the composite group homomorphism

$$G_1 \times G_2 \xrightarrow{\pi_1 \times \pi_2} \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \xrightarrow{\oplus} \mathrm{GL}(V_1 \oplus V_2),$$

is called the exterior sum of (V_1, π_1) and (V_2, π_2).

Spelling out the definition in terms of elements, we have

$$(\pi_1 \boxplus \pi_2)(g_1, g_2)(\boldsymbol{v}_1 \oplus \boldsymbol{v}_2) = \pi_1(g_1)(\boldsymbol{v}_1) \oplus \pi_2(g_2)(\boldsymbol{v}_2)$$

for $g_1 \in G_1$, $g_2 \in G_2$, $\boldsymbol{v}_1 \in V_1$ and $\boldsymbol{v}_2 \in V_2$.

For every group G, the diagonal map

$$G \xrightarrow{\Delta_G} G \times G$$

defined by $\Delta_G(g) = (g, g)$ is also a group homomorphism.

Definition 8. Let (V_1, π_1) and (V_2, π_2) be k-linear representations of the *same* group G. The k-linear representation $(V_1 \oplus V_2, \pi_1 \oplus \pi_2)$ of G, where $\pi_1 \oplus \pi_2$ is defined to be the composite group homomorphism

$$G \xrightarrow{\Delta_G} G \times G \xrightarrow{\pi_1 \boxplus \pi_2} \mathrm{GL}(V_1 \oplus V_2).$$

is called the sum of π_1 and π_2.[3]

Again, spelling out the definition in terms of elements, we have

$$(\pi_1 \oplus \pi_2)(g)(\boldsymbol{v}_1 \oplus \boldsymbol{v}_2) = \pi_1(g)(\boldsymbol{v}_1) \oplus \pi_2(g)(\boldsymbol{v}_2)$$

for $g \in G$, $\boldsymbol{v}_1 \in V_2$, and $\boldsymbol{v}_2 \in V_2$.

Remark 9. If (V_1, π_1) and (V_2, π_2) both are k-linear representations of the same group G, then it may seem as if there is not much difference between the representations $\pi_1 \boxplus \pi_2$ and $\pi_1 \oplus \pi_2$. However, there is a big difference, which is that the former is a representation of the group $G \times G$, while the latter is a representation of the much smaller group G.

Example 10. Let (V, π) be a k-linear representation of a group G. If $U_1, U_2 \subset V$ are π-invariant subspaces, then the canonical inclusion maps

$$U_1 \xrightarrow{j_1} V \xleftarrow{j_2} U_2$$

are intertwining between π_{U_i} and π, and hence, the induced map

$$U_1 \oplus U_2 \xrightarrow{j_1 + j_2} V$$

is intertwining between $\pi_{U_1} \oplus \pi_{U_2}$ and π. We recall that $j_1 + j_2$ is surjective if and only if $U_1 + U_2 = V$ and that $j_1 + j_2$ is injective if and only if $U_1 \cap U_2 = \{\mathbf{0}\}$. In particular, if $j_1 + j_2$ is bijective, then $\pi \simeq \pi_{U_1} \oplus \pi_{U_2}$.

We can now restate Theorems 12 and 13 from Lecture 2 as follows:

Theorem 11. *A finite dimensional k-linear representation (V, π) of a group G is completely reducible if and only if $\pi \simeq \pi_1 \oplus \cdots \oplus \pi_m$ with $\pi_1, \ldots, \pi_m$ irreducible.*

Theorem 12. *Let (V, π) be a k-linear representation of a group G, and suppose that $\pi \simeq \pi_1 \oplus \cdots \oplus \pi_m$ with $\pi_1, \ldots, \pi_m$ irreducible. If $U \subset V$ is π-invariant, then π_U is isomorphic to the sum of some of the π_i, and $\pi_{V/U}$ is isomorphic to the sum of the remaining π_i.*

Lemma 13. *Let (V, π) be a k-linear representation of a group G, and suppose that $V_1, \ldots, V_m \subset V$ are π-invariant subspaces such that the representations $\pi_{V_1}, \ldots, \pi_{V_m}$ are irreducible and pairwise non-isomorphic. In this case, the canonical map*

$$V_1 \oplus \cdots \oplus V_m \longrightarrow V$$

is injective, so $\pi_{V_1} \oplus \cdots \oplus \pi_{V_m}$ is a subrepresentation of π.

Proof. We argue by induction on $m \geq 0$, the case $m = 0$ being trivial. So we assume that the statement has been proved for $m < r$ and prove it for $m = r$. By the inductive hypothesis, the canonical map

$$V_1 \oplus \cdots \oplus V_{r-1} \longrightarrow V$$

[3] The book writes $\pi_1 + \pi_2$ instead of $\pi_1 \oplus \pi_2$.

is injective with image $V_1 + \cdots + V_{r-1}$, so the kernel of the canonical map

$$V_1 \oplus \cdots \oplus V_{r-1} \oplus V_r \longrightarrow V$$

is equal to $(V_1 + \cdots + V_{r-1}) \cap V_r$. Since π_{V_r} is irreducible, the kernel in question is nonzero if and only if $V_r \subset V_1 + \cdots + V_{r-1}$. However, by Theorem 12, this is not possible, because $\pi_{V_r} \not\simeq \pi_{V_i}$ for all $1 \leq i < r$. $\square$

We can now prove the following analogue of unique prime factorization for semisimple representations.

Theorem 14. *Let $\pi_1, \ldots, \pi_m$ and $\rho_1, \ldots, \rho_n$ be irreducible k-linear representations of a group G, and suppose that $\pi_1 \oplus \cdots \oplus \pi_m \simeq \rho_1 \oplus \cdots \oplus \rho_n$. In this case, $m = n$ and, up to a reordering, $\pi_i \simeq \rho_i$ for all $1 \leq i \leq m$.*

Proof. The proof is by induction on $m \geq 0$, the case $m = 0$ being trivial. So we assume that the statement has been proved for $m < r$ and prove it for $m = r$. We choose any $0 < s < r$ and consider the two subrepresentations

$$\pi_1 \oplus \cdots \oplus \pi_s, \pi_{s+1} \oplus \cdots \oplus \pi_r \subset \pi_1 \oplus \cdots \oplus \pi_r \simeq \rho_1 \oplus \cdots \oplus \rho_n.$$

Theorem 12 shows that $\pi_1 \oplus \cdots \oplus \pi_s$ is a sum of some of the ρ_i, and that $\pi_{s+1} \oplus \cdots \oplus \pi_r$ is the sum of the remaining ρ_i. Since we $s < r$ and $r - s < r$, it follows from the inductive hypothesis that, up to a reordering, $\pi_i \simeq \rho_i$ for $1 \leq i \leq s$ and for $s + 1 \leq i \leq r$. This proves the induction step, and hence, the theorem. $\square$

Tensor products of representations

We finally define tensor products of representations, and again there is both an exterior tensor product "$\boxtimes$" and an interior tensor product "$\otimes$." First, we recall that a tensor product of two k-vector spaces is a k-bilinear map

$$V_1 \times V_2 \xrightarrow{p_{V_1,V_2}} V_1 \otimes V_2$$

with the property that for every k-bilinear map

$$V_1 \times V_2 \xrightarrow{b} U$$

there is exists a unique k-linear map $\widehat{b} \colon V_1 \otimes V_2 \to U$ that makes the diagram

$$\begin{array}{ccc} V_1 \times V_2 & \xrightarrow{p_{V_1,V_2}} & V_1 \otimes V_2 \\ & {\scriptstyle b}\searrow \quad \swarrow{\scriptstyle \widehat{b}} & \\ & U & \end{array}$$

commute. We say that this property of the tensor product is its defining universal property. In particular, we conclude that if also $q_{V_1,V_2} \colon V_1 \times V_2 \to V_1 \mathbin{\widetilde{\otimes}} V_2$ is a tensor product of V_1 and V_2, then the unique k-linear maps

$$V_1 \otimes V_2 \underset{\widehat{p}}{\overset{\widehat{q}}{\rightleftarrows}} V_1 \mathbin{\widetilde{\otimes}} V_2$$

are each other's inverses. In this way, a tensor product of V_1 and V_2 is unique, up to unique isomorphism, so we often abuse language and call it *the* tensor product of V_1 and V_2. Given $\boldsymbol{v}_1 \in V_1$ and $\boldsymbol{v}_2 \in V_2$, we write[4]

$$\boldsymbol{v}_1 \otimes \boldsymbol{v}_2 = p_{V_1,V_2}(\boldsymbol{v}_1, \boldsymbol{v}_2) \in V_1 \otimes V_2.$$

We recall that if the families $(\boldsymbol{e}_i)_{i\in I}$ and $(\boldsymbol{f}_j)_{j\in J}$ are bases of V_1 and V_2, respectively, then the family $(\boldsymbol{e}_i \otimes \boldsymbol{f}_j)_{(i,j)\in I\times J}$ is a basis of $V_1 \otimes V_2$. In particular, we have

$$\dim_k(V_1 \otimes V_2) = \dim_k(V_1) \cdot \dim_k(V_2).$$

Suppose that $f_1 \colon W_1 \to V_1$ and $f_2 \colon W_2 \to V_2$ are two k-linear maps. It follows from the defining universal property of the tensor product, there is a unique k-linear map $f_1 \otimes f_2 \colon W_1 \otimes W_2 \to V_1 \otimes V_2$ that makes the diagram

$$\begin{array}{ccc} W_1 \times W_2 & \xrightarrow{p_{W_1,W_2}} & W_1 \otimes W_2 \\ \Big\downarrow{\scriptstyle f_1 \times f_2} & & \Big\downarrow{\scriptstyle f_1 \otimes f_2} \\ V_1 \times V_2 & \xrightarrow{p_{V_1,V_2}} & V_1 \otimes V_2 \end{array}$$

commute. Indeed, the map $p_{V_1,V_2} \circ (f_1 \times f_2)$ is k-bilinear. By the uniqueness of this assignment, we conclude that there is a well-defined map

$$\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \xrightarrow{\otimes} \mathrm{GL}(V_1 \otimes V_2)$$

that to (f_1, f_2) assigns $f_1 \otimes f_2$ and that this map is a group homomorphism.

Definition 15. Let (V_1, π_1) and (V_2, π_2) be k-linear representations of two groups G_1 and G_2, respectively. The k-linear representation $(V_1 \otimes V_2, \pi_1 \boxtimes \pi_2)$ of the product group $G_1 \times G_2$, where $\pi_1 \boxtimes \pi_2$ is the composite group homomorphism

$$G_1 \times G_2 \xrightarrow{\pi_1 \times \pi_2} \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \xrightarrow{\otimes} \mathrm{GL}(V_1 \otimes V_2),$$

is called the exterior tensor product of (V_1, π_1) and (V_2, π_2).[5]

Spelling out the definition in terms of pure tensors, we have

$$(\pi_1 \boxtimes \pi_2)(g_1, g_2)(\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) = \pi_1(g_1)(\boldsymbol{v}_1) \otimes \pi_2(g_2)(\boldsymbol{v}_2),$$

where $g_1 \in G_1$, $g_2 \in G_2$, $\boldsymbol{v}_1 \in V_1$, and $\boldsymbol{v}_2 \in V_2$.

Definition 16. Let (V_1, π_1) and (V_2, π_2) be k-linear representations of the *same* group G. The k-linear representation $(V_1 \otimes V_2, \pi_1 \otimes \pi_2)$, where $\pi_1 \otimes \pi_2$ is defined to be the composite group homomorphism

$$G \xrightarrow{\Delta_G} G \times G \xrightarrow{\pi_1 \boxtimes \pi_2} \mathrm{GL}(V_1 \otimes V_2).$$

is called the tensor product of π_1 and π_2.[6]

[4] The tensors of the form $\boldsymbol{v}_1 \otimes \boldsymbol{v}_2$ are called pure tensors. They are the tensors that belong to the image of the map $p_{V_1,V_2} \colon V_1 \times V_2 \to V_1 \otimes V_2$. Every tensor can be written as a sum of pure tensors, but it is almost never useful to do so, since the sum is not unique. A tensor that is not a pure tensor is said to be entangled. This is the source of entanglement in quantum mechanics.

[5] Confusingly, the book writes $\pi_1 \otimes \pi_2$ instead of $\pi_1 \boxtimes \pi_2$.

[6] The book writes $\pi_1\pi_2$ instead of $\pi_1 \otimes \pi_2$.

Again, spelling out the definition in terms of pure tensors, we have

$$(\pi_1 \otimes \pi_2)(g)(\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) = \pi_1(g)(\boldsymbol{v}_1) \otimes \pi_2(g)(\boldsymbol{v}_2).$$

Example 17. 1) We recall that sum and tensor product satisfy a distributive law in the sense that the canonical map

$$(U \otimes V_1) \oplus (U \otimes V_2) \longrightarrow U \otimes (V_1 \oplus V_2)$$

is an isomorphism. So if $\tau \colon G \to \mathrm{GL}(k^n)$ is the trivial k-linear representation of G on $V = k^n$, and if $\pi \colon G \to \mathrm{GL}(U)$ is any k-linear representation, then

$$\pi \otimes \tau \simeq \pi \otimes (k \oplus \cdots \oplus k) \simeq (\pi \otimes k) \oplus \cdots \oplus (\pi \otimes k) \simeq \pi \oplus \cdots \oplus \pi,$$

where there are n summands.

2) If U and V are right k-vector spaces, then there is a natural k-linear map

$$V \otimes U^* \xrightarrow{\alpha_{U,V}} \mathrm{Hom}_k(U, V)$$

defined by $\alpha(\boldsymbol{v} \otimes f)(\boldsymbol{u}) = \boldsymbol{v} \cdot f(\boldsymbol{u})$. It is an isomorphism if at least one of U and V is finite dimensional. That the map $\alpha_{U,V}$ is natural means that if $\varphi \colon U_2 \to U_1$ and $\psi \colon V_1 \to V_2$ are k-linear maps, then the diagram

$$\begin{array}{ccc} V_1 \otimes U_1^* & \xrightarrow{\alpha_{U_1,V_1}} & \mathrm{Hom}_k(U_1, V_1) \\ \downarrow{\scriptstyle \psi \otimes U_1^*} & & \downarrow{\scriptstyle \mathrm{Hom}(U_1, \psi)} \\ V_2 \otimes U_1^* & \xrightarrow{\alpha_{U_1,V_2}} & \mathrm{Hom}_k(U_1, V_2) \\ \downarrow{\scriptstyle V_2 \otimes \varphi^*} & & \downarrow{\scriptstyle \mathrm{Hom}(\varphi, V_2)} \\ V_2 \otimes U_2^* & \xrightarrow{\alpha_{U_2,V_2}} & \mathrm{Hom}_k(U_2, V_2) \end{array}$$

commutes. In particular, if (V, π) is a k-linear represenstation of G, then

$$\begin{array}{ccc} V \otimes V^* & \xrightarrow{\alpha_{V,V}} & \mathrm{Hom}_k(V, V) \\ \downarrow{\scriptstyle \pi(g) \otimes V^*} & & \downarrow{\scriptstyle \mathrm{Hom}(V, \pi(g))} \\ V \otimes V^* & \xrightarrow{\alpha_{V,V}} & \mathrm{Hom}_k(V, V) \\ \downarrow{\scriptstyle V \otimes \pi(g^{-1})^*} & & \downarrow{\scriptstyle \mathrm{Hom}(\pi(g^{-1}), V)} \\ V \otimes V^* & \xrightarrow{\alpha_{V,V}} & \mathrm{Hom}_k(V, V) \end{array}$$

commutes for all $g \in G$. The outer diagram is

$$\begin{array}{ccc} V \otimes V^* & \xrightarrow{\alpha_{V,V}} & \mathrm{End}_k(V) \\ \downarrow{\scriptstyle (\pi \otimes \pi^*)(g)} & & \downarrow{\scriptstyle \mathrm{Ad}(\pi(g))} \\ V \otimes V^* & \xrightarrow{\alpha_{V,V}} & \mathrm{End}_k(V). \end{array}$$

So if V is finite dimensional, then $\alpha_{V,V}$ is isomorphism of k-linear representations

$$\pi \otimes \pi^* \simeq \mathrm{Ad} \circ \pi,$$

where the right-hand side is the k-linear representation of G on V given by the composite group homomorphism

$$G \xrightarrow{\pi} \mathrm{GL}(V) \xrightarrow{\mathrm{Ad}} \mathrm{GL}(V).$$

In particular, if we take $G = \mathrm{GL}(V)$ and $\pi = \mathrm{id}$, then π and π^* are irreducible, but their tensor product $\pi \otimes \pi^*$ is not!

Remark 18. If π_1 and π_2 are irreducible representations, then it is an important problem called "scattering" to determine how $\pi_1 \otimes \pi_2$ decomposes as a sum

$$\pi_1 \otimes \pi_2 \simeq \rho_1 \oplus \cdots \oplus \rho_m$$

of irreducible representations. The name "scattering" comes from physics. Indeed, by colliding the elementary particles π_1 and π_2, one obtains the state $\pi_1 \otimes \pi_2$, which, in turn, decays to the collection of elementary particles $\rho_1, \ldots, \rho_m$.