SIX-FUNCTOR FORMALISM FOR $\mathrm{QCoh}([G\backslash X])$

Let (V, π) be a k-linear representation representation of a group G. If $f: G' \to G$ is a group homomorphism, then the composite group homomorphism

$$G' \xrightarrow{f} G \xrightarrow{\pi} \mathrm{GL}(V)$$

defines a k-linear representation of G' that we write $f^*(\pi)$ and call the *restriction* of π along f. We will show that, given a k-linear representation (V', π') of G', there are two ways to produce a k-linear representation of G. We write $f_!(\pi')$ and $f_*(\pi')$ for these two k-linear representation of G and call them *compact induction* of π' along f and *induction* of π' along f, respectively. However, to define and understand these, it is better to first generalize our notion of k-linear representation. So, in this lecture, I will assume some familiarity with categories, functors, natural transformations, and adjunctions. We have already encountered these in Lecture 5, when we discussed extension/restriction of scalars.

If G is a group, then we define a category BG, whose set of objects is the singleton set $1 = \{0\}$, and whose set of morphisms is

$$\mathrm{Map}(0, 0) = G.$$

We define the composition of morphisms in the category BG to be the product of these as elements of the group G, that is,

$$g \circ h = gh,$$

and we define the identity morphism of the unique object 0 in the category BG to be the identity element in the group G, that is,

$$\mathrm{id}_0 = e.$$

Let k be a field, and let Vect_k be the category, whose set of objects is the set of (small, right) k-vector spaces, and whose set of morphisms is the set of k-linear maps between such k-vector spaces. Composition of morphisms is defined to be composition of maps, and the identity morphism of V is defined to be the identity map id_V. Now, a k-linear representation (V, π) of G determines a functor

$$BG \xrightarrow{\pi} \mathrm{Vect}_k$$

that to the unique object 0 assigns the k-vector space $\pi(0) = V$ and that to the morphism $g: 0 \to 0$ assigns the k-linear map $\pi(g): V \to V$. Indeed, it is a functor, since for all morphisms $g, h: 0 \to 0$ in BG, we have

$$\pi(g \circ h) = \pi(gh) = \pi(g) \circ \pi(h),$$

and for the unique object 0 in BG, we have

$$\pi(\mathrm{id}_0) = \pi(e) = \mathrm{id}_V = \mathrm{id}_{\pi(0)}.$$

Conversely, a functor $\pi: BG \to \mathrm{Vect}_k$ determines a k-linear representation

$$G \xrightarrow{\pi} \mathrm{GL}(V),$$

where $V = \pi(0)$, and where $\pi(g)\colon V \to V$ is the k-linear map $\pi(g)\colon \pi(0) \to \pi(0)$. This map is invertible. Indeed, every morphism $g\colon 0 \to 0$ in BG is an isomorphism, and every functor takes isomorphisms to isomorphisms, but let us give the proof. Let $g\colon 0 \to 0$ be a morphism in BG. That $h\colon 0 \to 0$ is an inverse of g means that $g \circ h = \mathrm{id}_0$ and $h \circ g = \mathrm{id}_0$. Since $\pi\colon BG \to \mathrm{Vect}_k$ is a functor, we have

$$\pi(g) \circ \pi(h) = \pi(g \circ h) = \pi(\mathrm{id}_0) = \mathrm{id}_{\pi(0)},$$
$$\pi(h) \circ \pi(g) = \pi(h \circ g) = \pi(\mathrm{id}_0) = \mathrm{id}_{\pi(0)},$$

which shows that $\pi(g) \in \mathrm{GL}(V)$, as claimed.

We generalize this as follows. Let G be a group and recall that a left G-set is defined to be a pair (X, ρ) of a set X and a group homomorphism

$$G \xrightarrow{\ \rho\ } \mathrm{Aut}(X).$$

As we explained in the Lecture 8, we also write $g \cdot x$ or gx instead of $\rho(g)(x)$ and we say that G acts from the left on the set X. Given a left G-set (X, ρ), we define a category called the translation groupoid of (X, ρ) and denoted

$$[G \backslash X]$$

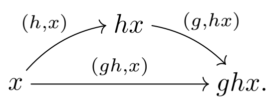
as follows. The set of objects is the set $[G \backslash X]_0 = X$, and the set of morphisms is the set $[G \backslash X]_1 = G \times X$. The source and target maps

$$[G \backslash X]_1 \overset{s}{\underset{t}{\rightrightarrows}} [G \backslash X]_0$$

are given by $s(g, x) = x$ and $t(g, x) = gx$, respectively, and the identity map

$$[G \backslash X]_0 \xrightarrow{\ e\ } [G \backslash X]_1$$

is given by $e(x) = (e, x)$. So, in other words, we view the pair (g, x) as a morphism from x to gx, and we define the identity morphism of x to be the pair (e, x). The composition of $(g, hx)\colon hx \to ghx$ and $(h, x)\colon x \to hx$ is $(gh, x)\colon x \to ghx$:



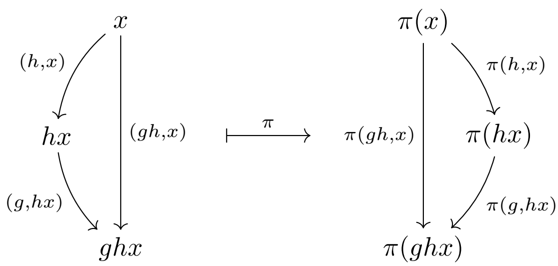
In the case of the trivial action of G on the set $1 = \{0\}$, we recover the category

$$BG = [G \backslash 1].$$

We now define a k-linear representation of $[G \backslash X]$ to be a functor

$$[G \backslash X] \xrightarrow{\ \pi\ } \mathrm{Vect}_k\,.$$

Such a functor assigns k-vector spaces and k-linear maps as indicated below.



The category $[G \backslash X]$ is a simple example of what is called a stack, and a functor $\pi \colon [G \backslash X] \to \mathrm{Vect}_k$ is also called a quasi-coherent sheaf on this stack. We write

$$\mathrm{QCoh}([G \backslash X]) = \mathrm{Fun}([G \backslash X], \mathrm{Vect}_k)$$

for the category, whose objects are the functors $\pi \colon [G \backslash X] \to \mathrm{Vect}_k$, and whose morphisms are natural transformations between such functors. So a morphism

$$\pi \xrightarrow{h} \pi'$$

is a family $(h_x)_{x \in X}$ of k-linear maps

$$\pi(x) \xrightarrow{h_x} \pi'(x)$$

such that for every $(g, x) \in G \times X$, the diagram

$$\begin{array}{ccc} \pi(x) & \xrightarrow{h_x} & \pi'(x) \\ \downarrow{\scriptstyle \pi(g,x)} & & \downarrow{\scriptstyle \pi'(g,x)} \\ \pi(gx) & \xrightarrow{h_{gx}} & \pi'(gx) \end{array}$$

commutes. In particular, the category

$$\mathrm{Rep}_k(G) = \mathrm{QCoh}(BG) = \mathrm{QCoh}([G \backslash 1])$$

is the category of k-linear representations and intertwining k-linear maps.

It happens rarely that categories are equal or even that they are isomorphic. Being equal or being isomorphic are not good notions for categories. (In fact, they are so-called "evil" notions, because they involve equality.) Instead, the notion of equivalence is a good notion. A functor

$$\mathcal{D} \xrightarrow{F} \mathcal{C}$$

is defined to be an equivalence, if there exists a functor

$$\mathcal{C} \xrightarrow{H} \mathcal{D}$$

in the opposite direction and natural transformations

$$F \circ H \xrightarrow{\epsilon} \mathrm{id}_{\mathcal{C}}$$

$$\mathrm{id}_{\mathcal{D}} \xrightarrow{\eta} H \circ F$$

such that for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$, the morphisms

$$(F \circ H)(c) \xrightarrow{\epsilon_c} \mathrm{id}_{\mathcal{C}}(c) = c$$

$$d = \mathrm{id}_{\mathcal{D}}(d) \xrightarrow{\eta_d} (H \circ F)(d)$$

in $\mathcal{C}$ and $\mathcal{D}$, respectively, are isomorphisms. In this situation, we say that ϵ and η are natural isomorphisms, and that H is a quasi-inverse of F. We note, however, that H is *not* uniquely determined by F.

Remark 1. If $F\colon \mathcal{D} \to \mathcal{C}$ is an equilvance (of categories), then it is always possible to choose $H\colon \mathcal{C} \to \mathcal{D}$ and $\epsilon\colon F \circ H \to \mathrm{id}_{\mathcal{C}}$ and $\eta\colon \mathrm{id}_{\mathcal{D}} \to H \circ F$ such that the following diagrams of natural transformations commute.

$$\begin{array}{ccccc} & & F \circ H \circ F & & \\ & \overset{F\circ\eta}{\nearrow} & & \overset{\epsilon\circ F}{\searrow} & \\ F & & \xrightarrow{\quad \mathrm{id}_F \quad} & & F \end{array} \qquad \begin{array}{ccccc} & & H \circ F \circ H & & \\ & \overset{\eta\circ H}{\nearrow} & & \overset{H\circ\epsilon}{\searrow} & \\ H & & \xrightarrow{\quad \mathrm{id}_H \quad} & & H \end{array}$$

In this situation, we say that ϵ and η satisfy the triangle identities and that the quadruple (F, G, ϵ, η) is an adjoint equivalence from $\mathcal{D}$ to $\mathcal{C}$.

Proposition 2. *Let G be a group, and let (X, ρ) be a transitive left G-set. Let $x \in X$, and let $G_x \subset G$ be the isotropy subgroup. The canonical inclusion functor*

$$BG_x = [G_x \backslash \{x\}] \xrightarrow{\;i\;} [G \backslash X]$$

is an equivalence.

Proof. To produce a quasi-inverse, we choose for all $y \in X$, an element $h_y \in G$ such that $y = h_y x$, and define

$$[G \backslash X] \xrightarrow{\;H\;} [G_x \backslash \{x\}],$$

to be the functor given on objects and morphisms by

$$\begin{array}{c} y \\ {\scriptstyle (g,y)}\downarrow \\ gy \end{array} \quad \xmapsto{\quad H \quad} \quad \begin{array}{c} x \\ \downarrow{\scriptstyle (h_{gy}^{-1} g h_y, x)} \\ x \end{array}$$

We further define $\epsilon\colon i \circ H \to \mathrm{id}_{[G \backslash X]}$ and $\eta\colon H \circ i \to \mathrm{id}_{[G \backslash \{x\}]}$ by

$$\begin{array}{ccc} (i \circ H)(y) & \xrightarrow{\;\epsilon_y\;} & y \\ \| & & \| \\ x & \xrightarrow{\;(h_y, x)\;} & y \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{\;\eta_x\;} & (H \circ i)(x) \\ \| & & \| \\ x & \xrightarrow{\;(h_x^{-1}, x)\;} & x. \end{array}$$

respectively. The family $\epsilon = (\epsilon_y)_{y \in X}$ is a natural transformation, since the diagram

$$\begin{array}{ccc} x & \xrightarrow{\;(h_y, x)\;} & y \\ \downarrow{\scriptstyle h_{gy}^{-1} g h_y} & & \downarrow{\scriptstyle (g,y)} \\ x & \xrightarrow{\;(h_{gy}, x)\;} & gy \end{array}$$

commutes for all $(g, y) \in G \times X$, and similarly, the family $\eta = (\eta_x)_{x \in \{x\}}$ is a natural transformation since the diagram

$$\begin{array}{ccc} x & \xrightarrow{\;(h_x^{-1}, x)\;} & x \\ \downarrow{\scriptstyle (g,x)} & & \downarrow{\scriptstyle (h_x^{-1} g h_x, x)} \\ x & \xrightarrow{\;(h_x^{-1}, x)\;} & x \end{array}$$

commutes for all $g \in G_x$. Both ϵ and η are automatically natural isomorphisms, since all morphisms in $[G \backslash X]$ and $[G_x \backslash \{x\}]$ are isomorphisms. $\square$

A category $\mathcal{G}$ is defined to be a groupoid if all morphisms in $\mathcal{G}$ are isomorphisms. The translation groupoid $[G \backslash X]$ of any left G-set (X, ρ) is indeed a groupoid.

Corollary 3. *In the situation of Proposition 2, the restriction along i,*

$$\mathrm{QCoh}([G \backslash X]) \xrightarrow{i^*} \mathrm{QCoh}([G_x \backslash \{x\}]) = \mathrm{Rep}_k(G_x),$$

which to π assigns $\pi \circ i$, is an equivalence.

Proof. If H is a quasi-inverse of i, then H^* is a quasi-inverse of i^*. □

Remark 4. Something better is true, namely, that, as opposed to the equivalence i, the equivalence i^* has a canonical quasi-inverse $i_! \simeq i_*$ given by the left or right Kan extension along i. Explicitly, the functors $i_!$ and i_* are given by

$$\begin{aligned} i_!(\pi)(y) &\simeq \varinjlim(\pi \mid BG_x \times_{[G \backslash X]} [G \backslash X]_{/y}) \\ &\simeq (\textstyle\bigoplus_{(h,x)\colon\, x \to y} \pi(x))/G_x \\ i_*(\pi)(y) &\simeq \varprojlim(\pi \mid BG_x \times_{[G \backslash X]} [G \backslash X]_{y/}) \\ &\simeq (\textstyle\prod_{(h,y)\colon\, y \to x} \pi(x))^{G_x}. \end{aligned}$$

It is the possibility of forming sums and products in Vect_k, which we cannot do in $[G \backslash X]$, that makes it possibly to define these functors.

Example 5. Let G be a group, and let $H \subset G$ be a subgroup. The pair (X, ρ) consisting of the set $X = G/H$ of left cosets of H in G and the group homomorphism $\rho\colon G \to \mathrm{Aut}(X)$ defined by $\rho(g)(g'H) = gg'H$, is a transitive left G-set. If we use Corollary 3 with $x = H = eH \in G/H$, then we find that

$$\mathrm{QCoh}([G \backslash (G/H)]) \xrightarrow{i^*} \mathrm{Rep}_k(H)$$

is an equivalence.

Let (X, ρ) be a left G-set, and let

$$X \xrightarrow{p} G \backslash X$$

be the canonical projection onto the set of orbits. (We remark that

$$G \backslash X \simeq \pi_0([G \backslash X])$$

is the set of isomorphism classes of objects in $[G \backslash X]$.) If we choose an element

$$x = s(\bar{x}) \in \bar{x} = G \cdot x \in G \backslash X$$

in each orbit, then we obtain an isomorphism of left G-sets

$$\textstyle\coprod_{\bar{x} \in G \backslash X} G/G_x \longrightarrow X$$

that to gG_x assigns $g \cdot x$. We note that this isomorphism is highly non-canonical, since it depends on the choice a section $s\colon G \backslash X \to X$ of $p\colon X \to G \backslash X$. Be that as it may, given this choice, we obtain equivalences

$$\textstyle\coprod_{\bar{x} \in G \backslash X} BG_x \longrightarrow \coprod_{\bar{x} \in G \backslash X} [G \backslash (G/G_x)] \longrightarrow [G \backslash X].$$

Finally, taking functors into Vect_k, we obtain the following result.

Proposition 6. *Let G be a group, and let (X,ρ) be a left G-set. A choice of representative $x \in \bar{x} \in G \backslash X$ of each orbit determines an equilvalence*

$$\mathrm{QCoh}([G\backslash X]) \longrightarrow \textstyle\prod_{\bar{x}\in G\backslash X} \mathrm{Rep}_k(G_x).$$

In Proposition 6, the big advantage of the left-hand side is that it only depends on the left G-set (X,ρ), whereas the right-hand side also depends on a choice[1] of section $s\colon G\backslash X \to X$ of the canonical projection $p\colon X \to G\backslash X$. We are now ready to define the compact induction and induction functors.

So let G be a group, and let $f\colon Y \to X$ be a G-equivariant map between left G-sets X and Y. We do not assume that G, X, or Y is finite. It induces a functor

$$[G\backslash Y] \xrightarrow{\ f\ } [G\backslash X],$$

which, by abuse of notation, we again denote by f, and that maps

$$\begin{array}{ccc} y & & f(y) \\ {\scriptstyle (g,y)}\big\downarrow & \overset{f}{\longmapsto} & \big\downarrow{\scriptstyle (g,f(y))} \\ gy & & f(gy) \end{array}$$

Since the map $f\colon Y \to X$ is G-equivariant, we have $f(gy) = gf(y)$, so this functor is well-defined. The functor f induces a functor

$$\mathrm{QCoh}([G\backslash X]) \xrightarrow{\ f^*\ } \mathrm{QCoh}([G\backslash Y])$$

that to π assigns $\pi \circ f$ and that we call the restriction along f. It admits both a left adjoint functor $f_!$ and a right adjoint functor f_* given by the left Kan extension along f and the right Kan extension along f, respectively. We call the functor $f_!$ compact induction along f, and we call the functor f_* induction along f. We now spell these out two functions out in detail. First, the functor

$$\mathrm{QCoh}([G\backslash Y]) \xrightarrow{\ f_!\ } \mathrm{QCoh}([G\backslash X])$$

is given by

$$\begin{array}{ccc} f_!(\tau)(x) & \xrightarrow{\ f_!(\tau)(g,x)\ } & f_!(\tau)(gx) \\ \Big\| & & \Big\| \\ \bigoplus_{f(y)=x} \tau(y) & \xrightarrow{\ \bigoplus \tau(g,y)\ } & \bigoplus_{f(y)=x} \tau(gy), \end{array}$$

where the two sums are indexed by

$$f^{-1}(x) = \{y \in Y \mid f(y) = x\},$$

and where we use that, since $f\colon Y \to X$ is G-equivariant, we have

$$\textstyle\bigoplus_{f(y')=gx} \tau(y') = \bigoplus_{f(y)=x} \tau(gy).$$

We define natural transformations $\epsilon = (\epsilon_\pi)$ and $\eta = (\eta_\tau)$ with[2]

$$f_!f^*(\pi) \xrightarrow{\ \epsilon_\pi\ } \pi \qquad\qquad \tau \xrightarrow{\ \eta_\tau\ } f^*f_!(\tau)$$

[1] In general, we need the axiom of choice to even know that it is possible to make this choice!
[2] We abbreviate and write $f_!f^*$ instead of $f_! \circ f^*$, etc.

as follows. The k-linear map

$$\begin{array}{ccc} f_!f^*(\pi)(x) & \xrightarrow{\epsilon_{\pi,x}} & \pi(x) \\ \| & & \| \\ \bigoplus_{f(y)=x} \pi(f(y)) & \xrightarrow{\nabla} & \pi(x) \end{array}$$

is the fold map (or co-diagonal), whose restriction to each summand is the identity map of $\pi(x) = \pi(f(y))$, and the k-linear map

$$\begin{array}{ccc} \tau(y) & \xrightarrow{\eta_{\tau,y}} & f^*f_!(\tau)(y) \\ \| & & \| \\ \tau(y) & \xrightarrow{i_y} & \bigoplus_{f(y')=f(y)} \tau(y') \end{array}$$

is the inclusion of the summand indexed by y. One verifies that ϵ and η are indeed well-defined natural transformations and that the triangle identities

$$\begin{array}{ccccc} & & f_!\,f^*f_! & & \\ & \nearrow^{f_!\eta} & & \searrow^{\epsilon f_!} & \\ f_! & & \xrightarrow{\mathrm{id}_{f_!}} & & f_! \end{array} \qquad \begin{array}{ccccc} & & f^*f_!\,f^* & & \\ & \nearrow^{\eta f^*} & & \searrow^{f^*\epsilon} & \\ f^* & & \xrightarrow{\mathrm{id}_{f^*}} & & f^* \end{array}$$

hold. As explained in Lecture 5, this immediately implies:

Theorem 7 (Frobenius reciprocity I)**.** *In the situation above, the maps*

$$\mathrm{Map}(f_!(\tau),\pi) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Map}(\tau, f^*(\pi))$$

defined by $\alpha(h) = f^*(h) \circ \eta_\tau$ and $\beta(k) = \epsilon_\pi \circ f_!(k)$ are each other's inverses.

Similarly, the functor

$$\mathrm{QCoh}([G\backslash Y]) \xrightarrow{f_*} \mathrm{QCoh}([G\backslash X])$$

is given by

$$\begin{array}{ccc} f_*(\tau)(x) & \xrightarrow{f_*(\tau)(g,x)} & f_*(\tau)(gx) \\ \| & & \| \\ \prod_{f(y)=x} \tau(y) & \xrightarrow{\prod \tau(g,y)} & \prod_{f(y)=x} \tau(gy), \end{array}$$

where the products are indexed by $f^{-1}(x)$ as before. The natural transformations $\epsilon = (\epsilon_\tau)$ and $\eta = (\eta_\pi)$ with

$$f^*f_*(\tau) \xrightarrow{\epsilon_\tau} \tau \qquad\qquad \pi \xrightarrow{\eta_\pi} f_*f^*(\pi)$$

as follows. The k-linear map

$$\begin{array}{ccc} f^*f_*(\tau)(y) & \xrightarrow{\epsilon_{\tau,y}} & \tau(y) \\ \| & & \| \\ \prod_{f(y')=f(y)} \tau(y') & \xrightarrow{p_y} & \tau(y) \end{array}$$

is the projection on the factor indexed by y, and the k-linear map

$$\begin{array}{ccc} \pi(x) & \xrightarrow{\eta_{\pi,x}} & f_*f^*(\pi)(x) \\ \| & & \| \\ \pi(x) & \xrightarrow{\Delta} & \prod_{f(y)=x} \pi(f(y)) \end{array}$$

is given by the diagonal map. One verifies that ϵ and η are well-defined natural transformations and that they satisfy the triangle identities:

$$\begin{array}{ccc} & f^*f_*f^* & \\ {\scriptstyle f^*\eta}\nearrow & & \searrow{\scriptstyle \epsilon f^*} \\ f^* & \xrightarrow{\mathrm{id}_{f^*}} & f^* \end{array} \qquad \begin{array}{ccc} & f_*f^*f_* & \\ {\scriptstyle \eta f_*}\nearrow & & \searrow{\scriptstyle f_*\epsilon} \\ f_* & \xrightarrow{\mathrm{id}_{f_*}} & f_* \end{array}$$

This gives the following result:

Theorem 8 (Frobenius reciprocity II)**.** *In the situation above, the maps*

$$\mathrm{Map}(f^*(\pi), \tau) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Map}(\pi, f_*(\tau))$$

defined by $\alpha(h) = f_*(h) \circ \eta_\pi$ *and* $\beta(k) = \epsilon_\tau \circ f^*(k)$ *are each other's inverses.*

There is a canonical natural transformation called the norm map

$$f_! \xrightarrow{\mathrm{Nm}_f} f_*.$$

In our description of $f_!$ and f_*, it is given by the canonical inclusion

$$\begin{array}{ccc} f_!(\tau)(x) & \xrightarrow{\mathrm{Nm}_{f,\tau,x}} & f_*(\tau)(x) \\ \| & & \| \\ \bigoplus_{f(y)=x} \tau(y) & \longrightarrow & \prod_{f(y)=x} \tau(y) \end{array}$$

of the sum in the product. (But a better definition of Nm_f is given in Lurie's Higher Algebra, Section 6.1.6.) We will say that a map $f\colon Y \to X$ is proper, if for all $x \in X$, the inverse image $f^{-1}(x) \subset Y$ is finite.

Theorem 9. *If* $f\colon Y \to X$ *is proper, then the norm map*

$$f_! \xrightarrow{\mathrm{Nm}_f} f_*.$$

is a natural isomorphism.

Proof. Indeed, finite sums and finite products of k-vector spaces agree. □

Finally, we will prove an important theorem called the base-change theorem. A commutative diagram of left G-sets and G-equivariant maps

(10)
$$\begin{array}{ccc} Y' & \xrightarrow{h'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{h} & X \end{array}$$

induces a diagram of categories and functors

$$\begin{array}{ccc}
\mathrm{QCoh}([G\backslash Y']) & \xleftarrow{h'^*} & \mathrm{QCoh}([G\backslash Y]) \\
\uparrow{\scriptstyle f'^*} & & \uparrow{\scriptstyle f^*} \\
\mathrm{QCoh}([G\backslash X']) & \xleftarrow{h^*} & \mathrm{QCoh}([G\backslash X])
\end{array}$$

that commutes, up to unique natural isomorphism. The diagram (10) is defined to be cartesian, if the map

$$Y' \xrightarrow{(f',h')} X' \times_X Y = \{(x',y) \in X' \times Y \mid h(x') = f(y)\} \subset X' \times Y$$

is a bijection. In this case, also the diagrams

$$\begin{array}{ccc}
\mathrm{QCoh}([G\backslash Y']) & \xleftarrow{h'^*} & \mathrm{QCoh}([G\backslash Y]) \\
\downarrow{\scriptstyle f'_* \text{ (resp. } f'_!)} & & \downarrow{\scriptstyle f_* \text{ (resp. } f_!)} \\
\mathrm{QCoh}([G\backslash X']) & \xleftarrow{h^*} & \mathrm{QCoh}([G\backslash X])
\end{array}$$

commute, up to specified natural isomorphisms. Here is a precise statement:

Theorem 11 (Base-change)**.** *If a diagram of G-sets and G-equivariant maps as in* (10) *is cartesian, then the following hold.*

(1) *The composite natural transformation*

$$h^* f_* \xrightarrow{\eta h^* f_*} f'_* f'^* h^* f_* \simeq f'_* h'^* f^* f_* \xrightarrow{f'_* h'^* \epsilon} f'_* h'^*$$

is a natural isomorphism.

(2) *The composite natural transformation*

$$f'_! h'^* \xrightarrow{f'_! h'^* \eta} f'_! h'^* f^* f_! \simeq f'_! f'^* h^* f_! \xrightarrow{\epsilon h^* f_!} h^* f_!$$

is a natural isomorphimsm.

Proof. We first remark that (1) and (2) are in fact equivalent statements. Indeed, the natural transformation $h^* f_* \to f'_* h'^*$ in (1), determines and is determined by a natural transformation $h'_! f'^* \to h_! f^*$, which, up to interchanging the role of f and h, precisely is the natural transformation in (2). So it suffices to prove (1). To this end, let $\tau \in \mathrm{QCoh}([G\backslash Y])$, and let $x' \in X'$. On the one hand, we have

$$h^* f_*(\tau)(x') = f_*(\tau)(h(x')) = \textstyle\prod_{f(y)=h(x')} \tau(y),$$

and, on the other hand, we have

$$f'_* h'^*(\tau)(x') = \textstyle\prod_{f'(y')=x'} h'^*(\tau)(y') = \prod_{f'(y')=x'} \tau(h'(y')),$$

and since the diagram (10) is cartesian, the two products agree. Finally, one checks that the composite map in the statement takes the factor indexed by (y,x') with $f(y) = h'(x')$ to the factor indexed by the unique $y' \in Y'$ such that $f'(y') = x'$ and $h'(y') = y$ by the identity map

$$\tau(y) \xrightarrow{\mathrm{id}} \tau(h'(y')).$$

So it is an isomorphism, which proves (1). ∎

In the next lecture, we consider the special case, where G is a finite group, where $H, K \subset G$ are two subgroups, and where (10) is the cartesian diagram

$$\begin{array}{ccc} G/H \times G/K & \xrightarrow{h'} & G/H \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ G/K & \xrightarrow{h} & G/G. \end{array}$$

Here, f and h are the unique maps (note that $G/G = \{G\}$ only has one element), and h' and f' are the canonical projections. The left G-sets G/H and G/K are both transitive, but $G/H \times G/K$ is not, unless either $H = G$ or $K = G$ or both. Proposition 6 gives a product decomposition of $\mathrm{QCoh}([G \backslash (G/H \times G/K)])$, once we fix a choice of representatives of the G-orbits in $G/H \times G/K$. As we will see, this turns out to be rather complicated!

Remark 12. The formulas for the left Kan extension $f_!$ and right Kan extension f_* that we have given above are based on the fact that the diagram of anima[3]

$$\begin{array}{ccc} Y & \longrightarrow & [G \backslash Y] \\ \downarrow & & \downarrow \\ X & \longrightarrow & [G \backslash X] \end{array}$$

is cartesian.

[3] In Lurie's Higher Topos Theory, anima are called "spaces." However, since these are nothing like topological spaces and are in fact discrete in nature, Clausen and Scholze have proposed to use the name anima or animated sets instead.