Algebra I/Introduction to Algebra V

Due: Please upload solutions to NUCT by Tuesday, June 1, 2021.

Problem 1. Let G be an abelian group, and let (V,π) be a finite dimensional irreducible real representation of G. Show that either $\dim_{\mathbb{R}}(V)=1$ or $\dim_{\mathbb{R}}(V)=2$.

Problem 2. Let $k=\overline{k}$ be an algebraically closed field,[1] let $\pi\colon G\to \mathrm{GL}(V)$ be a finite dimensional irreducible k-linear representation, and let

$$A=\mathrm{span}(\pi(g)\mid g\in G)\subset \mathrm{End}_k(V).$$

Show that $A=\mathrm{End}_k(V)$.

[Hint: Consider the representation $\rho\colon G\times G\to \mathrm{GL}(\mathrm{End}_k(V))$ defined by

$$\rho(g_1,g_2)(f)(\boldsymbol{x})=\pi(g_1)(f(\pi(g_2^{-1})(\boldsymbol{x}))).$$

Show that $A\subset \mathrm{End}_k(V)$ is ρ-invariant and that $\rho\simeq\pi\boxtimes\pi^*$. Then use a theorem.]

You are welcome to hand in the next problem, but you do not have to do so.

Problem 3 (Optional)**.** We consider the additive group $G=(\mathbb{R},+)$ of real numbers and the representation $\pi\colon G\to\mathrm{GL}(V)$ on $V=\mathbb{R}^2$ defined by

$$\pi(t)(\begin{pmatrix}x_1\\x_2\end{pmatrix})=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}.$$

Since π is irreducible, the ring $D=\mathrm{End}(\pi)\subset\mathrm{End}_{\mathbb{R}}(V)$ is a real division algebra, so either $D\simeq\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Determine which one it is.

[1] You are welcome to assume that $k=\mathbb{C}$.