SCHUR'S LEMMA AND ITS APPLICATIONS

Schur's lemma

Let (V_1, π_1) and (V_2, π_2) be k-linear representations of a group G and recall that a k-linear map $f \colon V_1 \to V_2$ is intertwining between π_1 and π_2 if the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ \big\downarrow{\scriptstyle \pi_1(g)} & & \big\downarrow{\scriptstyle \pi_2(g)} \\ V_1 & \xrightarrow{f} & V_2 \end{array}$$

commutes for all $g \in G$. We will write

$$\mathrm{Hom}(\pi_1, \pi_2) \subset \mathrm{Hom}_k(V_1, V_2)$$

for the subspace of intertwining k-linear maps.

Theorem 1 (Schur's lemma)**.** *Let (V_1, π_1) and (V_2, π_2) be two irreducible k-linear representations of a group G. A k-linear map $f \colon V_1 \to V_2$ that is intertwining between π_1 and π_2 is either an isomorphism or the zero map.*

Proof. It follows immediately from the fact that $f \colon V_1 \to V_2$ is intertwining between π_1 and π_2, that $\ker(f) \subset V_1$ is π_1-invariant and that $\mathrm{im}(f) \subset V_2$ is π_2-invariant. Therefore, since π_1 was assumed to irreducible, either $\ker(f) = \{\mathbf{0}\}$ or $\ker(f) = V_1$, and since π_2 was assumed to be irreducible, either $\mathrm{im}(f) = \{\mathbf{0}\}$ or $\mathrm{im}(f) = V_2$. □

Theorem 2. *Suppose that $((V_s, \pi_s))_{s \in S}$ is a finite family of pairwise non-isomorphic irreducible k-linear representations of a group G, and that $U \subset \bigoplus_{s \in S} V_s$ is a $\bigoplus_{s \in S} \pi_s$-invariant subspace. There exists a (unique) subset $T \subset S$ such that*

$$U = \bigoplus\nolimits_{t \in T} V_t \ \subset \ \bigoplus\nolimits_{s \in S} V_s.$$

Proof. Let $V = \bigoplus_{s \in S} V_s$, let $\pi = \bigoplus_{s \in S} \pi_s$, and let $i_s \colon V_s \to V$ be the canonical inclusion, which is intertwining between π_s and π. Theorem 12 in Lecture 4 shows that $U \subset V$ is the image of *some* injective k-linear map

$$\bigoplus\nolimits_{t \in T} V_t \xrightarrow{f} V$$

that is intertwining between $\bigoplus_{t \in T} \pi_t$ and π, and we wish to show that the map

$$\bigoplus\nolimits_{t \in T} V_t \xrightarrow{i = \sum_{t \in T} i_t} V$$

will do. Let $p_s\colon V \to V_s$ be the unique map such that $p_s \circ i_t\colon V_t \to V_s$ is the identity map of V_s if $s = t$ and the zero map if $s \neq t$. We consider the composite maps

$$\begin{array}{ccc} \bigoplus_{u\in T} V_u & \xrightarrow{\;f\;} & V \\ \uparrow{\scriptstyle i_t} & & \downarrow{\scriptstyle p_s} \\ V_t & \xrightarrow{\;f_{s,t}\;} & V_s \end{array}$$

for $s \in S$ and $t \in T$. Theorem 1 shows that $f_{s,t}$ is zero if $s \neq t$, so the diagram

$$\begin{array}{ccc} \bigoplus_{t\in T} V_t & \overset{f}{\searrow} & \\ \downarrow{\scriptstyle \bigoplus_{t\in T} f_{t,t}} & & V \\ \bigoplus_{t\in T} V_t & \underset{i}{\nearrow} & \end{array}$$

commutes. Moreover, the maps $f_{t,t}$ cannot be zero, since the top slanted map is injective, so Theorem 1 shows that the $f_{t,t}$ all are isomorphisms, and hence, the left-hand vertical map is an isomorphism. In particular,

$$U = \operatorname{im}(f) = \operatorname{im}(i),$$

as we wanted to prove. □

If (V, π) is an irreducible k-linear representation of G, then Schur's lemma shows, in particular, that the endomorphism ring

$$\operatorname{End}(\pi) \subset \operatorname{End}_k(V)$$

is a division algebra D over k. In general, every finite dimensional division algebra over k occurs as $\operatorname{End}(\pi)$ for a finite dimensional irreducible k-linear representation of some group G.[1] We now make the very simplifying assumption that

$$k = \bar{k}$$

is an algebraically closed field, so that, up to unique isomorphism, the only finite dimensional division algebra over k is $D = k$. In this case, Schur's lemma implies the following result, which is also known as Schur's lemma.

Theorem 3. *Let k be an algebraically closed field. If (V, π) is a finite dimensional irreducible k-linear representation of a group G, then the map*

$$k \xrightarrow{\;\eta\;} \operatorname{End}(\pi)$$

defined by $\eta(\lambda) = \lambda \cdot \mathrm{id}_V$ is a ring isomorphism.

Proof. The map η is injective, because V is nonzero, and to prove that it surjective, we let $f\colon V \to V$ be a k-linear map that is intertwining with respect to π. Since k is algebraically closed, the map f has an eigenvalue $\lambda \in k$, and since f is intertwining with respect to π, the eigenspace

$$\{\mathbf{0}\} \neq V_\lambda \subset V$$

is a π-invariant subspace. Since π is irreducible, we conclude that $V_\lambda = V$, or equivalently, that $f = \lambda \cdot \mathrm{id}_V$, which shows that η is surjective. □

[1] If D finite dimensional real division algebra, then $D \simeq \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

Corollary 4. *Let (V_1, π_1) and (V_2, π_2) be isomorphic finite dimensional irreducible k-linear representations of a group G. If k is algebraically closed, then*

$$\dim_k \operatorname{Hom}(\pi_1, \pi_2) = 1.$$

Proof. We choose any k-linear isomorphism $h\colon V_1 \to V_2$ that intertwines between π_1 and π_2. (Such an h exists by assumption.) Now, if $f\colon V_1 \to V_2$ is any k-linear map that intertwines between π_1 and π_2, then $f \circ h^{-1}\colon V_2 \to V_2$ is intertwining with respect to π_2. Hence, by Theorem 3, $f \circ h^{-1} = \lambda \cdot \mathrm{id}_{V_2}$ for a unique $\lambda \in k$, so we find that $f = \lambda \cdot h$. □

Remark 5. More precisely, Corollary 4 shows that the composition maps

$$\operatorname{Hom}(\pi_1, \pi_2) \times \operatorname{End}(\pi_1) \xrightarrow{\circ} \operatorname{Hom}(\pi_1, \pi_2)$$
$$\operatorname{End}(\pi_2) \times \operatorname{Hom}(\pi_1, \pi_2) \xrightarrow{\circ} \operatorname{Hom}(\pi_1, \pi_2)$$

simultaneously make $\operatorname{Hom}(\pi_1, \pi_2)$ a free right $\operatorname{End}(\pi_1)$-module of rank 1 and a free left $\operatorname{End}(\pi_2)$-module of rank 1. However, neither module has a preferred generator: There is no preferred way to compare π_1 and π_2.

In Theorem 2, we considered a finite sum of pairwise non-isomorphic irreducible representations. In the next result, we will consider the opposite situation of a finite sum of irreducible representations, all of which are isomorphic.

Theorem 6. *Let k be an algebraically closed field, and let (U, τ) and (V, π) be finite dimensional k-linear representations of a group G such that τ is trivial and π is irreducible. Given a $\pi \otimes \tau$-invariant subspace $W \subset V \otimes U$ with $(\pi \otimes \tau)_W$ irreducible, there exists (a non-unique) vector $\boldsymbol{u} \in U$ such that the map*

$$V \xrightarrow{i_{\boldsymbol{u}}} V \otimes U$$

defined by $i_{\boldsymbol{u}}(\boldsymbol{v}) = \boldsymbol{v} \otimes \boldsymbol{u}$ is an isomorphism from V onto W and is intertwining between π and $(\pi \otimes \tau)_W$.

Proof. The map $i_{\boldsymbol{u}}$ is clearly k-linear and intertwining between π and $\pi \otimes \tau$, so we must show that $\boldsymbol{u} \in U$ can be chosen such that $W = i_{\boldsymbol{u}}(V) \subset V \otimes U$. We have seen earlier that every irreducible subrepresentations of $\pi \otimes \tau$ is isomorphic to π. In particular, we may choose a k-linear isomorphism $h\colon W \to V$ that intertwines between $(\pi \otimes \tau)_W$ and π. Now, for every $f \in U^*$, we let

$$V \otimes U \xrightarrow{c_f} V$$

to be the unique k-linear map such that $c_f(\boldsymbol{v} \otimes \boldsymbol{u}) = \boldsymbol{v} \cdot f(\boldsymbol{u})$. It intertwines between $\pi \otimes \tau$ and π, and since k is algebraically closed, Corollary 4 shows that

$$c_f|_W = \lambda(f) \cdot h$$

for a unique $\lambda(f) \in k$. The map

$$U^* \xrightarrow{\lambda} k$$

that to f assigns $\lambda(f)$ is k-linear, so $\lambda \in U^{**}$. But the map

$$U \xrightarrow{\eta} U^{**}$$

is an isomorphism, since U is finite dimensional, so $\lambda = \eta(\boldsymbol{u})$ for a unique $\boldsymbol{u} \in U$. We claim that for this $\boldsymbol{u} \in U$, we have $i_{\boldsymbol{u}}(V) = W \subset V \otimes U$. To prove this, we choose a basis $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ of V and write $\boldsymbol{w} \in W$ as

$$\boldsymbol{w} = \textstyle\sum_{1 \leq i \leq n} \boldsymbol{v}_i \otimes \boldsymbol{u}_i$$

with $\boldsymbol{u}_i \in U$. (This way of writing $\boldsymbol{w}$ is unique, but we will not need this fact.) Now, for every $f \in U^*$, we have

$$\textstyle\sum_{1 \leq i \leq n} \boldsymbol{v}_i f(\boldsymbol{u}_i) = c_f(\boldsymbol{w}) = \lambda(f) \cdot h(\boldsymbol{w}) = f(\boldsymbol{u}) \cdot h(\boldsymbol{w}).$$

Therefore, if $f(\boldsymbol{u}) = 0$, then $f(\boldsymbol{u}_i) = 0$ for all $1 \leq i \leq n$, and hence,

$$\boldsymbol{u}_i \in \operatorname{span}_k(\boldsymbol{u}) \subset U.$$

This shows that $\{\boldsymbol{0}\} \neq W \subset i_{\boldsymbol{u}}(V) \subset V \otimes U$, and since $i_{\boldsymbol{u}}(V) \simeq V$ is irreducible, we conclude that $W = i_{\boldsymbol{u}}(V)$ as stated. Note that the vector $\boldsymbol{u} \in U$ depends on the choice of the isomorphism $h \colon W \to V$. □

Abelian groups

We will next show that if k is algebraically closed, then every finite dimensional k-linear irreducible representation of an abelian group is 1-dimensional. We have seen that the representation of the group $(\mathbb{R}, +)$ on $\mathbb{R}^2$ by rotation is irreducible, so the assumption that k be algebraically closed is necessary.

Theorem 7. *Let k be an algebraically closed field. If (V, π) is a finite dimensional irreducible k-linear representation of an abelian group A, then*

$$\dim_k(V) = 1.$$

Proof. Since A is abelian, we have

$$\pi(g) \circ \pi(h) = \pi(g \cdot h) = \pi(h \cdot g) = \pi(h) \circ \pi(g)$$

for all $g, h \in A$. Hence, for all $g \in A$, the k-linear map

$$V \xrightarrow{\pi(g)} V$$

is intertwining with respect to π, so by Schur's lemma, we have

$$\pi(g) = \lambda \cdot \mathrm{id}_V$$

for some $\lambda = \lambda(g) \in k$. But this implies that every subspace $W \subset V$ is π-invariant, and since V is irreducible, this shows that $\dim_k(V) = 1$. □

We recall that the abelianization of a group G is a group homomorphism

$$G \xrightarrow{p} G^{\mathrm{ab}}$$

with the property that for every group homomorphism $f \colon G \to A$ with A abelian, there exists a unique group homomorphism $f^{\mathrm{ab}} \colon G^{\mathrm{ab}} \to A$ such that $f = f^{\mathrm{ab}} \circ p$. This property characterizes the abelianization $p \colon G \to G^{\mathrm{ab}}$ uniquely, up to unique isomorphism under G. The group homomorphism p is surjective, and its kernel is the commutator subgroup $[G, G] \subset G$. In particular, any 1-dimensional k-linear representation $\pi \colon G \to \mathrm{GL}(V)$ of a group G determines and is determined by the 1-dimensional k-linear representation $\pi^{\mathrm{ab}} \colon G^{\mathrm{ab}} \to \mathrm{GL}(V)$ of G^{ab}. Moreover, if k is algebraically closed, then a finite dimensional k-linear representation of G^{ab} is 1-dimensional if and only if it is irreducible.

Example 8. The abelianization of the symmetric group Σ_n is the signature

$$\Sigma_n \xrightarrow{\text{sgn}} \{\pm 1\}.$$

Therefore, up to non-canonical isomorphism, the only 1-dimensional representations of Σ_n over an algebraically closed field k are the trivial representation and the sign representation.

Exterior tensor product

We have seen that tensor products of irreducible k-linear representations are typically not irreducible, even if k is algebraically closed. We now show that the exterior tensor product of irreducible k-linear representations is always irreducible.

Theorem 9. *Let k be an algebraically closed field. If both $\pi_1\colon G_1 \to \mathrm{GL}(V_1)$ and $\pi_2\colon G_2 \to \mathrm{GL}(V_2)$ are finite dimensional irreducible k-linear representations, then so is their exterior tensor product*

$$G_1 \times G_2 \xrightarrow{\pi_1 \boxtimes \pi_2} \mathrm{GL}(V_1 \otimes V_2).$$

Proof. We let $\{\mathbf{0}\} \neq W \subset V_1 \otimes V_2$ be a $\pi_1 \boxtimes \pi_2$-invariant subspace and must show that $W = V_1 \otimes V_2$. We note that

$$(\pi_1 \boxtimes \pi_2)(g_1, e) = (\pi_1 \otimes \tau)(g_1)$$
$$(\pi_1 \boxtimes \pi_2)(e, g_2) = (\tau \otimes \pi_2)(g_2),$$

so $W \subset V_1 \otimes V_2$ is both a $\pi_1 \otimes \tau$-invariant subspace of the k-linear representation $\pi_1 \otimes \tau$ of $G_1 \times \{e\} \subset G_1 \times G_2$ and a $\tau \otimes \pi_2$-invariant subspace of the k-representation $\tau \otimes \pi_2$ of $\{e\} \times G_2 \subset G_1 \times G_2$. Hence, by Theorem 6, there exists $U_2 \subset V_2$ and $U_1 \subset V_1$ such that both the (injective) maps induced by the canonical inclusions

$$V_1 \otimes U_2 \longrightarrow V_1 \otimes V_2 \longleftarrow U_1 \otimes V_2$$

have image W. So the square diagram of inclusions

$$\begin{array}{ccc} U_1 \otimes U_2 & \longrightarrow & V_1 \otimes U_2 \\ \downarrow & & \downarrow \\ V_1 \otimes U_2 & \longrightarrow & W \end{array}$$

is cocartesian, and the right-hand vertical map and the lower horizontal are both isomorphisms. This implies (by the five-lemma) that

$$(V_1/U_1) \otimes U_2 \simeq \{\mathbf{0}\} \simeq U_1 \otimes (V_2/U_2),$$

which, in turn, implies that $U_1 = V_1$ and $U_2 = V_2$. So $W = V_1 \otimes V_2$ as desired. □

Example 10. We show that, in Theorem 9, the assumption that k be algebraically closed is necessary. The representation $\pi\colon G = (\mathbb{R}, +) \to \mathrm{GL}(\mathbb{R}^2)$ defined by

$$\pi(t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is irreducible, but we claim that $\pi \boxtimes \pi$ is not. Indeed, if $\pi \boxtimes \pi$ were irreducible, then also $(\pi \boxtimes \pi)_{\mathbb{C}}$ would either be irreducible or a sum of two irreducible representations. But this contradicts Theorem 7, since $G \times G$ is abelian and

$$\dim_{\mathbb{C}}((\pi \boxtimes \pi)_{\mathbb{C}}) = \dim_{\mathbb{R}}(\pi \boxtimes \pi) = 4.$$

Let k be a field, and let G be a group. We recall that $k[G]$ is k-vector space of all functions $f\colon G \to k$ and that

$$G \xrightarrow{L,R} \mathrm{GL}(k[G])$$

are the left and right regular k-linear representations defined by

$$L(g)(f)(h) = f(g^{-1}h)$$
$$R(g)(f)(h) = f(hg).$$

Since the maps $R(g_1)$ and $L(g_2)$ commute, we obtain a representation

$$G \times G \xrightarrow{\mathrm{Reg}} \mathrm{GL}(k[G])$$

defined by $\mathrm{Reg}(g_1, g_2) = L(g_2) \circ R(g_1) = R(g_1) \circ L(g_2)$. We call this representation the two-sided regular representation. Spelling out the definition, we have

$$\mathrm{Reg}(g_1, g_2)(f)(h) = f(g_2^{-1} h g_1).$$

Given any k-linear representation

$$G \xrightarrow{\pi} \mathrm{GL}(V),$$

the k-linear map

$$V \otimes V^* \xrightarrow{\mu} k[G]$$

defined by $\mu(\boldsymbol{x} \otimes f)(h) = f(\pi(h)(\boldsymbol{x}))$ is intertwining between $\pi \boxtimes \pi^*$ and Reg. We define the space of matrix coefficients (or matrix elements) of π to be its image

$$M(\pi) = \mu(V \otimes V^*) \subset k[G].$$

The reason for this name is as follows. Suppose that V is finite dimensional. If we let $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ be a basis of V, let $(\boldsymbol{v}_1^*, \dots, \boldsymbol{v}_n^*)$ be the dual basis of V^*, and let

$$A(h) = (a_{ij}(h)) \in M_n(k)$$

be the matrix that represents $\pi(h)$ with respect to $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$, then

$$a_{ij}(h) = \mu(\boldsymbol{v}_j \otimes \boldsymbol{v}_i^*)(h).$$

This shows that that space of matrix coefficients is the subspace

$$M(\pi) = \mathrm{span}_k(a_{ij} \mid 1 \leq i, j \leq n) \subset k[G]$$

spanned by the functions $a_{ij}\colon G \to k$, whence the name. The reason that we do not take this formula as our definition of $M(\pi)$ is that it is not a priori clear that the subspace $\mathrm{span}_k(a_{ij}) \subset k[G]$ is independent of the choice of basis.

Theorem 11. *Let k be an algebraically closed field. If (V, π) is a finite dimensional irreducible k-linear representation of a group G, then*

$$\pi \boxtimes \pi^* \xrightarrow{\mu} \mathrm{Reg}_{M(\pi)}$$

is an isomorphism of k-linear representations of $G \times G$. In particular, the k-linear representation $\mathrm{Reg}_{M(\pi)}$ is irreducible.

Proof. The map $\mu\colon V \otimes V^* \to M(\pi)$ is surjective, by definition, and it intertwines between $\pi \boxtimes \pi^*$ and $\mathrm{Reg}_{M(\pi)}$. Since π is finite dimensional and irreducible, the same is true for π^*, and since k is algebraically closed, Theorem 9 shows that $\pi \boxtimes \pi^*$ is irreducible. Hence, the kernel of μ is either zero or all of $V \otimes V^*$. But it is easy to see that $\mu\colon V \otimes V^* \to k[G]$ is not the zero map. Indeed, choosing a basis of V as above, we see that $a_{ii}(e) = 1$, so $0 \neq a_{ii} \in M(\pi) = \mathrm{im}(\mu)$. So μ is injective. □

We list some consequences of Theorem 11:

(1) If k is algebraically closed and if π is an irreducible k-linear representation of finite dimension n, then
$$\dim_k(M(\pi)) = n^2.$$
(2) For π as in (1), we have $R_{M(\pi)} \simeq \pi \oplus \cdots \oplus \pi$ and $L_{M(\pi)} \simeq \pi^* \oplus \cdots \oplus \pi^*$, where there are n summands in both cases.
(3) If k is algebraically closed and if π_1 and π_2 are finite dimensional irreducible k-linear representations, then $\mathrm{Reg}_{M(\pi_1)} \simeq \mathrm{Reg}_{M(\pi_2)}$ implies that $\pi_1 \simeq \pi_2$.
(4) If k is algebraically closed and if $\pi_1, \dots, \pi_m$ are pairwise non-isomorphic finite dimensional irreducible k-linear representations of G, then
$$M(\pi_1) \oplus \cdots \oplus M(\pi_m) \longrightarrow k[G]$$
is injective.

Unitary representations

Our final application of Schur's lemma concerns unitary representations. A finite dimensional complex representation (V, π) of a group G is unitary if there exists a hermitian inner product $\langle -, - \rangle$ on V that is π-invariant in the sense that
$$\langle \pi(g)(\boldsymbol{x}), \pi(g)(\boldsymbol{y}) \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle,$$
for all $g \in G$ and $\boldsymbol{x}, \boldsymbol{y} \in V$, or equivalently, if the induced isomorphism
$$\bar{V} \xrightarrow{\ b\ } V^*$$
given by $b(\boldsymbol{x})(\boldsymbol{y}) = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ is intertwining between $\bar{\pi}$ and π^*. We will now show that a π-invariant hermitian inner product is unique, up to scaling.

Theorem 12. *Suppose that $\pi\colon G \to \mathrm{GL}(V)$ is a finite dimensional irreducible unitary representation. If both $\langle -, - \rangle_1$ and $\langle -, - \rangle_2$ are π-invariant hermitian inner products on V, then there exists a real number $\lambda > 0$ such that for all $\boldsymbol{x}, \boldsymbol{y} \in V$,*
$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_2 = \lambda \cdot \langle \boldsymbol{x}, \boldsymbol{y} \rangle_1.$$

Proof. We define $h\colon \bar{V} \to \bar{V}$ to be the composite isomorphism
$$\bar{V} \xrightarrow{\ b_1\ } V^* \xrightarrow{\ (b_2)^{-1}\ } \bar{V}.$$
By assumption, both b_1 and b_2 are intertwining with respect to $\bar{\pi}$ and π^*, so h is intertwining with respect to $\bar{\pi}$. Finally, since $\bar{\pi}$ is irreducible, Schur's lemma shows that $h = \lambda \cdot \mathrm{id}_{\bar{V}}$ for some $\lambda \in \mathbb{C}$, and $\lambda \neq 0$, because h is an isomorphism. Finally, for any $\boldsymbol{0} \neq \boldsymbol{x} \in V$, both $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_1$ and $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_2$ are positive real numbers, so λ is necessarily real and positive. □

Theorem 13. *Let (V, π) be a finite dimensional unitary representation of a group G, and suppose that $U_1, U_2 \subset V$ are π-invariant subspaces with the property that the representations $\pi_1 = \pi_{U_1}$ and $\pi_2 = \pi_{U_2}$ are non-isomorphic and irreducible. In this situation, the subspaces $U_1, U_2 \subset V$ are necessarily orthogonal with respect to any π-invariant hermitian inner product on V.*

Proof. We choose a π-invariant hermitian inner product $\langle -, - \rangle$ on V. Since $U_1 \subset V$ is π-invariant, so is its orthogonal complement $W_1 \subset V$ with respect to $\langle -, - \rangle$, and moreover, the composition of the canonical inclusion and the canonical projection

$$U_1 \xrightarrow{i_1} V \xrightarrow{q_1} V/W_1$$

is a complex linear isomorphism $h = q_1 \circ i_1$ that intertwines between π_{U_1} and π_{V/W_1}. The orthogonal projection $p\colon V \to U_1$ with respect to $\langle -, - \rangle$ is the composition

$$V \xrightarrow{q_1} V/W_1 \xrightarrow{h^{-1}} U_1,$$

so it is intertwining between π and π_1. Now, the composite map

$$U_2 \xrightarrow{i_2} V \xrightarrow{p_1} U_1$$

is intertwining between π_2 and π_1, and since these representations are assumed to be irreducible and non-isomorphic, it follows from Schur's lemma that the composite map is zero. This shows that the subspaces $U_1, U_2 \subset V$ are orthogonal with respect to $\langle -, - \rangle$, as stated. $\square$