EXTENSION AND RESTRICTION OF SCALARS

Let $f\colon A \to B$ be a ring homomorphism. If $N = (N, +, \cdot\,)$ is a right B-module, then we define a right A-module

$$f_*(N) = (N, +, \star)$$

with the same underlying set and addition, but with right scalar multiplication by elements $a \in A$ on elements $\boldsymbol{y} \in N$ given by

$$\boldsymbol{y} \star a = \boldsymbol{y} \cdot f(a).$$

Moreover, if $h\colon N_1 \to N_2$ is a B-linear map between right B-modules N_1 and N_2, then the same map is an A-linear map

$$f_*(N_1) \xrightarrow{f_*(h)=h} f_*(N_2)$$

between the right A-modules $f_*(N_1)$ and $f_*(N_2)$.

Conversely, if $M = (M, +, \cdot\,)$ is a right A-module, then we define

$$f^*(M) = (M \otimes_A B, +, \cdot\,)$$

to be the right B-module, where for $\boldsymbol{x} \in M$ and $b_1, b_2 \in B$,

$$(\boldsymbol{x} \otimes b_1) \cdot b_2 = \boldsymbol{x} \otimes (b_1 b_2).$$

If $g\colon M_1 \to M_2$ is an A-linear map, then we define

$$f^*(M_1) \xrightarrow{f^*(g)} f^*(M_2)$$

to be the unique B-linear map such that for $\boldsymbol{x} \in M$ and $b \in B$,

$$f^*(g)(\boldsymbol{x} \otimes b) = g(\boldsymbol{x}) \otimes b.$$

It is well-defined, because g is A-linear. Indeed, if $\boldsymbol{x} \in M$, $a \in A$, and $b \in B$, then

$$f^*(g)(\boldsymbol{x}a \otimes b) = g(\boldsymbol{x}a) \otimes b = g(\boldsymbol{x})a \otimes b = g(\boldsymbol{x}) \otimes f(a)b = f^*(g)(\boldsymbol{x} \otimes f(a)b).$$

We say that f^* is the extension of scalars along f, and we say that f_* is the restriction of scalars along f. They are functors

$$\mathrm{Mod}_A \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Mod}_B$$

between the respective categories of right modules and linear maps. Indeed, it follows immediately from the definitions that, as required,

$$\begin{aligned} f^*(\mathrm{id}_M) &= \mathrm{id}_{f^*(M)} \\ f^*(g_1 \circ g_2) &= f^*(g_1) \circ f^*(g_2) \end{aligned}$$

and that

$$\begin{aligned} f_*(\mathrm{id}_N) &= \mathrm{id}_{f_*(N)} \\ f_*(h_1 \circ h_2) &= f_*(h_1) \circ f_*(h_2). \end{aligned}$$

Example 1. If $\sigma\colon \mathbb{C} \to \mathbb{C}$ is the ring homomorphism given by complex conjugation,

$$\sigma(a + ib) = a - ib,$$

and if $V = (V, +, \cdot\,)$ is a right $\mathbb{C}$-vector space, then

$$\sigma_*(V) = (V, +, \star)$$

is the complex conjugate right $\mathbb{C}$-vector space $\bar{V}$.

Returning to the general situation above, we define the unit map

$$M \xrightarrow{\eta_M} f_* f^*(M)$$

by $\eta_M(\boldsymbol{x}) = \boldsymbol{x} \otimes 1$ and the counit map

$$f^* f_*(N) \xrightarrow{\epsilon_N} N$$

by $\epsilon_N(\boldsymbol{y} \otimes b) = \boldsymbol{y}b$. They are both natural transformations of functors, which means that if $g\colon M_1 \to M_2$ and $h\colon N_1 \to N_2$ are an A-linear map and a B-linear map, respectively, then the following diagrams commute:

$$\begin{array}{ccc} M_1 & \xrightarrow{\eta_{M_1}} & f_* f^*(M_1) \\ \big\downarrow g & & \big\downarrow f_* f^*(g) \\ M_2 & \xrightarrow{\eta_{M_2}} & f_* f^*(M_2) \end{array} \qquad \begin{array}{ccc} f^* f_*(N_1) & \xrightarrow{\epsilon_{N_1}} & N_1 \\ \big\downarrow f^* f_*(h) & & \big\downarrow h \\ f^* f_*(N_2) & \xrightarrow{\epsilon_{N_2}} & N_2 \end{array}$$

Moreover, for every right A-module M and every right B-module N, the diagrams

$$\begin{array}{ccc} f^*(M) & \xrightarrow{f^*(\eta_M)} & f^* f_* f^*(M) \\ & \diagdown\!\!\diagdown & \big\downarrow \epsilon_{f^*(M)} \\ & & f^*(M) \end{array} \qquad \begin{array}{ccc} f_*(N) & \xrightarrow{\eta_{f_*(N)}} & f_* f^* f_*(N) \\ & \diagdown\!\!\diagdown & \big\downarrow f_*(\epsilon_N) \\ & & f_*(N) \end{array}$$

commute. We refer to this by saying that η and ϵ satisfy the triangle identities and that the quadruple $(f^*, f_*, \epsilon, \eta)$ is an adjunction from Mod_B to Mod_A.

Proposition 2. *In the above situation, the maps*

$$\mathrm{Hom}_B(f^*(M), N) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Hom}_A(M, f_*(N))$$

defined by $\alpha(g) = f_*(g) \circ \eta_M$ *and* $\beta(h) = \epsilon_N \circ f^*(h)$ *are each other's inverses.*

Proof. By definition, the map $\alpha(g)$ is the composite map

$$M \xrightarrow{\eta_M} f_* f^*(M) \xrightarrow{f_*(g)} f_*(N)$$

so the map $(\beta \circ \alpha)(g) = \beta(\alpha(g))$ is the composition of the upper horizontal maps and right-hand vertical map in the following diagram:

$$\begin{array}{ccccc} f^*(M) & \xrightarrow{f^*(\eta_M)} & f^* f_* f^*(M) & \xrightarrow{f^* f_*(g)} & f^* f_*(N) \\ & \diagdown\!\!\diagdown & \big\downarrow \epsilon_{f^*(M)} & & \big\downarrow \epsilon_N \\ & & f^*(M) & \xrightarrow{\quad g \quad} & N \end{array}$$

But the left-hand triangle commutes by the triangle identities, and the right-hand square commutes by the naturality of ϵ. So we conclude that $(\beta \circ \alpha)(g) = g$, as desired. Similarly, the map $\beta(h)$ is defined to be the composite map

$$f^*(M) \xrightarrow{f^*(h)} f^* f_*(N) \xrightarrow{\epsilon_N} N$$

so the map $(\alpha \circ \beta)(h) = \alpha(\beta(h))$ is the composition of the left-hand vertical map and lower horizontal maps in the following diagram:

$$\begin{array}{ccccc}
M & \xrightarrow{\quad h \quad} & f_*(N) & & \\
\downarrow{\scriptstyle \eta_M} & & \downarrow{\scriptstyle \eta_{f_*(N)}} & \searrow{\scriptstyle =} & \\
f_* f^*(M) & \xrightarrow{f_* f^*(h)} & f_* f^* f_*(N) & \xrightarrow{f_*(\epsilon_N)} & f_*(N),
\end{array}$$

But the left-hand square commutes by the naturality of η, and the right-hand triangle commutes by the triangle identities, so we conclude that $(\alpha \circ \beta)(h) = h$. $\square$

Let $f \colon k \to k'$ be an extension of fields. If V is a k-vector space, then the map

$$\mathrm{GL}(V) \xrightarrow{f^*} \mathrm{GL}(f^*(V))$$

is a group homomorphism, because f^* is a functor. Hence, if (V, π) is a k-linear representation of a group G, then we obtain a k'-linear representation of G given by the pair $(f^*(V), f^*\pi)$, where $f^*\pi$ is the composite group homomorphism

$$G \xrightarrow{\pi} \mathrm{GL}(V) \xrightarrow{f^*} \mathrm{GL}(f^*(V))$$

We call $f^*\pi$ the k'-linear representation obtained from the k-linear representation π by extension of scalars along f.

Similarly, if V' is a k'-vector space, then the map

$$\mathrm{GL}(V') \xrightarrow{f_*} \mathrm{GL}(f_*(V'))$$

is a group homomorphism, because f_* is a functor. Hence, if (V', π') is a k'-linear representation of G, then we obtain a k-linear representation of G given by the pair $(f_*(V'), f_*\pi')$, where $f_*\pi'$ is the composite group homomorphism

$$G \xrightarrow{\pi'} \mathrm{GL}(V') \xrightarrow{f_*} \mathrm{GL}(f_*(V')).$$

We call $f_*\pi'$ the k-linear representation obtained from the k'-linear representation π' by restriction of scalars along f.

Remark 3. If $f \colon k \to k'$ is a field extension, then

$$\begin{aligned}
\dim_{k'}(f^*(V)) &= \dim_k(V) \\
\dim_k(f_*(V')) &= d \cdot \dim_{k'}(V'),
\end{aligned}$$

where $d = [k' : k]$ is the degree of the extension.

Theorem 4. *Let $f: k \to k'$ be a field extension. Two finite-dimensional k-linear representations (V_1, π_1) and (V_2, π_2) of a group G are isomorphic if and only if the k'-linear representations $(f^*(V_1), f^*\pi_1)$ and $(f^*(V_2), f^*\pi_2)$ are so.*

Proof. If $h: V_1 \to V_2$ is a k-linear isomorphism that is intertwining between π_1 and π_2, then $f^*(h): f^*(V_1) \to f^*(V_2)$ is a k'-linear isomorphism that is intertwining between $f^*\pi_1$ and $f^*\pi_2$. This proves the "only if" part of the statement.

To prove the "if" part of the statement, we assume that $f^*\pi_1 \simeq f^*\pi_2$ and prove that $\pi_1 \simeq \pi_2$. The proof that we give here uses that the field k is infinite. A different proof based on the Krull–Schmidt theorem works for all k. We first note that

$$\dim_k(V_1) = \dim_{k'}(f^*(V_1)) = \dim_{k'}(f^*(V_2)) = \dim_k(V_2),$$

where the middle equality holds by the assumption that $f^*\pi_1 \simeq f^*\pi_2$. So we may consider π_1 and π_2 to be matrix representations

$$G \xrightarrow{\pi_1, \pi_2} \mathrm{GL}_n(k).$$

Moreover, by viewing k as a subfield $k \subset k'$, namely, as the image of the extension $f: k \to k'$, we may consider $f^*\pi_1$ and $f^*\pi_2$ as the matrix representations

$$G \xrightarrow{\pi_1, \pi_2} \mathrm{GL}_n(k) \subset \mathrm{GL}_n(k').$$

Now, that $\pi_1 \simeq \pi_2$ means that there exists $C \in M_n(k)$ such that

(a) For all $g \in G$, $C \cdot \pi_1(g) = \pi_2(g) \cdot C$.
(b) The determinant $\det(C)$ is nonzero.

The requirement (a) is a system of linear equations of k in n^2 variables. By Gauss elimination, we know that a general solution has can be written uniquely as

$$C = t_1 C_1 + \cdots + t_m C_m$$

with $(C_1, \dots, C_m)$ a linearly independent family of vectors in the k-vector space $M_n(k)$ and with $(t_1, \dots, t_m)$ a family of scalars in the field k. The requirement (b) is the statement that there exists a family $(t_1, \dots, t_m)$ of scalars in k such that the value of the polynomial

$$p(x_1, \dots, x_m) = \det(x_1 C_1 + \cdots + x_m C_m) \in k[x_1, \dots, x_m]$$

at $(x_1, \dots, x_m) = (t_1, \dots, t_m)$ is nonzero. Similarly, that $f^*\pi_1 \simeq f^*\pi_2$ means that there exists $C' \in M_n(k')$ such that

(a') For all $g \in G$, $C' \cdot \pi_1(g) = \pi_2(g) \cdot C'$.
(b') The determinant $\det(C')$ is nonzero.

But (a') is the same system of linear equations as (a), so Gauss elimination tells us that a general solution $C' \in M_n(k')$ can be written uniquely as

$$C' = t'_1 C_1 + \cdots + t'_m C_m$$

with $(C_1, \dots, C_m)$ as before and with $(t'_1, \dots, t'_m)$ a family of scalars in the field k'. And (b') is the requirement that there exists a family $(t'_1, \dots, t'_m)$ of scalars in k' such that the value of the polynomial

$$p(x_1, \dots, x_m) \in k[x_1, \dots, x_m] \subset k'[x_1, \dots, x'_m]$$

is nonzero. Since k, and hence, k' is infinite, the k'-linear map

$$k'[x_1,\dots,x_m] \xrightarrow{\text{ev}} \mathrm{Map}((k')^m,k')$$

is injective, so our assumption that $f^*\pi_1 \simeq f^*\pi_2$ implies that the polynomial

$$p(x_1,\dots,x_m) \in k[x_1,\dots,x_m]$$

is nonzero. But, since k is infinite, the k-linear map

$$k[x_1,\dots,x_m] \xrightarrow{\text{ev}} \mathrm{Map}(k^m,k)$$

is injective, so we exists $(t_1,\dots,t_m) \in k^m$ such that $p(t_1,\dots,t_m) \neq 0$. This shows that $\pi_1 \simeq \pi_2$, as desired. □

Suppose that $f\colon k \to k'$ is a Galois extension with Galois group

$$\Gamma = \mathrm{Aut}_k(k').$$

If V is a k-vector space, then the group homomorphism

$$\Gamma \xrightarrow{\rho} \mathrm{GL}(f_*f^*(V))$$

given by the formula

$$\rho(\gamma)(\boldsymbol{x}\otimes b) = \boldsymbol{x}\otimes\gamma(b)$$

with $\boldsymbol{x} \in V$ and $b \in k'$ defines a k-linear representation of Γ on $f_*f^*(V)$. Suppose that we also have a k-linear representation

$$G \xrightarrow{\pi} \mathrm{GL}(V)$$

of some group G on V. In this situation, the group homomoprhism

$$G \xrightarrow{f_*f^*\pi} \mathrm{GL}(f_*f^*(V))$$

also defines a k-linear representation of the group G on $f_*f^*(V)$. We note that for all $g \in G$, the k-linear isomorphism $f_*f^*\pi(g)$ is intertwining with respect to γ. Similarly, for all $\gamma \in \Gamma$, the k-linear isomorphism $\rho(\gamma)$ is intertwining with respect to $f_*f^*(\pi)$. Indeed, for all $\gamma\in\Gamma$, $g \in G$, $\boldsymbol{x} \in V$, and $b \in k'$, we have

$$\begin{aligned}\rho(\gamma)(f_*f^*\pi(g)(\boldsymbol{x}\otimes b)) &= \rho(\gamma)(\pi(g)(\boldsymbol{x})\otimes b)\\ &= \pi(g)(\boldsymbol{x})\otimes\gamma(b)\\ &= f_*f^*\pi(g)(\boldsymbol{x}\otimes\gamma(b))\\ &= f_*f^*\pi(g)(\rho(\gamma)(\boldsymbol{x}\otimes b)).\end{aligned}$$

Equivalently, the map

$$G\times\Gamma \xrightarrow{\tau} \mathrm{GL}(f_*f^*(V))$$

given by

$$\tau(g,\gamma)(\boldsymbol{x}\otimes b) = \pi(g)(\boldsymbol{x})\otimes\gamma(b)$$

is a group homomorphism and defines a representation of the group $G\times\Gamma$ on the k-vector space $f_*f^*(V)$. It follows that the subspace

$$W = (f_*f^*(V))^\Gamma = \{\boldsymbol{y} \in f_*f^*(V) \mid \rho(\gamma)(\boldsymbol{y}) = \boldsymbol{y} \text{ for all } \gamma\in\Gamma\} \subset f_*f^*(V)$$

is $f_*f^*\pi$-invariant. Moreover, the unit map

$$V \xrightarrow{\eta} f_*f^*(V)$$

is intertwining between π and $f_*f^*\pi$ and induces a map

$$V \xrightarrow{\tilde{\eta}} W = (f_*f^*(V))^\Gamma$$

that is intertwining between π and $(f_*f^*\pi)_W$.

Theorem 5. *Let $f\colon k \to k'$ be a finite Galois extension with group $\Gamma = \mathrm{Gal}(k'/k)$. If (V,π) is a k-linear representation of a group G, then the map*

$$V \xrightarrow{\tilde{\eta}} W = (f_*f^*(V))^\Gamma$$

is an isomorphism between π and $(f_*f^*\pi)_W$.

Proof. By faithfully flat descent for modules, the diagram

$$V \xrightarrow{\eta_V} f_*f^*(V) \underset{\eta_{f_*f^*(V)}}{\overset{f_*f^*(\eta_V)}{\rightrightarrows}} f_*f^*f_*f^*(V)$$

is an equalizer. This only uses that $f\colon k \to k'$ is faithfully flat, which is true for every extension of fields. That the diagram is an equalizer means that the map η_V is injective and that its image is equal to the subspace

$$W' = \{\boldsymbol{y} \in f_*f^*(V) \mid f_*f^*(\eta_V)(\boldsymbol{y}) = \eta_{f_*f^*(V)}(\boldsymbol{y})\} \subset f_*f^*(V).$$

So we wish to show that $W = W'$ and write out the diagram above as

$$V \xrightarrow{\eta_V} V \otimes_k k' \underset{\eta_{f_*f^*(V)}}{\overset{f_*f^*(\eta_V)}{\rightrightarrows}} V \otimes_k k' \otimes_k k'$$

with $\eta_V(\boldsymbol{x}) = \boldsymbol{x} \otimes 1$, $f_*f^*(\eta_V)(\boldsymbol{x} \otimes b) = \boldsymbol{x} \otimes 1 \otimes b$, and $\eta_{f_*f^*(V)}(\boldsymbol{x} \otimes b) = \boldsymbol{x} \otimes b \otimes 1$. The assumption that $f\colon k \to k'$ is a finite Galois extension with group Γ implies that the ring homomorphism

$$k' \otimes_k k' \xrightarrow{h} \textstyle\prod_{\gamma \in \Gamma} k'$$

with γth component $h_\gamma(b_1 \otimes b_2) = b_1\gamma(b_2)$ is an isomorphism. Thus the subspace $W' \subset V \otimes_k k'$ is equal to the equalizer of the two composite maps

$$V \otimes_k k' \underset{\eta_{f_*f^*(V)}}{\overset{f_*f^*(\eta_V)}{\rightrightarrows}} V \otimes_k k' \otimes_k k' \xrightarrow{V \otimes h} \textstyle\prod_{\gamma \in \Gamma} V \otimes_k k'.$$

Finally, the γth components of the two composite maps are given by

$$((V \otimes h_\gamma) \circ f_*f^*(\eta_V))(\boldsymbol{x} \otimes b) = \boldsymbol{x} \otimes \gamma(b)$$
$$((V \otimes h_\gamma) \circ \eta_{f_*f^*(V)})(\boldsymbol{x} \otimes b) = \boldsymbol{x} \otimes b,$$

which shows that $W = W'$ as desired. □

This was rather abstract! Let us now specialize to the case

$$k = \mathbb{R} \xrightarrow{f} k' = \mathbb{C}$$

which is Galois with group $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}, \sigma\}$, where $\sigma\colon \mathbb{C} \to \mathbb{C}$ is complex conjugation. If V is a real vector space, then it is common to write

$$V_{\mathbb{C}} = f^*(V)$$

and call it the complexification of V. If V' is a complex vector space, then it is also common to abuse of notation and write V' for the real vector space $f_*(V')$. This is very confusing, however, since V' is a complex vector space, whereas $f_*(V')$ is a real vector space.

If V is a real vector space, then so is $f_*(V_{\mathbb{C}})$, and we have the $\mathbb{R}$-linear map

$$f_*(V_{\mathbb{C}}) \xrightarrow{\rho(\sigma)} f_*(V_{\mathbb{C}})$$

where $\sigma \in \Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is complex conjugation. We will also refer to this map as complex conjugation, and given $\boldsymbol{y} \in f_*(V_{\mathbb{C}})$, we write

$$\bar{\boldsymbol{y}} = \rho(\sigma)(\boldsymbol{y}).$$

If we write $\boldsymbol{y} = \sum \boldsymbol{x}_i \otimes z_i$ with $\boldsymbol{x}_i \in V$ and $z_i \in \mathbb{C}$, then $\bar{\boldsymbol{y}} = \sum \boldsymbol{x}_i \otimes \bar{z}_i$.

If $W \subset V_{\mathbb{C}}$ is a complex subspace, then so is its image

$$\overline{W} = \rho(\sigma)(W) \subset V_{\mathbb{C}}$$

under complex conjugation. Indeed, if $\bar{\boldsymbol{y}} = \rho(\sigma)(\boldsymbol{y}) \in \overline{W}$ and $z \in \mathbb{C}$, then also

$$\bar{\boldsymbol{y}} \cdot z = \rho(\sigma)(\boldsymbol{y}) \cdot z = \rho(\sigma)(\boldsymbol{y} \cdot \bar{z}) \in \overline{W}.$$

Lemma 6. *Let V be a real vector space, and let $W \subset V_{\mathbb{C}}$ be a complex subspace of its complexification. The following are equivalent.*

(1) *The complex subspaces $W, \overline{W} \subset V_{\mathbb{C}}$ are equal.*
(2) *There exists a real subspace $U \subset V$ such that $W = U_{\mathbb{C}} \subset V_{\mathbb{C}}$.*

Proof. It is clear that (2) implies (1), so we assume (1) holds and prove (2). The unit map $\eta_V \colon V \to f_*(V_{\mathbb{C}})$ is $\mathbb{R}$-linear, and we define

$$U = \eta_V^{-1}(f_*(W)) \subset V.$$

By Proposition 2, the $\mathbb{R}$-linear map

$$U \xrightarrow{\eta_V|_U} f_*(W)$$

determines and is determined by the $\mathbb{C}$-linear map

$$U_{\mathbb{C}} = f^*(U) \xrightarrow{\beta(\eta_V|_U)} W,$$

and we claim that the latter map is an isomorphism. It is injective, because the diagram commutes and because the left-hand vertical map is injective.[1]

$$\begin{array}{ccc} f^*(U) & \xrightarrow{\beta(\eta_V|_U)} & W \\ \big\downarrow & & \big\downarrow \\ f^*(V) & = & V_{\mathbb{C}} \end{array}$$

To prove that it is also surjective, let $\boldsymbol{y} \in W$. We have $\bar{\boldsymbol{y}} \in \overline{W}$, so by (1), we also have $\bar{\boldsymbol{y}} \in W$. It follows that both $\boldsymbol{u} = \frac{1}{2}(\boldsymbol{y} + \bar{\boldsymbol{y}})$ and $\boldsymbol{v} = \frac{1}{2i}(\boldsymbol{y} - \bar{\boldsymbol{y}})$ belong to W. But $\bar{\boldsymbol{u}} = \boldsymbol{u}$ and $\bar{\boldsymbol{v}} = \boldsymbol{v}$, so by Theorem 5, we have

$$\boldsymbol{u}, \boldsymbol{v} \in \mathrm{im}(V \xrightarrow{\eta_V} f_*(V_{\mathbb{C}})).$$

[1] Here we use that $f \colon \mathbb{R} \to \mathbb{C}$ is flat, as is any field extension. Indeed, extension of scalars along a ring homomorphism $f \colon A \to B$ preserves monomorphisms if and only if $f \colon A \to B$ is flat.

and since also $\boldsymbol{u}, \boldsymbol{v} \in W$, we have

$$\boldsymbol{u}, \boldsymbol{v} \in \operatorname{im}(U \xrightarrow{\eta_V|_U} f_*(W)).$$

But this shows that

$$\boldsymbol{y} = \boldsymbol{u} + i\boldsymbol{v} \in \operatorname{im}(U_{\mathbb{C}} = f^*(U) \xrightarrow{\beta(\eta_V|_U)} W)$$

as desired. □

Example 7. We recall the real representation $(\mathbb{R}^2, \pi)$ of the additive group of real numbers $G = (\mathbb{R}, +)$ defined by

$$\pi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

It is an irreducible representation, but its complexification $\pi_{\mathbb{C}}$ is not. Indeed, in the basis $(\boldsymbol{e}_1 + i\boldsymbol{e}_2, \boldsymbol{e}_1 - i\boldsymbol{e}_2)$ of $(\mathbb{R}^2)_{\mathbb{C}} \simeq \mathbb{C}^2$, we have

$$\pi_{\mathbb{C}}(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.$$

Theorem 8. *Let $\pi\colon G \to \mathrm{GL}(V)$ be a real representation.*

(1) *If $\pi_{\mathbb{C}}$ is irreducible, then so is π.*
(2) *If π is irreducible, then either $\pi_{\mathbb{C}}$ is irreducible or a sum of two irreducible representations, which are each other's conjugate.*
(3) *The represention π is semisimple if and only if $\pi_{\mathbb{C}}$ is so.*

Proof. (1) If $U \subset V$ is π-invariant, then $U_{\mathbb{C}} \subset V_{\mathbb{C}}$ is $\pi_{\mathbb{C}}$-invariant. So $U_{\mathbb{C}}$ is equal to either $\{\mathbf{0}\}$ or $V_{\mathbb{C}}$, which shows that U is equal to either $\{\mathbf{0}\}$ or V as desired.

(2) Let $W \subset V_{\mathbb{C}}$ be a $\pi_{\mathbb{C}}$-invariant subspace with $\pi_{\mathbb{C},W}$ irreducible. In this situation,

$$W \cap \overline{W}, W + \overline{W} \subset V_{\mathbb{C}}$$

are both $\pi_{\mathbb{C}}$-invariant subspaces, and since

$$\overline{W \cap \overline{W}} = \overline{W} \cap \overline{\overline{W}} = \overline{W} \cap W$$
$$\overline{W + \overline{W}} = \overline{W} + \overline{\overline{W}} = \overline{W} + W,$$

it follows from Lemma 6 that both are complexifications of real subspaces of V. By the assumption that V is irreducible, the only possibilities are that

(i) $W \cap \overline{W} = W + \overline{W} = \{\mathbf{0}\}$,
(ii) $W \cap \overline{W} = \{\mathbf{0}\} \subset W + \overline{W} = V_{\mathbb{C}}$, or
(iii) $W \cap \overline{W} = W + \overline{W} = V_{\mathbb{C}}$.

In case (i), we have $W = \{\mathbf{0}\}$, in (ii), the map $W \oplus \overline{W} \to V_{\mathbb{C}}$ induced by the canonical inclusions is an isomorphism; and in case (iii), we have $W = V_{\mathbb{C}}$. This proves (2). Finally, (3) follows immediately from (1) and (2). □

Example 9. Let $\pi\colon \Sigma_3 \to \mathrm{GL}(\mathbb{R}^3)$ be the standard (permutation) representation of the symmetric group. The subspaces

$$V_1 = \{\boldsymbol{x} \in \mathbb{R}^3 \mid x_1 = x_2 = x_3\} \subset \mathbb{R}^3$$
$$V_2 = \{\boldsymbol{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\} \subset \mathbb{R}^3$$

are π-invariant, and moreover, the representations $\pi_1 = \pi_{V_1}$ and $\pi_2 = \pi_{V_2}$ are both irreducible and π_2 is faithful. We claim that $\pi_{2,\mathbb{C}}$ is irreducible. If not, then it is a sum of two irreducible representations, and since

$$\dim_{\mathbb{C}}(V_{2,\mathbb{C}}) = \dim_{\mathbb{R}}(V_2) = 2,$$

each of these two irreducible representations much be 1-dimensional. But $\pi_{2,\mathbb{C}}$ is again faithful,[2] so this would give an injective group homomorphism

$$\Sigma_3 \longrightarrow \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}),$$

which is impossible, since Σ_3 is non-abelian, while $\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$ is abelian.

[2] This is true, because $f\colon \mathbb{R} \to \mathbb{C}$ is faithful