Algebra I/Introduction to Algebra V: Representation Theory

Due: Please upload solutions to NUCT by Tuesday, April 20, 2021.

Problem 1. (1) Let G be a group and let $\pi\colon G \to \mathbb{C}^\times$ be a one-dimensional complex representation of a group G. Show that if an element $g \in G$ has (finite) order n, then $\pi(g) \in \mathbb{C}^\times$ is an nth root of unity.

(2) Let G be a cyclic group of order n. Show that, up to isomorphism, G admits exactly n one-dimensional complex representations.

[Hint: First construct n one-dimensional complex representations $\pi_i\colon G \to \mathbb{C}^\times$, $0 \leq i < n$, such that $\pi_i \simeq \pi_j$ implies that $i = j$. Next show that if $\pi\colon G \to \mathbb{C}^\times$ is any one-dimensional complex representation, then $\pi \simeq \pi_i$ for some $0 \leq i < n$.]

Problem 2. Let $f\colon (V_1, \pi_1) \to (V_2, \pi_2)$ be an intertwining map between k-linear representations of a group G. Show that the following are equivalent:

(i) The intertwining map $f\colon (V_1, \pi_1) \to (V_2, \pi_2)$ is an isomorphism.[1]
(ii) The map $f\colon V_1 \to V_2$ is a bijection.

[Hint: To prove that (ii) implies (i), show that if $g = f^{-1}\colon V_2 \to V_1$, then the fact that f is k-linear and intertwining between π_1 and π_2 implies that g is k-linear and intertwining between π_2 and π_1.]

[1] By definition, this means that there exists an intertwining map $g\colon (V_2, \pi_2) \to (V_1, \pi_1)$ such that $g \circ f = \mathrm{id}_{V_1}$ and $f \circ g = \mathrm{id}_{V_2}$.