Algebra I/Introduction to Algebra V: Representation Theory

Due: Please upload solutions to NUCT by Tuesday, April 27, 2021.

Problem 1. Let $\pi\colon G \to GL(V)$ be a representation, let $U \subset V$ be a π-invariant subspace, and suppose that $W_1 \subset V$ and $W_2 \subset V$ are two π-invariant complements of U. Show that $\pi_{W_1} : G \to GL(W_1)$ and $\pi_{W_2} : G \to GL(W_2)$ are isomorphic.

[Hint: Show that both are canonically isomorphic to a common representation.]

Problem 2. Let $G = (\mathbb{C}, +)$ be a group of complex numbers under addition, let $A \in M_n(\mathbb{C})$ be a complex $n \times n$-matrix, and consider the representation

$$G \xrightarrow{\ \pi\ } GL(V)$$

on $V = \mathbb{C}^n$ given by $\pi(s) = e^{sA}$. Suppose that the characteristic polynomial

$$\chi_A(t) = \det(A - tE)$$

has n distinct roots $\lambda_1, \dots, \lambda_n$. Find all π-invariant subspaces of V.

[Hint: There are 2^n of them.]