

Algebra I/Introduction to Algebra V: Representation Theory

Due: Please upload solutions to NUCT by Tuesday, May 18, 2021.

Problem 1. If $(V, +, \cdot)$ is a (right) complex vector space, then we consider the new (right) complex vector space $(V, +, \star)$, where for $\mathbf{x} \in V$ and $z \in \mathbb{C}$,

$$\mathbf{x} \star z = \mathbf{x} \cdot \bar{z}.$$

Here $\bar{z} = a - ib$ is the complex conjugate of $z = a + ib$. We call $(V, +, \star)$ the conjugate vector space of $(V, +, \cdot)$. As usual, we abuse notation and write V instead of $(V, +, \cdot)$ and \bar{V} instead of $(V, +, \star)$.

(a) Prove that the groups $\mathrm{GL}(\bar{V})$ and $\mathrm{GL}(V)$ are equal.

Let (V, π) be a complex representation of a group G . The conjugate representation is the complex representation (\bar{V}, π) . Here we use that $\mathrm{GL}(\bar{V}) = \mathrm{GL}(V)$. It is common to abbreviate (V, π) by π and (\bar{V}, π) by $\bar{\pi}$. This is confusing, because the maps $\pi: G \rightarrow \mathrm{GL}(V)$ and $\bar{\pi}: G \rightarrow \mathrm{GL}(\bar{V})$ are equal!

Suppose that π is a unitary representation and let $\langle -, - \rangle: V \times V \rightarrow \mathbb{C}$ be a hermitian inner product¹ such that $\langle \pi(g)(\mathbf{x}), \pi(g)(\mathbf{y}) \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$, for all $g \in G$ and $\mathbf{x}, \mathbf{y} \in V$.

(b) Show that the map $b: \bar{V} \rightarrow V^*$ defined by $b(\mathbf{x})(\mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$ is intertwining between $\bar{\pi}$ and π^* . Here (V^*, π^*) is the dual representation of (V, π) .

We remark that, if V is finite dimensional, then $b: \bar{V} \rightarrow V^*$ is an isomorphism of complex vector spaces. So it follows from (b) that for every finite dimensional unitary representation π , we have $\bar{\pi} \simeq \pi^*$.

¹In particular, for $\mathbf{x}, \mathbf{y} \in V$ and $z, w \in \mathbb{C}$, we have $\langle \mathbf{x} \cdot z, \mathbf{y} \cdot w \rangle = \bar{z} \langle \mathbf{x}, \mathbf{y} \rangle w$.