CHARACTER THEORY FOR FINITE GROUPS

Let us first show that, for every field k an irreducible k-linear representation of a finite group G is necessarily finite dimensional.

Lemma 1. *Let k be a field. If a k-linear representation (V, π) of a finite group G is irreducible, then the k-vector space V is finite dimensional.*

Proof. Let (V, π) be an irreducible k-linear representation of G. Since V is nonzero, there exists a nonzero vector $\boldsymbol{x} \in V$, so the subspace $W \subset V$ spanned by the family $(\pi(g)(\boldsymbol{x}))_{g \in G}$ is nonzero. But it is also π-invariant, so $W = V$, by the assumption that π is irreducible. Since G is finite, the family $(\pi(g)(\boldsymbol{x}))_{g \in G}$ is a finite family, so $W = V$ is a finite generated, and hence, finite dimensional k-vector space. □

So let G be a finite group. Because of Lemma 1, we will only consider finite dimensional k-linear representations of G. We will also assume that k is algebraically closed. Since G is finite, a basis of $k[G]$ is given by the family $(\delta_x)_{x \in G}$, where

$$\delta_x(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

We let (V, π) be a finite dimensional k-linear representation of G and recall that the subspace of matrix coefficients

$$M(\pi) \subset k[G]$$

is defined to be the image of the map

$$V \otimes V^* \xrightarrow{\mu} k[G]$$

given by $\mu(\boldsymbol{x} \otimes \varphi)(h) = \varphi(\pi(h)(\boldsymbol{x}))$. It is a Reg-invariant subspace, where

$$G \times G \xrightarrow{\text{Reg}} \text{GL}(k[G])$$

is the two-sided regular representation of $G \times G$ on V defined by

$$\text{Reg}(g_1, g_2)(f)(h) = f(g_2^{-1} h g_1).$$

Since k is algebraically closed, Schur's lemma implies the following statements, which we proved last time.

(1) If π is irreducible, then $\text{Reg}_{M(\pi)} \simeq \pi \boxtimes \pi^*$.
(2) If π_1 and π_2 are irreducible, then $\pi_1 \simeq \pi_2$ if and only if $M(\pi_1) = M(\pi_2)$.
(3) If $\pi_1, \dots, \pi_m$ are pairwise non-isomorphic and irreducible, then the map

$$M(\pi_1) \oplus \cdots \oplus M(\pi_m) \longrightarrow k[G]$$

induced by the canonical inclusions is injective.

Theorem 2. *If G is a finite group and if k is an algebraically closed field, then G has at most $|G|$ pairwise non-isomorphic irreducible k-linear representations.*

Proof. It follows from (3) that

$$q \leq \dim_k(M(\pi_1) \oplus \cdots \oplus M(\pi_q)) \leq \dim_k(k[G]) = |G|,$$

which proves the theorem. □

Theorem 3. *Let G be a finite group, and let k be an algebraically closed field of characteristic zero. If $\pi_1, \ldots, \pi_q$ are representatives of the isomorphism classes of irreducible k-linear representations, then the map*

$$M(\pi_1) \oplus \cdots \oplus M(\pi_q) \longrightarrow k[G]$$

induced by the canonical inclusions is an isomorphism.

Proof. The map is injective by (3) above, so it remains to prove that it is also surjective. Let $R\colon G \to \mathrm{GL}(k[G])$ be the right regular representation, which, we recall, is defined by $R(g)(f)(h) = f(hg)$. We claim that

$$M(R) = k[G].$$

Indeed, let $\epsilon\colon k[G] \to k$ be the k-linear map defined by $\epsilon(f) = f(e)$. So $\epsilon \in k[G]^*$ and for all $f \in k[G]$, the calculation

$$\mu(f \otimes \epsilon)(h) = \epsilon(R(h)(f)) = f(e \cdot h) = f(h)$$

shows that $f = \mu(f \otimes \epsilon) \in M(R)$. Now, since G is a finite group, whose order $|G|$ is not divisible by the characteristic of k, it follows from Maschke's theorem that every finite dimensional k-linear representation of G is semisimple. So

$$R \simeq \pi_1^{n_1} \oplus \cdots \oplus \pi_q^{n_q}.$$

But if ρ and τ are any finite dimensional k-linear representations of G, then

$$M(\rho \oplus \tau) = M(\rho) + M(\tau) \subset k[G].$$

Therefore, we conclude that

$$k[G] = M(R) \subset M(\pi_1) + \cdots + M(\pi_q) \subset k[G],$$

which shows the surjectivity of the map in the statement. □

Addendum 4. *Let G be a finite group, and let k be an algebraically closed field of characteristic zero. If $(V_1, \pi_1), \ldots, (V_q, \pi_q)$ are representatives of the isomorphism classes of irreducible k-linear representations, then*

$$|G| = n_1^2 + \cdots + n_q^2,$$

where $n_i = \dim_k(V_i)$.

Proof. By Theorem 3, we have

$$|G| = \dim_k(k[G]) = \dim_k(M(\pi_1)) + \cdots + \dim_k(M(\pi_q)),$$

and $\dim_k(M(\pi_i)) = n_i^2$. Indeed, since k is algebraically closed and π_i irreducible, the map $\mu_{\pi_i}\colon V_i \otimes V_i^* \to M(\pi_i)$ is an isomorphism, and $\dim_k(V_i^*) = \dim_k(V_i) = n_i$. □

Example 5. Let k be algebraically closed of characteristic zero.

1) A finite abelian group A has precisely $|A|$ pairwise non-isomorphic irreducible k-linear representations, all of which 1-dimensional.

2) Let $G = \Sigma_3$. We have found three pairwise non-isomorphic irreducible k-linear representations of G, namely,

(i) the 1-dimensional trivial representation τ,
(ii) the 1-dimensional sign representation sgn, and
(iii) the 2-dimensional representation π of G on

$$V = \{\boldsymbol{x} \in k^3 \mid x_1 + x_2 + x_3 = 0\} \subset k^3$$

defined by

$$\pi(\sigma)(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \begin{pmatrix} x_{\sigma(1)} \\ x_{\sigma(2)} \\ x_{\sigma(3)} \end{pmatrix}.$$

Since we have

$$|G| = 6 = 1^2 + 1^2 + 2^2 = \dim_k(\tau)^2 + \dim_k(\text{sgn})^2 + \dim_k(\pi)^2,$$

we conclude from Addendum 4 that, up to non-canonical isomorphism, we have found all irreducible representations of G.

We next prove a lemma concerning matrix coefficients.

Lemma 6. *Let G be a group, let k be a field, and let (V, π) be a finite dimensional k-linear representation of G. There is a commutative diagram*

$$\begin{array}{ccc} \operatorname{End}_k(V) & \xrightarrow{\;\mu'\;} & k[G] \\ & {\scriptstyle\alpha}\nwarrow \quad \nearrow{\scriptstyle\mu} & \\ & V \otimes V^* & \end{array}$$

with $\mu'(f)(h) = \operatorname{tr}(\pi(h)(f))$ and $\alpha(\boldsymbol{x} \otimes \varphi)(\boldsymbol{y}) = \boldsymbol{x} \cdot \varphi(\boldsymbol{y})$, and moreover, the map α is an isomorphism. Accordingly, the subspace of matrix coefficients

$$M(\pi) \subset k[G]$$

is equal to the common image of μ and μ'.

Proof. Let $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ be a basis of V, and let $(\boldsymbol{v}_1^*, \dots, \boldsymbol{v}_n^*)$ be the dual basis of the dual k-vector space V^*. For $g \in G$, let

$$A(g) = (a_{ij}(g)) \in M_n(k)$$

be the matrix that represents $\pi(g)$ with respect to the basis $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ of V. As we have calculated before, we have

$$\mu(\boldsymbol{v}_j \otimes \boldsymbol{v}_i^*)(g) = a_{ij}(g).$$

Now, we have $\alpha(\boldsymbol{v}_j \otimes \boldsymbol{v}_i^*) = \boldsymbol{v}_j \cdot \boldsymbol{v}_i^*$, and

$$(\pi(g) \circ \boldsymbol{v}_j \cdot \boldsymbol{v}_i^*)(\boldsymbol{v}_k) = \begin{cases} \pi(g)(\boldsymbol{v}_j) & \text{if } i = k, \\ \boldsymbol{0} & \text{if } i \neq k, \end{cases}$$

and therefore, we find that also

$$(\mu' \circ \alpha)(\boldsymbol{v}_j \otimes \boldsymbol{v}_i^*)(g) = \operatorname{tr}(\pi(g) \circ \boldsymbol{v}_j \cdot \boldsymbol{v}_i^*) = a_{ij}(g),$$

which shows that indeed $\mu = \mu' \circ \alpha$ as stated. Finally, the map α is an isomorphism, since it maps the basis $(\boldsymbol{v}_j \otimes \boldsymbol{v}_i^*)_{1 \le i,j \le n}$ of the k-vector space $V \otimes V^*$ to the basis $(\boldsymbol{v}_j \cdot \boldsymbol{v}_i^*)_{1 \le i,j \le n}$ of the k-vector space $\operatorname{End}_k(V)$. $\square$

Remark 7. In the situation of Lemma 6, the maps $\mu = \mu_\pi$ and $\mu' = \mu'_\pi$ depend on the k-linear representation (V, π), where as the map $\alpha = \alpha_V$ only depends on the k-vector space V.

Definition 8. Let k be a field, and let G be a group. If (V, π) is a finite dimensional k-linear representation of G, then its character

$$\chi_\pi \in k[G]$$

is the function defined by $\chi_\pi(g) = \mathrm{tr}(\pi(g))$.

We note that χ_π belongs to the subspace of matrix coefficients. More precisely,

$$\chi_\pi = \mu'_\pi(\mathrm{id}_V) \in M(\pi) \subset k[G].$$

The main result of this lecture is that for G finite and k an algebraically closed field of characteristic zero, the character χ_π determines π, up to non-canonical isomorphism. We first record some properties of the character.

Proposition 9. *Let k be a field and let G be a group. The character of finite dimensional k-linear representations of G has the following properties.*

(1) *If $\pi_1 \simeq \pi_2$, then $\chi_{\pi_1} = \chi_{\pi_2}$.*
(2) *The character of the dual of a representation is given by*

$$\chi_{\pi^*}(g) = \chi_\pi(g^{-1}).$$

(3) *The character of a sum of representations is given by*

$$\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}.$$

(4) *The character of a tensor product of representations is given by*

$$\chi_{\pi_1 \otimes \pi_2} = \chi_{\pi_1} \cdot \chi_{\pi_2}.$$

(5) *For all $g, h \in G$, $\chi_\pi(ghg^{-1}) = \chi_\pi(h)$.*

Proof. (1) That $\pi_1 \simeq \pi_2$ means that there exists a k-linear isomorphism $h \colon V_1 \to V_2$ such that $\pi_2(g) = h \circ \pi_1(g) \circ h^{-1}$ for all $g \in G$. But then

$$\chi_{\pi_2}(g) = \mathrm{tr}(\pi_2(g)) = \mathrm{tr}(h \circ \pi_1(g) \circ h^{-1}) = \mathrm{tr}(\pi_1(g)) = \chi_{\pi_1}(g).$$

(2) By definition, $\pi^*(g) = \pi(g^{-1})^*$, so

$$\chi_{\pi^*}(g) = \mathrm{tr}(\pi^*(g)) = \mathrm{tr}(\pi(g^{-1})^*) = \mathrm{tr}(\pi(g^{-1})) = \chi_\pi(g^{-1}).$$

(3) This follows immediately from the fact that $\mathrm{tr}(f_1 \oplus f_2) = \mathrm{tr}(f_1) + \mathrm{tr}(f_2)$.
(4) Since $\mathrm{tr}(f_1 \otimes f_2) = \mathrm{tr}(f_1) \cdot \mathrm{tr}(f_2)$, we have, more generally, that

$$\chi_{\pi_1 \boxtimes \pi_2}(g_1, g_2) = \chi_{\pi_1}(g_1) \cdot \chi_{\pi_2}(g_2),$$

so restricting along $\Delta \colon G \to G \times G$, the stated formula follows.
(5) This follows from the fact that $\mathrm{tr}(f_1 \circ f_2) = \mathrm{tr}(f_2 \circ f_1)$. □

Definition 10. Let k be a field, and let G be a finite group. A function $f \colon G \to k$ is central if $f(ghg^{-1}) = f(h)$ for all $g, h \in G$.

It is clear that the subset of $k[G]$ that consists of the central functions is a k-linear subspace. We denote this subspace by[1]

$$Z(k[G]) \subset k[G].$$

The explanation for this notation is as follows. The k-vector space $k[G]$ becomes a k-algebra under the convolution product $*$ defined by

$$(f_1 * f_2)(g) = \textstyle\sum_{h_1 h_2 = g} f_1(h_1) f_2(h_2),$$

and moreover, the map $\mu'_\pi \colon \mathrm{End}_k(V) \to k[G]$ is a k-algebra homomorphism with respect to the composition product on $\mathrm{End}_k(V)$ and the convolution product on $k[G]$ in the sense that μ'_π is k-linear and

$$\mu'_\pi(f_1 \circ f_2) = \mu'_\pi(f_1) * \mu'_\pi(f_2)$$

for all $f_1, f_2 \in \mathrm{End}_k(V)$. Now, the subspace $Z(k[G]) \subset k[G]$ is the center of the k-algebra $(k[G], +, *)$ in the sense that $f \in Z(k[G])$ if and only if

$$f * h = h * f$$

for all $h \in k[G]$.

Lemma 11. *Let G be a finite group, and let k be an algebraically closed field of characteristic zero. If π is an irreducible k-linear representation of G, then*

$$Z(k[G]) \cap M(\pi) = \mathrm{span}_k(\chi_\pi).$$

Proof. We use that by Lemma 6, we have

$$M(\pi) = \{\mu'_\pi(f) \mid f \in \mathrm{End}_k(V)\} \subset k[G],$$

where $\mu'_\pi(f)(g) = \mathrm{tr}(\pi(g) \circ f)$. Since π is irreducible, the map μ_π, and hence, also the map μ'_π is injective, so the induced map

$$\mathrm{End}_k(V) \xrightarrow{\mu'_\pi} M(\pi)$$

is an isomorphism. Now, we calculate that

$$\begin{aligned}
\mu'_\pi(f)(ghg^{-1}) &= \mathrm{tr}(\pi(ghg^{-1}) \circ f) \\
&= \mathrm{tr}(\pi(g) \circ \pi(h) \circ \pi(g)^{-1} \circ f) \\
&= \mathrm{tr}(\pi(h) \circ \pi(g)^{-1} \circ f \circ \pi(g)) \\
&= \mu'_\pi(\pi(g)^{-1} \circ f \circ \pi(g))(h),
\end{aligned}$$

which shows that the function $\mu'_\pi(f)$ is central if and only if for all $g \in G$, we have

$$f = \pi(g)^{-1} \circ f \circ \pi(g).$$

So $\mu'_\pi(f)$ is central if and only if $f \colon V \to V$ is π-invariant. By Schur's lemma, $f \colon V \to V$ is π-invariant if and only if $f = c \cdot \mathrm{id}_V$ for some nonzero $c \in k$. But

$$\mu'_\pi(c \cdot \mathrm{id}_V) = c \cdot \mu'_\pi(\mathrm{id}_V) = c \cdot \chi_\pi,$$

which proves the lemma. □

[1] The book write $k[G]^\#$ for this subspace.

Theorem 12. *Let k be an algebraically closed field of characteristic zero, let G be a finite group, and let $\pi_1, \dots, \pi_q$ be representatives of the isomorphism classes of irreducible k-linear representations of G. In this situation, the family*

$$(\chi_{\pi_1}, \dots, \chi_{\pi_q})$$

of their characters is a basis of the k-vector space $Z(k[G])$.

Proof. This follows immediately form Theorem 3 and Lemma 11. □

The promised main result concerning characters is the following corollary.

Corollary 13. *In the situation of Theorem 12, the following hold:*

(1) *The dimension of the k-vector space $Z(k[G])$ is equal to the number of isomorphism classes of irreducible k-linear representations of G, which, in turn, is equal to the number of conjugacy classes of elements of G.*
(2) *The isomorphism class of any finite dimensional k-linear representation π of G (not necessarily irreducible) is determined by its character χ_π.*

Proof. (1) The fact that the dimension of the k-vector space $Z(k[G])$ is equal to the number of isomorphism classes of irreducible k-linear representations follows immediately from Theorem 12. But $Z(k[G])$ is defined to be the k-vector spaces of central functions $f\colon G \to k$, and a function $f\colon G \to k$ is central if and only if it factors through the canonical projection $p\colon G \to G\backslash G^{\mathrm{ad}}$ onto the set of orbits for action by G on itself by conjugation. Hence, the dimension of $Z(k[G])$ is also equal to the cardinality of $G\backslash G^{\mathrm{ad}}$.
(2) Since π is semisimple, we have $\pi \simeq \pi_1^{m_1} \oplus \dots \oplus \pi_q^{m_q}$, so by Proposition 9,

$$\chi_\pi = m_1\chi_{\pi_1} + \dots + m_q\chi_{\pi_q}.$$

But k has characteristic zero, so the unique ring homomorphism $\mathbb{Z} \to k$ is injective, and therefore, this identity in $k[G]$ determines the integers $m_1, \dots, m_q$.[2] □

Let us use this result to determine the isomorphism classes of irreducible complex representations of the symmetric group $G = \Sigma_4$, which has order $|G| = 24$. We recall that the cycle-type of a permutation of n letters is the partition of n obtained from counting the number of elements in cycles.

Lemma 14. *The map that to a permutation $\sigma \in \Sigma_n$ assigns its cycle type induces a bijection of the set of conjugacy classes of elements in Σ_n onto the set of cycle-types of permutations of n letters.*

Proof. Indeed, a conjugation of a permutation corresponds to a relabelling of the elements in $\{1, 2, \dots, n\}$. □

For $n = 4$, there are five cycle-types, namely

$$1+1+1+1,\ 2+1+1,\ 2+2,\ 3+1, \text{ and } 4,$$

and the permutations

$$e,\ (12),\ (12)(34),\ (123), \text{ and } (1234)$$

[2] If instead the characteristic of k were a prime number ℓ, then the identity in $k[G]$ would only determine the congruence classes of the integers $m_1, \dots, m_q$ modulo ℓ.

represent the corresponding conjugacy classes of elements in $G = \Sigma_4$. Hence,

$$\dim_{\mathbb{C}}(Z(\mathbb{C}[G])) = 5,$$

and there are five isomorphism classes of irreducible complex representations of G. We know three of these already, namely,

(i) the 1-dimensional trivial representation $\pi_1 = \tau$,
(ii) the 1-dimensional sign representation $\pi_2 = \mathrm{sgn}$, and
(iii) the 3-dimensional representation π_3 of G on

$$V_3 = \{\boldsymbol{x} \in \mathbb{C}^4 \mid z_1 + z_2 + z_3 + z_4 = 0\} \subset \mathbb{C}^4$$

defined by

$$\pi_3(\sigma)(\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}) = \begin{pmatrix} z_{\sigma(1)} \\ z_{\sigma(2)} \\ z_{\sigma(3)} \\ z_{\sigma(4)} \end{pmatrix}.$$

By Addendum 4, the dimensions n_4 and n_5 of the remaining two irreducible complex representations π_4 and π_5 satisfy

$$24 = 1^2 + 1^2 + 3^2 + n_4^2 + n_5^2,$$

which implies that $n_4 = 3$ and $n_5 = 2$. We claim that

$$\pi_4 \simeq \pi_2 \otimes \pi_3.$$

To prove this, we must show that $\pi_2 \otimes \pi_3$ is irreducible and not isomorphic to π_3. Now, the representation $\pi_2 \otimes \pi_3$ is irreducible, because

$$\mathrm{sgn} \otimes \pi_2 \otimes \pi_3 = \mathrm{sgn} \otimes \mathrm{sgn} \otimes \pi_3 \simeq \pi_3,$$

and because π_3 is irreducible, and to show that $\pi_2 \otimes \pi_3$ is not isomorphic to π_3, it suffices by Corollary 13 to show that

$$\chi_{\pi_2 \otimes \pi_3} = \chi_{\pi_2} \cdot \chi_{\pi_3} = \mathrm{sgn} \cdot \chi_{\pi_3} \neq \chi_{\pi_3}.$$

To prove this, it will suffice to find $\sigma \in G$ such that $\mathrm{sgn}(\sigma) = -1$ and $\chi_{\pi_3}(\sigma) \neq 0$. To this end, we consider $\pi = \pi_1 \oplus \pi_3$, which has

$$\chi_\pi = \chi_{\pi_1} + \chi_{\pi_3} = 1 + \chi_{\pi_3}.$$

But π is isomorphic to the standard permutation representation of G on $\mathbb{C}^4$, so the matrix that represents $\pi((12))$ with respect to the standard basis of $\mathbb{C}^4$ is

$$A((12)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so $\chi_\pi((12)) = 2$, which shows that $\chi_{\pi_3}((12)) = 1 \neq 0$ as desired. This proves the claim that $\pi_4 \simeq \pi_2 \otimes \pi_3$.

What about the remaining 2-dimensional irreducible complex representation π_5? In general, for every set X, there is a group homomorphism

$$\mathrm{Aut}(X) \xrightarrow{\iota_X} \mathrm{Aut}(\mathcal{P}(X))$$

from the group of permutations of the set X to the group of permutation of its power set $\mathcal{P}(X)$ defined by

$$\iota_X(\sigma)(U) = \{\sigma(x) \in X \mid x \in U\} \subset X.$$

Hence, given a (left) action $\rho\colon G \to \mathrm{Aut}(X)$ by a group G on a set X, we get the induced action $\iota_X \circ \rho\colon G \to \mathrm{Aut}(\mathcal{P}(X))$ of G on $\mathcal{P}(X)$. We let $X = \{1, 2, 3, 4\}$, and let $\rho\colon G \to \mathrm{Aut}(X)$ be the identity map and consider the action

$$G \xrightarrow{\iota_{\mathcal{P}(X)} \circ \iota_X \circ \rho} \mathrm{Aut}(\mathcal{P}(\mathcal{P}(X)))$$

on the iterated power set $\mathcal{P}(\mathcal{P}(X))$. It leaves the subset

$$Y = \{\{\{1, 2\}, \{3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}, \{\{1, 4\}, \{2, 3\}\}\} \subset \mathcal{P}(\mathcal{P}(X))$$

with three elements invariant, so we obtain a group homomorphism

$$G \xrightarrow{p} \mathrm{Aut}(Y) \simeq \Sigma_3.$$

Clearly, the kernel of p is the (necessarily normal) subgroup

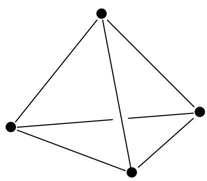
$$N = \{e, (12)(34), (13)(24), (14)(23)\} \subset G,$$

so comparing orders, we conclude that p is surjective. Hence, the 2-dimensional irreducible complex representation (V, π) of Σ_3 defines the 2-dimensional irreducible complex representation $(V, \pi \circ p)$ of G, and this is π_5.

Remark 15. Geometrically, we can picture the group homomorphism

$$G \xrightarrow{p} \mathrm{Aut}(Y)$$

above as follows. We may view G as the group of symmetries of the tetrahedron



It has 6 edges, and hence, 3 pairs of an edge and its opposing edge. Now the map p takes a permutation of the 4 vertices to the induced permutation of these 3 pairs of opposing edges.