Algebra I/Introduction to Algebra V

Due: Please upload solutions to NUCT by Tuesday, July 13, 2021.

Problem 1. Let $k = \mathbb{R}$ (resp. $\mathbb{C}$, resp. $\mathbb{H}$), and let $\sigma\colon k \to k^{\mathrm{op}}$ be the identity map (resp. complex conjugation, resp. quaternionic conjugation). We first consider $M_n(k)$ a right k-vector space with sum given by matrix sum and with right scalar multiplication given by $(x_{ij}) \cdot a = (x_{ij} \cdot a)$. The zero vector in this vector space is given by the zero matrix O.

(i) Show that the map $\langle -,- \rangle\colon M_n(k) \times M_n(k) \to k$ defined by

$$\langle X, Y \rangle = \mathrm{tr}(X^*Y)$$

is a hermitian inner product on $M_n(k)$.

We next consider $M_n(k)$ a real vector space by restriction of scalars along $\mathbb{R} \to k$, and we consider the norm $\|X\| = \sqrt{\langle X, X \rangle}$ on this real vector space induced by the hermitian inner product in (i). We give each of the classical groups $G \subset M_n(k)$[1] the subspace topology induced by the metric topology on $M_n(k)$.

(ii) Show that each of the classical groups $G \subset M_n(k)$ is contained in the sphere of radius $\sqrt{n}$ centered at O.
(iii) Show that each of the classical groups $G \subset M_n(k)$ is a closed subset of $M_n(k)$.
(iv) Conclude that each of the classical groups G is compact.

[1] For $G = O(n)$ or $SO(n)$, $k = \mathbb{R}$; for $G = U(n)$ or $SU(n)$, $k = \mathbb{C}$, and for $G = \mathrm{Sp}(n)$, $k = \mathbb{H}$.