SCHUR ORTHOGONALITY

Transitive group actions

Before we get to Schur orthogonality, we will finish some left-over business from last week. We recall that a left G-set is defined to be a pair (X, ρ) of a set X and a group homomorphism $\rho\colon G \to \mathrm{Aut}(X)$, and we say that ρ is a left action by G on X. As is common, we will abbreviate and write $g \cdot x$ or simply gx for $\rho(g)(x)$, where $g \in G$ and $x \in X$. We define the isotropy subgroup (or stabilizer) at $x \in X$ for the left action by G on X to be the subgroup

$$G_x = \{g \in G \mid gx = x\} \subset G,$$

and we define the orbit through $x \in X$ of the left action by G on X to be the subset

$$G \cdot x = \{gx \in X \mid g \in G\} \subset X.$$

Moreover, there is a well-defined bijection

$$G/G_x \xrightarrow{p_x} G \cdot x$$

from the set of left cosets of the isotropy subgroup $G_x \subset G$, which is typically not normal, and onto the orbit $G \cdot x \subset X$ defined by $p_x(hG_x) = hx$. The map p_x is equivariant with respect to the action of G on G/G_x by left multiplication and by the action of G on $G \cdot x \subset X$ obtained by restriction of the action by G on X. Indeed, given $hG_x \in G/G_x$ and $g \in G$, we find that

$$p_x(g \cdot hG_x) = p_x(ghG_x) = gh \cdot x = g \cdot hx = g \cdot p_x(hG_x)$$

as required. The orbits of the action by G on X are the equivalence classes of the equivalence relation $R \subset X \times X$ defined by the image of the map

$$G \times X \xrightarrow{(\mu, p)} X \times X$$

where $\mu\colon G \times X \to X$ is given by $\mu(g, x) = \rho(g)(x)$, and where $p\colon G \times X \to X$ is the canonical projection. We write

$$G\backslash X = \{G \cdot x \in \mathcal{P}(X) \mid x \in X\}$$

for the set of orbits. If there is only one orbit, in which case $G\backslash X = \{X\}$, then we say that the action by G on X is transitive. Equivalently, the action by G on X is transitive if for all $x, y \in X$, there exists $g \in G$ such that $y = gx$.

If two elements $x, y \in X$ belong to the same orbit for the left action by G on X, then their isotropy subgroups $G_x, G_y \subset G$ are conjugate, albeit not canonically so. Indeed, if we choose $g \in G$ such that $y = gx$, then the map

$$G_x \xrightarrow{c_g} G_y$$

defined by $c_g(h) = ghg^{-1}$ is a group isomorphism. We remark that this isomorphism depends on the choice of $g \in G$ with $y = gx$.

If $H \subset G$ is a subgroup, then we define the subset of H-fixed points for the left action by G on X to be the subset

$$X^H = \{x \in X \mid hx = x \text{ for all } h \in H\} \subset X.$$

It is generally not a G-invariant subset, so the action by G on X does generally not restrict to an action by G on X^H. However, we claim that the action by G on X restricts to an action by the normalizer subgroup

$$N_G(H) = \{g \in G \mid gHg^{-1} \subset H\} \subset G$$

on X^H. Indeed, if $g \in N_G(H)$, then for all $h \in H$, there exists some $h' \in H$ such that $hg = gh'$, and therefore, if $x \in X^H$, then $hgx = gh'x = gx$, which shows that also $gx \in X^H$. Let $\rho_H \colon N_G(H) \to \operatorname{Aut}(X^H)$ denote this action. By definition, this group homomorphism maps every element of $H \subset N_G(H)$ to the identity map of X^H, so it factors (uniquely) as the composition

$$\begin{array}{ccc} N_G(H) & \xrightarrow{\rho_H} & \operatorname{Aut}(X^H) \\ & {\scriptstyle p_H}\searrow \quad \nearrow{\scriptstyle \bar{\rho}_H} & \\ & W_G(H) & \end{array}$$

of the canonical projection p_H of $N_G(H)$ onto the quotient

$$W_G(H) = N_G(H)/H$$

and a left action $\bar{\rho}_H$ of $W_G(H)$ on X^H. The group $W_G(H)$ is called the Weyl group of H in G.

Example 1. 1) The group $G = O(3)$ of orthogonal 3×3-matrices acts on

$$S^2 = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \|\boldsymbol{x}\| = 1\} \subset \mathbb{R}^3$$

by left multiplication. The action is transitive, and the "North Pole"

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in S^2$$

has isotropy subgroup

$$G_{\boldsymbol{x}} = \left\{ \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \in O(3) \mid Q \in O(2) \right\} \subset G.$$

Hence, writing $G_{\boldsymbol{x}} = O(2)$, by abuse of notation, we have a canonical bijection

$$O(3)/O(2) \xrightarrow{p_{\boldsymbol{x}}} S^2.$$

This bijection is in fact a homeomorphism.

(2) Let k be a field. The action by $\mathrm{GL}_m(k)$ on $M_{m,n}(k)$ be left multiplication is not transitive, except in trivial cases. The theorem in linear algebra, which we call Gauss elimination, states that the map

$$\{A \in M_{m,n}(k) \mid A \text{ is on reduced echelon form}\} \longrightarrow \mathrm{GL}_m(k) \backslash M_{m,n}(k)$$

that to A assigns the orbit $\mathrm{GL}_m(k) \cdot A$ is a bijection. Indeed, we learn in linear algebra that two matrices $B, C \in M_{m,n}(k)$ belong to the same orbit for the action by $\mathrm{GL}_m(k)$ on $M_{m,n}(k)$ if and only if B can be transformed to C by means of row operations, and that every orbit for the action by $\mathrm{GL}_m(k)$ on $M_{m,n}(k)$ contains exactly one matrix on reduced echelon form.[1]

Let G be a group. If $H \subset G$ is a subgroup, then the action by G on G/H by left multiplication is transitive, and conversely, if a left action by G on a set X is transitive, then we have the G-equivariant bijection $p_x \colon G/G_x \to X$, once we choose an element $x \in X$. So every transitive left G-set is non-canonically isomorphic to G/H for some subgroup $H \subset G$. Let k be a field. By analogy with the two-sided regular representation of $G \times G$ on $k[G]$, we have the k-linear representation

$$W_G(H) \times G \xrightarrow{\bar{\rho}} \mathrm{GL}(k[G/H])$$

defined by

$$\bar{\rho}(g_1H, g_2)(\bar{f})(gH) = \bar{f}(g_2^{-1}gg_1H).$$

Moreover, from the general discussion above, we find that the two-sided regular representation of $G \times G$ on $k[G]$ restricts to a k-linear representation

$$W_G(H) \times G \xrightarrow{\rho} \mathrm{GL}(k[G]^{H \times \{e\}}),$$

where we use that the canonical projection

$$W_{G \times G}(H \times \{e\}) \longrightarrow W_G(H) \times G$$

is an isomorphism of groups. This representation is given by

$$\rho(g_1H, g_2)(f)(g) = f(g_2^{-1}gg_1).$$

In this situation, we have the following result.

Lemma 2. *Let $p \colon G \to G/H$ be the canonical projection. The map*

$$k[G/H] \xrightarrow{p^*} k[G]^{H \times \{e\}}$$

defined by $p^*(f) = f \circ p$ is a k-linear isomorphism that is intertwining with respect to $\bar{\rho}$ and ρ.

Proof. The right-hand side is the set consisting of the functions $f \colon G \to k$ such that $f(gh) = f(g)$ for all $g \in G$ and $h \in H$. But for every such function, there exists a unique function $\bar{f} \colon G/H \to k$ such that $f = \bar{f} \circ p$. So the map p^* is a bijection, and it is clear that it is k-linear and intertwining with respect to $\bar{\rho}$ and ρ. ☐

If $H \subset G$ is a subgroup of a group G, and if (V, π) is a k-linear representation of G, then we write (V^H, π^H) for the k-linear representation of $W_G(H)$ on V^H, where

$$W_G(H) \xrightarrow{\pi^H} \mathrm{GL}(V^H)$$

is given by $\pi^H(gH)(\boldsymbol{x}) = \pi(g)(\boldsymbol{x})$.

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[1] Challenge problem: Let $A \in M_{m,n}(k)$ be a matrix on reduced echelon form. Determine the isotropy subgroup $\mathrm{GL}_m(k)_A \subset \mathrm{GL}_m(k)$.

Theorem 3. *Let G be a finite group, let $H \subset G$ be a subgroup, and let k be an algebraically closed field of characteristic zero. Let $(V_1, \pi_1), \dots, (V_q, \pi_q)$ be representatives of the isomorphism classes of the irreducible k-linear representations of G. In this situation, the isomorphism*

$$\pi_1 \boxtimes \pi_1^* \oplus \cdots \oplus \pi_q \boxtimes \pi_q^* \xrightarrow{\mu} \mathrm{Reg}$$

of k-linear representations of $G \times G$ restricts to an isomorphism

$$\pi_1^H \boxtimes \pi_1^* \oplus \cdots \oplus \pi_q^H \boxtimes \pi_q^* \xrightarrow{\mu^{H\times\{e\}}} \mathrm{Reg}^{H\times\{e\}}$$

of k-linear representations of $W_G(H) \times G$.

Proof. In general, if Γ is a group and $K \subset \Gamma$ is a subgroup, then an isomorphism of k-linear representations of Γ induces an isomorphism of the k-linear representations of $W_\Gamma(K)$ obtained by taking K-fixed points. We apply this to $\Gamma = G \times G$ and $K = H \times \{e\}$. □

Let G be a finite group, and let (X, ρ) be a transitive left G-set. We will use Theorem 3 to determine the structure of the left regular representation

$$G \xrightarrow{L} k[X]$$

which, we recall, is given by $L(g)(f)(x) = f(\rho(g)^{-1}(x))$.

Corollary 4. *Let G be a finite group, let (X, ρ) be a transitive left G-set, and let $H = G_x \subset G$ be the isotropy subgroup of an element $x \in X$. Let k be an algebraically closed field of characteristic zero, let $(V_1, \pi_1), \dots, (V_q, \pi_q)$ be representatives of the isomorphism classes of irreducible k-linear representations of G, and let $m_i = \dim_k(V_i^H)$. In this situation, there exists a non-canonical isomorphism*

$$L \simeq \textstyle\bigoplus_{i=1}^q \pi_i^{m_i}$$

with $L \colon G \to \mathrm{GL}(k[X])$ the left regular representation of G on $k[X]$.

Proof. The map $p_x \colon G/H \to X$ defined by $p_x(gH) = gx$ is an isomorphism of left G-sets. Moreover, Lemma 2 and Theorem 3 give k-linear isomorphisms

$$k[G/H] \xrightarrow{p^*} k[G]^{H\times\{e\}} \xleftarrow{\mu} \textstyle\bigoplus_{i=1}^q V_i^H \times V_i^*,$$

which are intertwining with respect to the respective representations of the group $W_G(H) \times G$ on these k-vector spaces. In particular, they are also intertwining with respect to the subgroup $G = \{H\} \times G \subset W_G(H) \times G$. Therefore, we conclude that the representation π_i^* appears with multiplicity $\dim_k(\pi_i^H)$ in L. But the dual representations $\pi_1^*, \dots, \pi_q^*$ also form a set of representatives of the isomorphism classes of irreducible k-linear representations of G, so we may equivalently conclude that the representation π_i appears with multiplicity $\dim_k((\pi_i^*)^H)$ in L. Thus, it remains to prove that $\dim_k((\pi_i^*)^H)$ and $\dim_k(\pi_i^H)$ are equal.

More generally, for every finite dimensional k-linear representation (V, π) of G, we will prove that $\dim_k((V^*)^H) = \dim_k(V^H)$. The composition

$$H \longrightarrow G \xrightarrow{\pi} \mathrm{GL}(V)$$

of the canonical inclusion and the representation π is a finite dimensional k-linear representation of H, and hence, it decomposes as a sum

$$\pi \simeq \rho_1^{m_1} \oplus \cdots \oplus \rho_r^{m_r}$$

of irreducible k-linear representations of H. It follows that

$$\pi^* \simeq (\rho_1^*)^{m_1} \oplus \cdots \oplus (\rho_r^*)^{m_r}.$$

Exactly one of $\rho_1, \dots, \rho_r$ is a trivial (1-dimensional) representation of H, and exactly one of $\rho_1^*, \dots, \rho_r^*$ is a trivial (1-dimensional) representation of H. Moreover, ρ_i is trivial if and only if ρ_i^* is trivial. Reordering, if necessary, we can assume that ρ_1 and ρ_1^* are trivial. But then

$$\pi^H \simeq \pi_1^{m_1} \simeq (\pi_1^*)_1^m \simeq (\pi^*)^H,$$

so their dimensions agree, as we wanted to show. □

Remark 5. In addition to the choice of an element $x \in X$, the isomorphism in Corollary 4 depends on a choice of basis of V_i^H for all $1 \leq i \leq q$, and therefore, it is non-canonical.

Schur orthogonality

We now let $k = \mathbb{C}$ be the complex numbers and continue to let G be a finite group. Given a finite dimensional complex representation of G, we have defined the associated subspace of matrix coefficients

$$M(\pi) \subset \mathbb{C}[G]$$

to be the common image of the maps μ_π and μ'_π in the diagram

$$\begin{array}{ccc} \operatorname{End}_{\mathbb{C}}(V) & \xrightarrow{\mu'_\pi} & \mathbb{C}[G] \\ {\scriptstyle \alpha_V}\nwarrow^{\sim} & & \nearrow{\scriptstyle \mu_\pi} \\ & V \otimes V^* & \end{array}$$

which are defined by $\mu_\pi(\boldsymbol{x} \otimes \varphi) = \varphi(\pi(g)(\boldsymbol{x}))$ and $\mu'_\pi(h)(g) = \operatorname{tr}(\pi(g) \circ h)$. The map α_V in the diagram is defined by $\alpha_V(\boldsymbol{x} \otimes \varphi)(\boldsymbol{y}) = \boldsymbol{x} \cdot \varphi(\boldsymbol{y})$ and is an isomorphism. We also saw that if $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ is a basis of V and $(\boldsymbol{v}_1^*, \dots, \boldsymbol{v}_n^*)$ is the dual basis of V^*, then the family $(\mu_\pi(\boldsymbol{v}_j \otimes \boldsymbol{v}_i^*))_{1 \leq i,j \leq n}$ always generates $M(\pi)$, and if π is irreducible, then the family is a basis of $M(\pi)$.

Let $\langle -, - \rangle$ be a hermitian inner product on V. I will use the convention that

$$\langle \boldsymbol{x} \cdot z, \boldsymbol{y} \cdot w \rangle = \bar{z} \cdot \langle \boldsymbol{x}, \boldsymbol{y} \rangle \cdot w,$$

which is the opposite of the convention used in the book. It determines and is determined by the $\mathbb{C}$-linear isomorphism

$$\overline{V} \xrightarrow{b} V^*$$

defined by $b(\boldsymbol{x})(\boldsymbol{y}) = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$. So we may also identify $M(\pi) \subset \mathbb{C}[G]$ with the image of the composite map

$$V \otimes \overline{V} \xrightarrow{V \otimes b} V \otimes V^* \xrightarrow{\mu_\pi} \mathbb{C}[G],$$

which is given by

$$(\mu \circ (V \otimes b))(\boldsymbol{x} \otimes \boldsymbol{y})(g) = b(\boldsymbol{y})(\pi(g)(\boldsymbol{x})) = \langle \boldsymbol{y}, \pi(g)(\boldsymbol{x}) \rangle.$$

So if $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ is a basis of V that is orthonormal with respect to $\langle -, - \rangle$, then the matrix $A(g) = (a_{ij}(g)) \in M_{n,n}(\mathbb{C})$ that represents $\pi(g)\colon V \to V$ with respect to this basis has entries given by

$$a_{ij}(g) = \langle \boldsymbol{v}_i, \pi(g)(\boldsymbol{v}_j) \rangle.$$

The hermitian inner product $\langle -, - \rangle$ on V gives rise to a hermitian inner product $\langle -, - \rangle_{\mathrm{Frob}}$ on $\mathrm{End}_{\mathbb{C}}(V)$ called the Frobenius inner product. To define it, we recall that given a complex linear map $h\colon V \to V$, its adjoint with respect to $\langle -, - \rangle$ is the unique complex linear map $h^*\colon V \to V$ such that

$$\langle h^*(\boldsymbol{x}), \boldsymbol{y} \rangle = \langle \boldsymbol{x}, h(\boldsymbol{y}) \rangle$$

for all $\boldsymbol{x}, \boldsymbol{y} \in V$. Equivalently, the adjoint with respect to $\langle -, - \rangle$ is the unique complex linear map $h^*\colon V \to V$ that makes the diagram

$$\begin{array}{ccc} \overline{V} & \xrightarrow{\ b\ } & V^* \\ \big\downarrow h^* & & \big\downarrow h^* \\ \overline{V} & \xrightarrow{\ b\ } & V^* \end{array}$$

commute. Here, the right-hand vertical map is given by $h^*(\varphi)(\boldsymbol{x}) = \varphi(h(\boldsymbol{x}))$. Now, for $h_1, h_2 \in \mathrm{End}_{\mathbb{C}}(V)$, the Frobenius inner product is defined by

$$\langle h_1, h_2 \rangle_{\mathrm{Frob}} = \mathrm{tr}(h_1^* \circ h_2).$$

It is a hermitian inner product, which, we stress, depends on the choice of the hermitian inner product $\langle -, - \rangle$ on V.

Definition 6. Let G be a finite group. The Schur inner product on $\mathbb{C}[G]$ is the hermitian inner product given by

$$\langle f_1, f_2 \rangle_{\mathrm{Sch}} = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)}\, f_2(g).$$

It is straightforward to verify that the Schur inner product is invariant with respect to the two-sided regular representation of $G \times G$ on $\mathbb{C}[G]$ in the sense that

$$\langle \mathrm{Reg}(g_1, g_2)(f_1), \mathrm{Reg}(g_1, g_2)(f_2) \rangle_{\mathrm{Sch}} = \langle f_1, f_2 \rangle_{\mathrm{Sch}}$$

for all $(g_1, g_2) \in G \times G$ and $f_1, f_2 \in \mathbb{C}[G]$.

Theorem 7 (Schur orthogonality)**.** *Let G be a finite group.*

(a) *If π_1 and π_2 are non-isomorphic irreducible complex representations of G, then their subspaces of matrix coefficients*

$$M(\pi_1), M(\pi_2) \subset \mathbb{C}[G]$$

are orthogonal with respect to the Schur inner product.

(b) *If (V, π) is an irreducible complex representation of G that is unitary with respect to an hermitian inner product $\langle -, - \rangle$ on V, then*

$$\langle \mu'_\pi(h_1), \mu'_\pi(h_2) \rangle_{\mathrm{Sch}} = \frac{1}{n} \cdot \langle h_1, h_2 \rangle_{\mathrm{Frob}}$$

for all $h_1, h_2 \in \mathrm{End}_{\mathbb{C}}(V)$, where $n = \dim_{\mathbb{C}}(V)$.

Proof. (a) We wish to prove that the composition

$$M(\pi_1) \xrightarrow{\;i\;} \mathbb{C}[G] \xrightarrow{\;p\;} M(\pi_2)$$

of the canonical inclusion and the orthogonal projection with respect to the Schur inner product $\langle -, - \rangle_{\mathrm{Sch}}$ is the zero map. But the map is intertwining with respect to $\mathrm{Reg}_{M(\pi_1)}$ and $\mathrm{Reg}_{M(\pi_2)}$, and we have proved before that, as complex representations of $G \times G$, $\mathrm{Reg}_{M(\pi_1)}$ and $\mathrm{Reg}_{M(\pi_2)}$ are irreducible and non-isomorphic. So Schur's lemma proves that the map is zero, as desired.

(b) The representation $\pi \colon G \to \mathrm{GL}(V)$ gives rise to a representation

$$G \times G \xrightarrow{\;\rho\;} \mathrm{GL}(\mathrm{End}_{\mathbb{C}}(V))$$

defined by $\rho(g_1, g_2)(h) = \pi(g_1) \circ h \circ \pi(g_2)^{-1}$, and we claim that the map

$$\mathrm{End}_{\mathbb{C}}(V) \xrightarrow{\;\mu'_\pi\;} \mathbb{C}[G]$$

is intertwining between ρ and Reg. Indeed, we have

$$\begin{aligned}
\mu'_\pi(\rho(g_1, g_2)(h))(g) &= \mathrm{tr}(\pi(g) \circ \pi(g_1) \circ h \circ \pi(g_2)^{-1}) \\
&= \mathrm{tr}(\pi(g_2)^{-1} \circ \pi(g) \circ \pi(g_1) \circ h) \\
&= \mathrm{tr}(\pi(g_2^{-1} g g_1) \circ h) \\
&= \mathrm{Reg}(g_1, g_2)(\mu'_\pi(h))(g).
\end{aligned}$$

Since π is irreducible, the map μ'_π is injective, and hence, defines an isomorphism

$$\mathrm{End}_{\mathbb{C}}(V) \xrightarrow{\;\mu'_\pi\;} M(\pi)$$

that is intertwining between ρ and $\mathrm{Reg}_{M(\pi)}$. Now, we have two hermitian inner products on $\mathrm{End}_{\mathbb{C}}(V)$, namely, the Frobenius inner product $\langle -, - \rangle_{\mathrm{Frob}}$ and, via the isomorphism μ'_π, the Schur inner product $\langle -, - \rangle'_{\mathrm{Sch}}$ defined by

$$\langle h_1, h_2 \rangle'_{\mathrm{Sch}} = \langle \mu'_\pi(h_1), \mu'_\pi(h_2) \rangle,$$

and both are ρ-invariant. But $\rho \simeq \pi \boxtimes \pi^*$ is irreducible, so Theorem 12 in Lecture 6 shows that there exists a positive real number c such that

$$\langle h_1, h_2 \rangle'_{\mathrm{Sch}} = c \cdot \langle h_1, h_2 \rangle_{\mathrm{Frob}}$$

for all $h_1, h_2 \in \mathrm{End}_{\mathbb{C}}(V)$. It remains to determine the constant c.

We choose a basis $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ of V that is orthonormal with respect to the given hermitian inner product. Since π is unitary with respect to $\langle -, - \rangle$, the matrix

$$A(g) = (a_{ij}(g)) \in M_n(\mathbb{C})$$

that represents $\pi(g) \colon V \to V$ with respect to the basis $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ is a unitary matrix. Therefore, we find that

$$\langle a_{ij}, a_{kl} \rangle_{\mathrm{Sch}} = \frac{1}{|G|} \sum_{g \in G} \overline{a_{ij}(g)}\, a_{kl}(g) = \frac{1}{|G|} \sum_{g \in G} a_{ji}(g^{-1})\, a_{kl}(g),$$

where the second identity holds, because the matrix $A(g)$ is unitary. This formula gives us the idea to consider the sum

$$\sum_{1\le i\le n} \langle \alpha_V(\boldsymbol{v}_j \otimes \boldsymbol{v}_i^*), \alpha_V(\boldsymbol{v}_l \otimes \boldsymbol{v}_i^*)\rangle'_{\mathrm{Sch}} = \sum_{1\le i\le n} \langle \mu_\pi(\boldsymbol{v}_j \otimes \boldsymbol{v}_i^*), \mu_\pi(\boldsymbol{v}_l \otimes \boldsymbol{v}_i^*)\rangle_{\mathrm{Sch}}$$

$$= \frac{1}{|G|}\sum_{1\le i\le n}\sum_{g\in G} a_{ji}(g^{-1})a_{il}(g) = \frac{1}{|G|}\sum_{g\in G}\sum_{1\le i\le n} a_{ji}(g^{-1})a_{il}(g) = \delta_{jl},$$

where the last identity holds, because $A(g^{-1}) = A(g)^{-1}$. By comparison,

$$\sum_{1\le i\le n} \langle \alpha_V(\boldsymbol{v}_j \otimes \boldsymbol{v}_i^*), \alpha_V(\boldsymbol{v}_l \otimes \boldsymbol{v}_i^*)\rangle_{\mathrm{Frob}} = \sum_{1\le i\le n} \mathrm{tr}(\alpha_V(\boldsymbol{v}_j \otimes \boldsymbol{v}_i^*)^* \circ \alpha_V(\boldsymbol{v}_l \otimes \boldsymbol{v}_i^*))$$

$$= \sum_{1\le i\le n} \mathrm{tr}(\alpha_V(\boldsymbol{v}_i \otimes \boldsymbol{v}_j^*) \circ \alpha_V(\boldsymbol{v}_l \otimes \boldsymbol{v}_i^*)) = \sum_{1\le i\le n} \delta_{jl} = n\cdot\delta_{jl},$$

so we find that $c = \frac{1}{n}$, as stated. Let us explain the second and third identity in this calculation. The matrix B that represents $\alpha_V(\boldsymbol{v}_j \otimes \boldsymbol{v}_i^*)\colon V \to V$ with respect to the basis $(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n)$ has only one nonzero entry, namely, $b_{ij} = 1$. Since the basis $(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n)$ is orthonormal with respect to $\langle -,-\rangle$, the matrix that represents the adjoint map $\alpha_V(\boldsymbol{v}_j \otimes \boldsymbol{v}_i^*)^*\colon V \to V$ with respect to this basis is the adjoint matrix $C = B^*$, whose only nonzero entry is $c_{ji} = 1$. Similarly, the only nonzero entry in the matrix D that represents $\alpha_V(\boldsymbol{v}_l \otimes \boldsymbol{v}_i^*)$ with respect to $(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n)$ is $d_{il} = 1$. Finally, the matrix that represents $\alpha_V(\boldsymbol{v}_j \otimes \boldsymbol{v}_i^*)^* \circ \alpha_V(\boldsymbol{v}_l \otimes \boldsymbol{v}_i^*)\colon V \to V$ with respect to the basis $(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n)$ is $F = C\cdot D$, whose only nonzero entry is $f_{jl} = 1$, so its trace is indeed δ_{jl} as stated. ☐

Corollary 8. *Let G be a finite group, and let $Z(\mathbb{C}[G]) \subset \mathbb{C}[G]$ be the sub-$\mathbb{C}$-vector space consisting of the central functions. If $\pi_1,\dots,\pi_q$ are representatives of the irreducible complex representations of G, then the basis of $Z(\mathbb{C}[G])$ given by the family of their characters $(\chi_{\pi_1},\dots,\chi_{\pi_q})$ is orthonormal with respect to the Schur inner product.*

Proof. Theorem 7 (a) is precisely the statement that $(\chi_{\pi_1},\dots,\chi_{\pi_q})$ is orthogonal with respect to the Schur inner product. Moreover, we have,

$$\langle \chi_{\pi_i}, \chi_{\pi_i}\rangle_{\mathrm{Sch}} = \langle \mu'_{\pi_i}(\mathrm{id}_{V_i}), \mu'_{\pi_i}(\mathrm{id}_{V_i})\rangle'_{\mathrm{Sch}} = \frac{1}{n_i}\langle \mathrm{id}_{V_i}, \mathrm{id}_{V_i}\rangle_{\mathrm{Frob}},$$

where the second identity is Theorem 7 (b), and by definition

$$\langle \mathrm{id}_{V_i}, \mathrm{id}_{V_i}\rangle_{\mathrm{Frob}} = \mathrm{tr}(\mathrm{id}_{V_i}^* \circ \mathrm{id}_{V_i}) = \mathrm{tr}(\mathrm{id}_{V_i} \circ \mathrm{id}_{V_i}) = \mathrm{tr}(\mathrm{id}_{V_i}) = n_i,$$

which shows that $\langle \chi_{\pi_i}, \chi_{\pi_i}\rangle_{\mathrm{Sch}} = 1$, as desired. ☐

Corollary 9. *Let G be a finite group, and let $\pi_1,\dots,\pi_q$ be representatives of the irreducible complex representations of G. If (V,π) is any finite dimensional complex representation of G, then there is a non-canonical isomorphism*

$$\pi \simeq \pi_1^{m_1} \oplus \cdots \oplus \pi_q^{m_q},$$

where $m_i = \langle \chi_\pi, \chi_{\pi_i}\rangle_{\mathrm{Sch}}$.

Proof. By Corollary 13 from Lecture 7, it suffices to show that

$$\chi_\pi = m_1\chi_{\pi_1} + \cdots + m_q\chi_{\pi_q}$$

with m_i as stated. But this follows immediately from Corollary 8. ☐