Algebra I/Introduction to Algebra V: Representation Theory

Due: Please upload solutions to NUCT by Tuesday, May 11, 2021.

Problem 1. Let $\pi\colon G \to \mathrm{GL}(V)$ be a unitary representation of the group G on a finite dimensional complex vector space V. Prove that for all g, the eigenvalues of the linear automorphism $\pi(g)\colon V \to V$ have absolute value 1.

Problem 2. Let $G = \{1, \zeta, \zeta^2\}$ be a cyclic group of order 3, and let $\pi\colon G \to \mathrm{GL}_2(\mathbb{R})$ be the representation of G on $\mathbb{R}^2$ defined by

$$\pi(\zeta) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Find an inner product on $\mathbb{R}^2$ that is invariant with respect to π in the sense that

$$\langle \pi(g)(\boldsymbol{x}), \pi(g)(\boldsymbol{y}) \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$$

for all $g \in G$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$.