FIVE LECTURES ON CATEGORY THEORY

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1. Category theory

As it is currently formalized, mathematics builds on the notion of a set. This means that all mathematical objects are sets. We agree that sets are undefinable but that they satisfy a list of basic assumptions (the ZFC axioms[1]) and all of mathematics is built by combining these basic assumptions. At present, however, a revolution is underway which, in my opinion, eventually will replace sets as the building blocks of mathematics by the new notion of anima (or animated sets).

The purpose of this series of lectures is to being to explain what anima are and what are they good for. This can only be a partial explanation for two reasons. First, we do not yet understand the true nature of anima. In particular, we do not know a list of basic assumptions that they must safisfy or even a language in which to express these basic assumptions. (Lurie has constructed a theory of anima within set theory, and while this gives a workable and powerful theory, it does not answer these questions.) Second, the full-fledged theory of ∞-categories is too much to cover in these lectures. So we will settle for the theory of 1-categories, which is good enough to amply display the difference with set theory and illustrate the salient features of the new theory.

The main distinction with set theory is that equality is not a meaningful notion in the new setting. We cannot say that two objects x and y are equal. Instead, we must explicitly say how to compare x and y. This is a big difference! For to say that $x = y$ is a *property*, whereas to provide a comparison $f\colon y \to x$ is *structure*. We already know this phenomenon well from many parts of mathematics. To wit, in linear algebra, it is not meaningful to ask if two vector spaces V and W are equal. Instead we should produce a linear map $f\colon W \to V$ and show that it is an isomorphism. The map $f\colon W \to V$ tells us *how* to translate between V and W and not only that a translation is possible. Obviously, knowing how to translate is much more useful than knowing that a translation is possible.

Let us now make some precise definitions. Since we have nothing better available at the moment, we will define the notion of a category within set theory.

Definition 1.1. A category to is a sextuple

$$\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, s, t, e, \circ)$$

consisting of a set $\mathcal{C}_0$, whose elements are called the objects of $\mathcal{C}$, a set $\mathcal{C}_1$, whose elements are called the morphisms of $\mathcal{C}$, two maps $s, t\colon \mathcal{C}_1 \to \mathcal{C}_0$ that to a morphism assign its source and target, a map $e\colon \mathcal{C}_0 \to \mathcal{C}_1$ that to an object assigns its identity morphism, and a map $\circ\colon \mathcal{C}_2 \to \mathcal{C}_1$, from the subset

$$\mathcal{C}_2 = \{(f,g) \in \mathcal{C}_1 \times \mathcal{C}_1 \mid s(f) = t(g)\} \subset \mathcal{C}_1 \times \mathcal{C}_1$$

of composable morphisms to the set of morphisms that to a pair (f,g) of composable morphisms assigns their composition $f \circ g$, subject to the following axioms:

(C1) For all $(f,g) \in \mathcal{C}_2$, $s(f \circ g) = s(g)$ and $t(f \circ g) = t(f)$.
(C2) For all $f \in \mathcal{C}_1$, $f \circ e(s(f)) = f = e(t(f)) \circ f$.

[1] For example, the first axiom reads $\forall x \forall y [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y]$, which means that if two sets x and y have the same elements, then they are equal; and the second axiom reads $\exists x \forall y \neg (y \in x)$, which means that there exists a set x, which has no elements. By the first axiom, this set x is unique, and we denote it by $\emptyset$ and call it the empty set.

(C3) For all $(f,g,h) \in \mathcal{C}_3$, $(f \circ g) \circ h = f \circ (g \circ h)$, where $\mathcal{C}_3 \subset \mathcal{C}_1 \times \mathcal{C}_1 \times \mathcal{C}_1$ is the subset of (f,g,h) with $s(f) = t(g)$ and $s(g) = t(h)$.

The axioms (C1)–(C3) formalize the behavior of maps that we are used to. If $\mathcal{C}$ is a category and if $f \in \mathcal{C}_1$ is a morphism, then we will write $f \colon y \to x$ to indicate that $s(f) = y$ and $t(f) = x$. Also, if $x \in \mathcal{C}_0$ is an object, then we will write $\mathrm{id}_x \colon x \to x$ for its identity morphism $e(x) \in \mathcal{C}_1$. With this notation, we indicate that composable morphisms f and g have composition $h = f \circ g$ by saying that the diagram

$$\begin{tikzcd} & x & \\ z \arrow[ur, "h"] \arrow[rr, "g"] & & y \arrow[ul, "f"'] \end{tikzcd}$$

commutes. This expresses the content of (C1), which is that g and h have the same source, while f and h have the same target. Similarly, the axiom (C2) expresses that for every morphism f, the diagrams

$$\begin{tikzcd} & x & \\ y \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_y"] & & y \arrow[ul, "f"'] \end{tikzcd} \qquad \begin{tikzcd} & x & \\ y \arrow[ur, "f"] \arrow[rr, "f"] & & x \arrow[ul, "\mathrm{id}_x"'] \end{tikzcd}$$

commute, whereas the axiom (C3) expresses that if f, g, and h are three composable morphisms, then the tetrahedral diagram

$$\begin{tikzcd} & x & \\ w \arrow[ur, "c"] \arrow[rr, "b"] \arrow[dr, "h"'] & & y \arrow[ul, "f"'] \\ & z \arrow[uu, "a"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes.

Example 1.2. (1) If X is a set, then the sextuple $\mathcal{C} = (X, X, \mathrm{id}_X, \mathrm{id}_X, \mathrm{id}_X, \circ)$, where $\circ \colon \mathcal{C}_2 \to \mathcal{C}_1$ is the map that to $(x,x) \in \mathcal{C}_2$ assigns $x \in \mathcal{C}_1$, is a category. In this category, the only morphisms are the identity morphisms $\mathrm{id}_x \colon x \to x$. We say that such a category is a static category. It is common to abuse notation and denote the category $\mathcal{C}$ by X. If $X = \emptyset$, then we call $\mathcal{C}$ the empty category.

(2) If G is a group, then the sextuple $\mathcal{C} = (1, G, p, p, e, \mu)$, where $1 = \{0\}$ is the first non-empty ordinal, $p \colon G \to 1$ is the unique map, $e \colon 1 \to G$ is the map that to $0 \in 1$ assigns the identity element $e = e(0) \in G$, and $\mu \colon G \times G \to G$ is the map that to $(g,h) \in G \times G$ assigns their product $\mu(g,h) = gh \in G$, is a category. This category is denoted by BG.

These two examples are both rather extreme. In the first example, we have many objects (assuming that X has many elements), but we cannot compare objects, unless they are equal. And in the second example, we have a single object, but we now have many different ways of comparing this object to itself (assuming that G has many elements). We wish to consider some more familiar examples, such as the category of all sets and the category of all k-vector spaces. But here we run into the problem that set theory does not allow us to form a set of all sets or a set of all k-vector spaces. This problem has nothing to do with category theory. It is the word "all" that is the problem. So we introduce the word "small" to counter it.[2] We will assume that small sets satisfy the ZFC axioms.

Example 1.3. We define the category Set of small sets as follows. The set of objects is the set Set_0 of all small sets. Let $X, Y \in \mathrm{Set}_0$, and let $\mathcal{P}(Y \times X) \in \mathrm{Set}_0$ be the power set of their product. We recall that $(Y, X, \Gamma_f) \in \mathrm{Set}_0$ is defined to be a map from Y to X if $\Gamma_f \in \mathcal{P}(Y \times X)$ and if for all $y \in Y$, there exists a unique $x \in X$ such that $(y, x) \in \Gamma_f$. In this situation, we write $f\colon Y \to X$ to indicate that (Y, X, Γ_f) is a map from Y to X and we write $f(y) = x$ to indicate that $(y, x) \in \Gamma_f$. Now, the set of morphisms is the set

$$\mathrm{Set}_1 = \{(Y, X, \Gamma_f) \in \mathrm{Set}_0 \mid (Y, X, \Gamma_f) \text{ is a map from } Y \text{ to } X\},$$

and the maps $s, t\colon \mathrm{Set}_1 \to \mathrm{Set}_0$ are defined by $s(Y, X, \Gamma_f) = Y$ and $t(Y, X, \Gamma_f) = X$. The map $e\colon \mathrm{Set}_0 \to \mathrm{Set}_1$ is defined by $e(X) = (X, X, \Delta)$, where $\Delta \subset X \times X$ is the diagonal, and finally, the map $\circ\colon \mathrm{Set}_2 \to \mathrm{Set}_1$ is defined by

$$(Y, X, \Gamma_f) \circ (Z, Y, \Gamma_g) = (Z, X, \Gamma_h),$$

where $(z, x) \in \Gamma_h$ if and only if $(y, x) \in \Gamma_f$ and $(z, y) \in \Gamma_g$. So in more familiar notation, we have $h(z) = f(g(z))$.

As these examples illustrate, making definions within set theory is cumbersome and forces us to include a lot of *structure* that we are not really interested in after all. For example, we do usually not think of a map of sets $f\colon Y \to X$ in terms of a triple (Y, X, Γ_f) of two sets and a graph of a function. Instead, category theory allows us to make definitions in terms of *properties* that we actually are interested in. Let us now illustrate this with the definition of a product.

Definition 1.4. Let $\mathcal{C}$ be a category. A product of a pair (x_1, x_2) of objects in $\mathcal{C}$ is a triple $(y, p_1\colon y \to x_1, p_2\colon y \to x_2)$ of an object y and two morphisms $p_i\colon y \to x_i$ in $\mathcal{C}$ with the *property* that if $(z, f_1\colon z \to x_1, f_2\colon z \to x_2)$ is any such triple, then there exists a *unique* morphism $f\colon z \to y$ such that the diagram

$$\begin{tikzcd}
 & z \arrow[dl, bend right, "f_1"'] \arrow[d, "f"] \arrow[dr, bend left, "f_2"] & \\
x_1 & y \arrow[l, "p_1"'] \arrow[r, "p_2"] & x_2
\end{tikzcd}$$

commutes.

A product of a pair of objects is not unique. Instead, it is unique, up to unique isomorphism. But this is a good thing! It means that we have exactly managed to ignore the unnecessary structure that set theory forced us to consider. Let us now prove this uniqueness statement. First, we define a morphism $f\colon y \to x$ in a category $\mathcal{C}$ to be an isomorphism if there exists a morphism $g\colon x \to y$ in the opposite direction such that the composite morphisms $f \circ g\colon x \to x$ and $g \circ f\colon y \to y$ are equal to the the identity morphisms $\mathrm{id}_x\colon x \to x$ and $\mathrm{id}_y\colon y \to y$, respectively. In this case, we say that g is an inverse of f.[3]

[2] If α is an ordinal in our model of ZFC set theory, then we can form the set V_α consisting of the sets of rank $< \alpha$. The set V_κ is itself a model of ZFC set theory if and only if κ is a strongly inaccessible cardinal. Since $\kappa \notin V_\kappa$, we cannot prove within ZFC set theory that such a κ exists.

[3] The inverse of a morphism is unique.

Proposition 1.5. *Let $\mathcal{C}$ be a category, and let (x_1, x_2) be a pair of objects in $\mathcal{C}$. If both $(y, p_1 \colon y \to x_1, p_2 \colon y \to x_2)$ and $(y', p'_1 \colon y' \to x_1, p'_2 \colon y' \to x_2)$ are products of (x_1, x_2), then the unique morphisms $p' \colon y' \to y$ and $p \colon y \to y'$ are each other's inverses.*

Proof. We show that $p' \circ p = \mathrm{id}_y$; the proof that $p \circ p' = \mathrm{id}_{y'}$ is analogous. By the definition of a product, the diagram

$$\begin{tikzcd} x_1 \arrow[d, "\mathrm{id}_{x_1}"] & y \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "p"] & x_2 \arrow[d, "\mathrm{id}_{x_2}"] \\ x_1 \arrow[d, "\mathrm{id}_{x_1}"] & y' \arrow[l, "p'_1"'] \arrow[r, "p'_2"] \arrow[d, "p'"] & x_2 \arrow[d, "\mathrm{id}_{x_2}"] \\ x_1 & y \arrow[l, "p_1"'] \arrow[r, "p_2"] & x_2 \end{tikzcd}$$

commutes. It follows (by the associativity of composition) that the diagram

$$\begin{tikzcd} x_1 \arrow[d, "\mathrm{id}_{x_1}"] & y \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "p' \circ p"] & x_2 \arrow[d, "\mathrm{id}_{x_2}"] \\ x_1 & y \arrow[l, "p_1"'] \arrow[r, "p_2"] & x_2 \end{tikzcd}$$

commutes. But so does the diagram

$$\begin{tikzcd} x_1 \arrow[d, "\mathrm{id}_{x_1}"] & y \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\mathrm{id}_y"] & x_2 \arrow[d, "\mathrm{id}_{x_2}"] \\ x_1 & y \arrow[l, "p_1"'] \arrow[r, "p_2"] & x_2, \end{tikzcd}$$

and therefore, it follows from the uniqueness statement in the definition of a product that $p' \circ p = \mathrm{id}_y$, as we wanted to prove. ☐

When we say that a product of (x_1, x_2) is unique, up to unique isomorphism, we mean that Proposition 1.5 holds. It shows that, while two products of (x_1, x_2) may not be equal, we have a unique way to compare one to the other. This is all we care about. Indeed, if I make calculations in one and you make calculations in another, then we can compare our calculations. Because of this, we agree to write

$$(x_1 \times x_2, p_1 \colon x_1 \times x_2 \to x_1, p_2 \colon x_1 \times x_2 \to x_2)$$

for any product of (x_1, x_2) with the understanding that this product is defined, up to unique isomorphism, only. Moreover, given $(z, f_1 \colon z \to x_1, f_2 \colon z \to x_2)$, we write

$$z \xrightarrow{(f_1, f_2)} x_1 \times x_2$$

for the unique morphism f that makes the diagram in Definition 1.4 commute.

In a category $\mathcal{C}$, it may or may not be true that every pair of objects admits a product. It every pair of objects in $\mathcal{C}$ does admit a product, then we say that $\mathcal{C}$ admits binary products.

Proposition 1.6. *The category* Set *of small sets admits binary products.*

Proof. We give one possible definition of the product of a pair (X_1, X_2) of small sets within set theory. We may view the pair (X_1, X_2) as a map $X\colon \{1,2\} \to \mathrm{Set}_0$, and in this situation, the axioms of set theory allow us to form the union $\bigcup_{i\in\{1,2\}} X_i$. We now define the set-theoretic product of (X_1, X_2) to be the subset

$$X_1 \times X_2 \subset \mathrm{Map}(\{1,2\}, \textstyle\bigcup_{i\in\{1,2\}} X_i)$$

of those maps $x\colon \{1,2\} \to \bigcup_{i\in\{1,2\}} X_i$ such that $x(i) \in X_i$ for all $i \in \{1,2\}$. Given such a map x, we also write x_i instead of $x(i)$ and (x_1, x_2) instead of x, and for all $i \in \{1,2\}$, we define $p_i\colon X_1 \times X_2 \to X_i$ to be the map that to (x_1, x_2) assigns x_i. We now claim that the triple

$$(X_1 \times X_2, p_1\colon X_1 \times X_2 \to X_1, p_2\colon X_1 \times X_2 \to X_2)$$

is a product of (X_1, X_2) in Set. Indeed, given

$$(Z, f_1\colon Z \to X_1, f_2\colon Z \to X_2),$$

the map $f\colon Z \to X_1 \times X_2$ defined by $f(z) = (f_1(z), f_2(z))$ makes the diagram

$$\begin{tikzcd} & Z \arrow[dl, bend right, "f_1"'] \arrow[d, "f"] \arrow[dr, bend left, "f_2"] & \\ X_1 & X_1 \times X_2 \arrow[l, "p_1"'] \arrow[r, "p_2"] & X_2 \end{tikzcd}$$

commute and is unique with this property. ∎

Let us notice that in Definition 1.4 and Propositions 1.5 and 1.6, the only property of the index set $\{1,2\}$ that we used was that it was a small set. So let us upgrade the definition and the propositions to small products in general. As is common, given a map of sets $x\colon I \to X$, we will also say that x is an I-indexed family of elements in X and write $(x_i)_{i\in I}$ instead of $x\colon I \to X$, where $x_i = x(i)$.

Definition 1.7. Let $\mathcal{C}$ be a category. A product of a family $(x_i)_{i\in I}$ of objects in $\mathcal{C}$ is a pair $(y, (p_i\colon y \to x_i)_{i\in I})$ of an object y and a family $(p_i\colon y \to x_i)_{i\in I}$ of morphisms with the *property* that if $(z, (f_i\colon z \to x_i)_{i\in I})$ is any such pair, then there exists a *unique* morphism $f\colon z \to y$ such that $f_i = p_i \circ f\colon z \to x_i$ for all $i \in I$.

The proof of Proposition 1.5 generalizes to show that, if it exists, then a product of a family of objects in a category is unique, up to unique isomorphism. So if $(x_i)_{i\in I}$ is a family of object in a category $\mathcal{C}$, then we agree to write

$$(\textstyle\prod_{i\in I} x_i, (p_i\colon \prod_{j\in I} x_j \to x_i))$$

for any product of this family, again with the understanding that this product is only well-defined, up to unique isomorphism. Given $(z, (f_i\colon z \to x_i)_{i\in I})$, we write

$$z \xrightarrow{(f_i)_{i\in I}} \textstyle\prod_{i\in I} x_i$$

for the unique morphism f such that $f_i = p_i \circ f$ for all $i \in I$. We say that $\mathcal{C}$ admits all products, if every family of objects in $\mathcal{C}$ admits a product, and we say that $\mathcal{C}$ admits all small products, if every family of objects in $\mathcal{C}$ indexed by a small set does so.

Addendum 1.8. *The category* Set *admits all small products.*

Proof. Let $X\colon I \to \mathrm{Set}_0$ be a family of small sets indexed by a small set I. The axioms of set theory allow us to form the union $\bigcup_{i\in I} X_i$, and since I and each of the X_i are small, so is this union. Hence, as in the proof of Proposition 1.6, we define the set-theoretic product of $(X_i)_{i\in I}$ to be the subset

$$\prod_{i\in I} X_i \subset \mathrm{Map}(I, \bigcup_{i\in I} X_i)$$

of those maps $x\colon I \to \bigcup_{i\in I} X_i$ such that $x(i) \in X_i$ for all $i \in I$, and we define

$$\prod_{j\in I} X_j \xrightarrow{p_i} X_i$$

to be the map that to $(x_j)_{j\in I}$ assigns $x_i \in X_i$. Now, the pair

$$(\prod_{i\in I} X_i, (p_i\colon \prod_{j\in I} X_j \to X_i)_{i\in I})$$

is a product in Set of the family of objects $(X_i)_{i\in I}$. □

Remark 1.9. The category Set does not admit all products. More generally, if $\mathcal{C}$ is a category, and if $\mathcal{C}$ admits products by families of objects $(x_i)_{i\in I}$, where I has the same cardinality κ as the set $\mathcal{C}_1$ of morphisms, then the set of morphisms between any two objects in $\mathcal{C}$ cannot have more that one element. Indeed, suppose that (x, y) is a pair of objects in $\mathcal{C}$ such that the set $\mathrm{Map}(y, x)$ of morphisms $f\colon y \to x$ has at least two elements, and suppose that a product $(z, (p_g\colon z \to x)_{g\in\mathcal{C}_1})$ of the (constant) family of objects $(x)_{g\in\mathcal{C}_1}$ exists. Now, on the one hand, we have that $\mathrm{Map}(y, z) \subset \mathcal{C}_1$, so the cardinality of $\mathrm{Map}(y, z)$ is at most κ, and on the other hand, the definition of product shows that the maps $p_g\colon z \to x$ induce a bijection

$$\mathrm{Map}(y, z) \longrightarrow \prod_{g\in\mathcal{C}_1} \mathrm{Map}(y, x),$$

so the cardinality of $\mathrm{Map}(y, z)$ is at least 2^κ, because $\mathrm{Map}(y, x)$ has cardinality at least 2. So the assumption implies that $\kappa \geq 2^\kappa$, which, by Cantor's diagonal argument, is impossible.

Example 1.10. It is always good to consider extreme cases. In Definition 1.7, we can take $I = \emptyset$ to be the empty set and $(x_i)_{i\in I}$ to be the empty family corresponding to the unique map $x\colon \emptyset \to \mathcal{C}_0$. By definition, a product of the empty family is a pair $(y, p\colon \emptyset \to \mathcal{C}_1)$ of an object y and a map $p\colon \emptyset \to \mathcal{C}_1$, such that if also $(z, q\colon \emptyset \to \mathcal{C}_1)$ is such a pair, then there exists a unique map $f\colon z \to y$ such that $q = p \circ f$. The condition $q = p \circ f\colon \emptyset \to \mathcal{C}_1$ is automatically satisfied, since such a map is unique. So to give a product of the empty family is equivalent to giving an object y with the property that for every object z, there exists a unique map $f\colon z \to y$. We say that an object y with this property is a *terminal* or *final* object. It is common to write 1 or $*$ for a terminal object. The terminal objects in Set are the small sets that have precisely one element.

The notion of a product is a special example of the more general notion of a limit, which we will discuss in detail in the next lecture. However, we will now discuss the special case of a base-change.

Definition 1.11. Let $\mathcal{C}$ be a category, and let $(f\colon y \to x, g\colon x' \to x)$ be a pair of morphism in $\mathcal{C}$ with common target. A pullback of (f, g) is a pair of morphisms $(g'\colon y' \to y, f'\colon y' \to x')$ in $\mathcal{C}$ with common source such that the diagram

$$\begin{tikzcd} y' \arrow[r, "g'"] \arrow[d, "f'"] & y \arrow[d, "f"] \\ x' \arrow[r, "g"] & x \end{tikzcd}$$

commutes and such that if $(b\colon z \to y, a\colon z \to x')$ is any such pair, then there exists a unique morphism $h\colon z \to y'$ such that $a = f' \circ h$ and $b = g' \circ h$.

We also express the fact that (g', f') is a pullback of (f, g) by saying that f' is a base-change of f along g and that g' is a base-change of g along f or by saying that the diagram in Definition 1.11 is cartesian. A pullback is unique, up to unique isomorphism, and we also write $(p_1\colon y \times_x x' \to y, p_2\colon y \times_x x' \to x')$ for any pullback of (f, g). We say that a category admits pullbacks if every pair of morphisms with a common target therein admits a pullback.

Proposition 1.12. *The category* Set *admits pullbacks.*

Proof. Given a pair $(f\colon Y \to X, g\colon X' \to X)$ of maps with common target, let

$$Y' = Y \times_X X' = \{(y, x') \in Y \times X' \mid f(y) = g(x')\} \subset Y \times X',$$

and let $g'\colon Y' \to Y$ and $f'\colon Y' \to X'$ be the maps that to (y, x') assign y and x', respectively. Then (g', f') is a pullback of (f, g). $\square$

Example 1.13. Let $\mathcal{C}$ be a small category. Given a pair (y, x) of objects in $\mathcal{C}$, we define the set of morphisms in $\mathcal{C}$ from y to x to be the pullback

$$\begin{tikzcd} \mathrm{Map}(y, x) \arrow[r, "(y,x)'"] \arrow[d, "(s,t)'"] & \mathcal{C}_1 \arrow[d, "(s,t)"] \\ 1 \arrow[r, "(y,x)"] & \mathcal{C}_0 \times \mathcal{C}_0 \end{tikzcd}$$

in Set. Here 1 denotes a terminal object in Set the lower horizontal map takes the unique element of 1 to the element $(y, x) \in \mathcal{C}_0 \times \mathcal{C}_0$. The base-change $(s, t)'$ of (s, t) along (y, x) is the unique map to the terminal object. Moreover, the diagram

$$\begin{tikzcd} \mathcal{C}_2 \arrow[r, "t'"] \arrow[d, "s'"] & \mathcal{C}_1 \arrow[d, "s"] \\ \mathcal{C}_1 \arrow[r, "t"] & \mathcal{C}_0, \end{tikzcd}$$

where $s'(f, g) = g$ and $t'(f, g) = f$, is cartesian.

There is another feature of category-theoretical definitions, which is that if we reverse all the arrows, then we get a definition of a "dual" concept. In the case of the product, the dual concept is called a coproduct.

Definition 1.14. Let $\mathcal{C}$ be a category. A coproduct of a family $(x_s)_{s\in S}$ of objects in $\mathcal{C}$ is a pair $(y, (i_s\colon x_s \to y)_{s\in S})$ of an object y and a family $(i_s\colon x_s \to y)_{s\in S}$ of morphisms such that if $(z, (f_s\colon x_s \to z)_{s\in S})$ is any such pair, then there exists a *unique* morphism $f\colon y \to z$ such that $f_s = f \circ i_s\colon x_s \to z$ for all $s \in S$.

A coproduct of $(x_s)_{s\in S}$ is unique, up to unique isomorphism, and we write

$$(\textstyle\coprod_{s\in S} x_s, (i_s\colon x_s \to \coprod_{t\in S} x_t)_{s\in S})$$

for any coproduct of $(x_s)_{s\in S}$, and given $(z, (f_s\colon x_s \to z)_{s\in S})$. We also write

$$\textstyle\coprod_{s\in S} x_s \xrightarrow{\sum_{s\in S} f_s} z$$

for the unique morphism f such that $f_s = f \circ i_s$ for all $s \in S$. The summation symbol does not (necessarily) indicate an actual sum.

Example 1.15. A coproduct of the empty family in a category $\mathcal{C}$ determines and is determined by an object y with the property that for every object x in $\mathcal{C}$, there is a unique map $f\colon y \to x$. We say that an object y with this property is an *initial* object. It is common to write 0 or $\emptyset$ for an initial object. The category Set has a unique initial object, namely, the empty set $\emptyset$.

Proposition 1.16. *The category* Set *admits small coproducts.*

Proof. The coproduct in the category Set is given by disjoint union, which is defined set theoretically as follows. If $(X_s)_{s\in S}$ is a family of sets, then we let

$$\textstyle\coprod_{s\in X} X_s = \{(x,s) \in (\bigcup_{s\in S} X_s) \times S \mid x \in X_s \subset \bigcup_{t\in S} X_t\},$$

and let $i_s\colon X_s \to \coprod_{t\in S} X_t$ be the map defined by $i_s(x) = (x,s)$. Given a pair $(Z, (f_s\colon X_s \to Z)_{s\in S})$, the map $f\colon \coprod_{s\in S} X_s \to Z$ defined by $f(x,s) = f_s(x)$ satisfies $f_s = f \circ i_s$ for all $s \in S$ and is unique with this property. $\square$

We will next prove a theorem, which states that fiber products and coproducts in the category Set interact in a particular way. The fact that they do so is quite special and is part of what it means for Set to be a Grothendieck topos. The same theorem holds for the ∞-category of anima.

Theorem 1.17. *The category* Set *has the following properties.*

(1) *Coproducts are universal: If $(f_s\colon Y_s \to X)_{s\in S}$ is a family of maps, and if $(f'_s\colon Y'_s \to X')_{s\in S}$ is the family of maps obtained by base-change along a map $g\colon X' \to X$, then the diagram*

$$\begin{tikzcd} \coprod_{s\in S} Y'_s \arrow[r] \arrow[d] & \coprod_{s\in S} Y_s \arrow[d] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

is cartesian.

(2) *Coproducts are disjoint: If $(f_s\colon Y_s \to X)_{s\in S}$ is a family of maps, then for all $s \in S$, the diagram*

$$\begin{tikzcd} X_s \arrow[r, "\mathrm{id}_{X_s}"] \arrow[d, "\mathrm{id}_{X_s}"'] & X_s \arrow[d, "i_s"] \\ X_s \arrow[r, "i_s"] & \coprod_{u\in S} X_u \end{tikzcd}$$

is cartesian, and for all $s, t \in S$ with $s \neq t$, the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X_s \arrow[d, "i_s"] \\ X_t \arrow[r, "i_t"] & \coprod_{u\in S} X_u \end{tikzcd}$$

is cartesian.

Proof. We first prove (1). For every $s \in S$, the map $i_s\colon Y_s \to \coprod_{t\in S} Y_t$ induces a map $i_s \times_X X'\colon Y_s \times_X X' \to (\coprod_{t\in S} Y_t) \times_X X'$, and the family of these maps indexed by $s \in S$, in turn, induces the map

$$\textstyle\coprod_{s\in S}(Y_s \times_X X') \xrightarrow{\sum_{s\in S}(i_s\times_X X')} (\coprod_{s\in S} Y_s) \times_X X',$$

which we wish to prove is a bijection. To this end, we will use the explict model of the fiber product and the coproduct from Propositions 1.12 and 1.16. The elements of the left-hand side are tuples $((y, x'), s)$ with $s \in S$, $y \in Y_s$, and $x' \in X'$ such that $f_s(y) = g(x')$, and the elements of the right-hand side are tuples $((y, s), x')$ with $s \in S$, $y \in Y_s$, and $x' \in X'$ such that $f_s(y) = g(x')$. Moreover, the map in question takes $((y, x'), s)$ to $((y, s), x')$, so it is indeed a bijection, as stated.

We next prove (2). In the model for the coproduct provided by Proposition 1.16, the map $i_s\colon X_s \to \coprod_{u\in S} X_u$ takes $x \in X_s$ to $(x, s) \in (\bigcup_{u\in S} X_u) \times S$, and in the model for the fiber product provided by Proposition 1.12, the pullback of (i_s, i_t) is given by subset of $X_s \times X_t$ consisting of the pairs (x_1, x_2) such that $i_s(x_1) = i_t(x_2)$, or equivalently, such that $(x_1, s) = (x_2, t)$. If $s = t$, then this equality holds if and only if $x_1 = x_2$, which shows that the first diagram in (2) is cartesian, and if $s \neq t$, then the equality does not hold for any $(x_1, x_2) \in X_s \times X_t$, which shows that also the second diagram in (2) is cartesian. □

We will use Theorem 1.17 to give an equivalent definition of a category that only requires us to specify each of the individual morphism sets $\mathrm{Map}(y, x)$ and not the full morphism set $\mathcal{C}_1$, which is often easier. This description, however, is based on the following lemma, which has no analogue for anima.

Lemma 1.18. *For every set S, the map*

$$\textstyle\coprod_{s\in S}\{s\} \longrightarrow S$$

induced by the canonical inclusions is a bijection.

Proof. This is a rephrasing of the (ZFC) axiom of extensionality, which states that two sets agree if and only if the have the same elements. □

Corollary 1.19. *Let $\mathcal{C}$ be a small category. The map*

$$\coprod_{(y,x)\in\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Map}(y,x) \longrightarrow \mathcal{C}_1$$

induced by the canonical inclusions is a bijection.

Proof. For every $(y,x) \in \mathcal{C}_0 \times \mathcal{C}_0$, the diagram

$$\begin{tikzcd} \mathrm{Map}(y,x) \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d, "(s,t)"] \\ \{(y,x)\} \arrow[r] & \mathcal{C}_0 \times \mathcal{C}_0 \end{tikzcd}$$

is cartesian, so Theorem 1.17 (1) shows that the induced diagram

$$\begin{tikzcd} \coprod_{(y,x)\in\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Map}(y,x) \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d, "(s,t)"] \\ \coprod_{(y,x)\in\mathcal{C}_0\times\mathcal{C}_0}\{(y,x)\} \arrow[r] & \mathcal{C}_0 \times \mathcal{C}_0 \end{tikzcd}$$

is cartesian. But the lower horizontal map is a bijection, by Lemma 1.18, and hence, so is the upper horizontal map. □

Corollary 1.20. *Let $\mathcal{C}$ be a small category. There is a cartesian diagram*

$$\begin{tikzcd} \coprod_{(z,y,x)\in\mathcal{C}_0\times\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Map}(y,x) \times \mathrm{Map}(z,y) \arrow[r, "t'"] \arrow[d, "s'"] & \coprod_{(u,x)\in\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Map}(u,x) \arrow[d, "s"] \\ \coprod_{(z,v)\in\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Map}(z,v) \arrow[r, "t"] & \mathcal{C}_0 \end{tikzcd}$$

in which the maps are defined as follows: The (u,x)th component of the map s takes every element of $\mathrm{Map}(u,x)$ to $u \in \mathcal{C}_0$, the (z,v)th component of the map t takes every element of $\mathrm{Map}(z,v)$ to $v \in \mathcal{C}_0$, and finally, the (z,y,x)th component of the maps t' and s' are given by the composite maps

$$\mathrm{Map}(y,x) \times \mathrm{Map}(z,y) \xrightarrow{p_1} \mathrm{Map}(y,x) \xrightarrow{i_{(y,x)}} \coprod_{(v,u)\in\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Map}(v,u),$$

$$\mathrm{Map}(y,x) \times \mathrm{Map}(z,y) \xrightarrow{p_2} \mathrm{Map}(z,y) \xrightarrow{i_{(z,y)}} \coprod_{(w,v)\in\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Map}(w,v)$$

respectively.

Proof. Applying Theorem 1.17 (1) twice, we conclude that the canonical maps

$$\begin{aligned} &\coprod_{(u,x)\in\mathcal{C}_0\times\mathcal{C}_0} \coprod_{(v,z)\in\mathcal{C}_0\times\mathcal{C}_0} (\mathrm{Map}(u,x) \times_{\mathcal{C}_0} \mathrm{Map}(z,v)) \\ &\longrightarrow \coprod_{(u,x)\in\mathcal{C}_0\times\mathcal{C}_0} (\mathrm{Map}(u,x) \times_{\mathcal{C}_0} (\coprod_{(z,v)\in\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Map}(z,v))) \\ &\longrightarrow (\coprod_{(u,x)\in\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Map}(u,x)) \times_{\mathcal{C}_0} (\coprod_{(z,v)\in\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Map}(z,v))) \end{aligned}$$

are bijections. Now we use Lemma 1.18 for $S = \mathrm{Map}(u,x)$ and $S = \mathrm{Map}(z,v)$ and conclude again from Theorem 1.17 (1) that the map

$$\coprod_{(f,g)\in\mathrm{Map}(u,x)\times\mathrm{Map}(z,v)} (\{f\} \times_{\mathcal{C}_0} \{g\}) \longrightarrow \mathrm{Map}(x,u) \times_{\mathcal{C}_0} \mathrm{Map}(z,v)$$

induced by the canonical inclusions is a bijection. Similarly, using Lemma 1.18 for $S = \mathcal{C}_0$, we see from Theorem 1.17 (2) that if $(f,g) \in \mathrm{Map}(u,x) \times \mathrm{Map}(z,v)$, then the fiber product $\{f\} \times_{\mathcal{C}_0} \{g\}$ is canonically bijective to $\{(f,g)\}$, if $u = v$, and to $\emptyset$, otherwise. It follows that the canonical map

$$\mathrm{Map}(y,x) \times \mathrm{Map}(z,y) \longrightarrow \mathrm{Map}(y,x) \times_{\mathcal{C}_0} \mathrm{Map}(z,y)$$

is a bijection, and that $\mathrm{Map}(u,x) \times_{\mathcal{C}_0} \mathrm{Map}(z,v)$ is the empty if $u \neq v$. $\square$

By the uniqueness of fiber product, there is a commutative diagram

$$\begin{tikzcd}
\coprod_{(z,y,x)\in\mathcal{C}_0\times\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Map}(y,x) \times \mathrm{Map}(z,y) \arrow[r] \arrow[d] & \mathcal{C}_2 \arrow[d, "\circ"] \\
\coprod_{(z,x)\in\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Map}(z,x) \arrow[r] & \mathcal{C}_1
\end{tikzcd}$$

where the bottom horizontal map is the isomorphism of Corollary 1.19, and where the top horizontal map is the unique isomorphism of fiber products determined by Example 1.13 and Corollary 1.20. The (z,y,x)th component of the left-hand vertical map is the composition of a "pointwise" composition map

$$\mathrm{Map}(y,x) \times \mathrm{Map}(z,y) \xrightarrow{\ \circ\ } \mathrm{Map}(z,x)$$

and of the canonical inclusion

$$\mathrm{Map}(z,x) \longrightarrow \coprod_{(v,u)\in\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Map}(v,u).$$

Hence, up to unique isomorphism, the composition map $\circ\colon \mathcal{C}_2 \to \mathcal{C}_1$ determines and is determined by the family of pointwise composition maps. In a similarly way, the identity map $e\colon \mathcal{C}_0 \to \mathcal{C}_1$ determines and is determined by pointwise identity maps $e\colon \{x\} \to \mathrm{Map}(x,x)$, that is, by an element $\mathrm{id}_x \in \mathrm{Map}(x,x)$ for all $x \in \mathcal{C}_0$.

Corollary 1.21. *A category $\mathcal{C}$ determines and is determined by a set $\mathrm{ob}(\mathcal{C})$ of objects; for every pair (y,x) of objects, a set $\mathrm{Map}(y,x)$ of morphisms from y to x; for every triple (z,y,x) of objects, a map $\circ\colon \mathrm{Map}(y,x) \times \mathrm{Map}(z,y) \to \mathrm{Map}(z,x)$; and for every object x, an identity map $\mathrm{id}_x \in \mathrm{Map}(x,x)$; such that for every pair (y,x) of objects and every $f \in \mathrm{Map}(y,x)$, $\mathrm{id}_x \circ f = f = f \circ \mathrm{id}_y$; and for every quadruple (w,z,y,x) of objects and every triple $(h,g,f) \in \mathrm{Map}(w,z) \times \mathrm{Map}(z,y) \times \mathrm{Map}(y,x)$, $f \circ (g \circ h) = (f \circ g) \circ h$.*

Proof. This follows from the discussion above. $\square$

Example 1.22. Let us use Corollary 1.21 to define the category of small k-vector spaces, where k is a field. Recall that a (right) k-vector space is a triple $(V,+,\cdot)$ of a set V and two maps $+\colon V \times V \to V$ and $\cdot\colon V \times k \to V$ satisfying the vector space axioms. So a k-vector space consists of a set V with some additional *structure* on that set, namely, the vector sum and the scalar multiplication. If $(V,+,\cdot)$ and $(W,+,\cdot)$ are k-vector spaces, then a map $f\colon W \to V$ is k-linear, if it has the *property* that it preserves the additional structure. So we define

$$\mathrm{Map}((W,+,\cdot),(V,+,\cdot)) \subset \mathrm{Map}(W,V)$$

to be the subset consisting of the maps $f\colon W \to V$ that are k-linear. We now check that the composition of two k-linear maps is a k-linear map and that the identity map is a k-linear map. So given three k-vector spaces $(W,+,\cdot)$, $(V,+,\cdot)$, and $(U,+,\cdot)$, the pointwise composition of maps of sets

$$\mathrm{Map}(V,U)\times\mathrm{Map}(W,V)\xrightarrow{\circ}\mathrm{Map}(W,U)$$

restricts to a pointwise composition of k-linear maps

$$\mathrm{Map}((V,+,\cdot),(U,+,\cdot))\times\mathrm{Map}((W,+,\cdot),,(V,+,\cdot))$$

$$\xrightarrow{\circ}\mathrm{Map}((W,+,\cdot),(U,+,\cdot))$$

and $\mathrm{id}_V\in\mathrm{Map}((V,+,\cdot))$. So the set $\mathrm{ob}(\mathrm{Vect}_k)$ of small (right) k-vector spaces with the above sets of morphisms, composition, and identity maps define a category Vect_k that we call the category of small (right) k-vector spaces.

Remark 1.23. Theorem 1.17 does *not* hold for the category Vect_k of small right k-vector spaces: the category Vect_k is not a topos.

Exercise 1.24. Let $f\colon y\to x$ be a morphism in a category $\mathcal{C}$. Show that if an inverse $g\colon x\to y$ of $f\colon y\to x$ exists, then it is unique.

Exercise 1.25. Let G be a group, and let BG be the category from Example 2 in the lecture notes. Show that if G is non-trivial, then BG does not admit products. [Hint: Show that no triple $(0,g_1: 0\to 0,g_2: 0\to 0)$ can be a product of $(0,0)$.]

Exercise 1.26. We consider a commutative diagram

$$\begin{tikzcd}
y'' \arrow[r, "h'"] \arrow[d, "f''"] & y' \arrow[r, "g'"] \arrow[d, "f'"] & y \arrow[d, "f"] \\
x'' \arrow[r, "h"] & x' \arrow[r, "g"] & x
\end{tikzcd}$$

in a category $\mathcal{C}$.

(a) Show that if the left-hand square and the right-hand square both are cartesian, then so is the outer square.
(b) Show that if the right-hand square and the outer square both are cartesian, then so is the left-hand square.
(c) Give an example, where the left-hand square and the outer square both are cartesian, but where the right-hand square is not cartesian.

We essentially have two methods for building new sets out of old ones, namely, by forming sets of solutions to systems of equations or by gluing sets together. The first method is formalized into the notion of a *limit*, and the second is formalized into the notion of a *colimit*, and these notions are the subject of this lecture.

We first give a proper definition of a diagram in a category. A diagram is the same as a functor, and functors are the morphisms between categories, in the same way as k-linear maps are the morphisms between k-vector spaces.

Definition 2.1. Let $K = (K_0, K_1, s, t, e, \circ)$ and $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, s, t, e, \circ)$ be categories. A functor $p\colon K \to \mathcal{C}$ is a pair of maps $(p_0\colon K_0 \to \mathcal{C}_0, p_1\colon K_1 \to \mathcal{C}_1)$ such that the following diagrams commute.

$$
\begin{tikzcd}
K_1 \arrow[r, "p_1"] \arrow[d, "s"] & \mathcal{C}_1 \arrow[d, "s"] \\
K_0 \arrow[r, "p_0"] & \mathcal{C}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
K_1 \arrow[r, "p_1"] \arrow[d, "t"] & \mathcal{C}_1 \arrow[d, "t"] \\
K_0 \arrow[r, "p_0"] & \mathcal{C}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
K_1 \arrow[r, "p_1"] & \mathcal{C}_1 \\
K_0 \arrow[r, "p_0"] \arrow[u, "e"] & \mathcal{C}_0 \arrow[u, "e"]
\end{tikzcd}
\qquad
\begin{tikzcd}
K_2 \arrow[r, "p_2"] \arrow[d, "\circ"] & \mathcal{C}_2 \arrow[d, "\circ"] \\
K_1 \arrow[r, "p_1"] & \mathcal{C}_1
\end{tikzcd}
$$

Here, in the last diagram, the map p_2 is defined by $p_2(f, g) = (p_1(f), p_1(g))$. It is well-defined by the commutativity of the first two diagrams.

Let $p\colon K \to \mathcal{C}$ be a functor, let (j, i) be a pair of objects in K, and let (y, x) be the pair of objects in $\mathcal{C}_0$, where $x = p_0(i)$ and $y = p_0(j)$. The commutativity of the first two diagrams in Definition 2.1 expresses that $p_1\colon K_1 \to \mathcal{C}_1$ restricts to a map

$$\mathrm{Map}(j, i) \xrightarrow{p_{1,(i,j)}} \mathrm{Map}(y, x)$$

between the indicated sets of morphisms, and the commutativity of the remaining two diagrams expresses that the following hold:

(i) For every object i in K,
$$p_{1,(i,i)}(\mathrm{id}_i) = \mathrm{id}_x,$$
where $x = p_0(i)$.

(ii) For every triple (i, j, k) of objects in K, diagram
$$
\begin{tikzcd}
\mathrm{Map}(j,i) \times \mathrm{Map}(k,j) \arrow[r, "\circ"] \arrow[d, "p_{1,(i,j)} \times p_{1,(j,k)}"] & \mathrm{Map}(k,i) \arrow[d, "p_{1,(i,k)}"] \\
\mathrm{Map}(y,x) \times \mathrm{Map}(z,y) \arrow[r, "\circ"] & \mathrm{Map}(z,x),
\end{tikzcd}
$$
where $x = p_0(i)$, $y = p_0(j)$, and $z = p_0(k)$, commutes.

Conversely, a map $p_0\colon K_0 \to \mathcal{C}_0$ and a family of maps $(p_{1,(i,j)})_{(i,j) \in K_0 \times K_0}$ as above that satisfy (i)–(ii) determine a unique functor $p\colon K \to \mathcal{C}$.

As is common, in the following we will now abuse notation and write p for the maps p_0, p_1, and $p_{1,(i,j)}$. Let us now prove the simple, but very useful, fact that every functor preserves isomorphisms.

Lemma 2.2. *Let $p\colon K \to \mathcal{C}$ be a functor. If the morphisms $a\colon j \to i$ and $b\colon i \to j$ in K are each other's inverses, then so are the morphisms $p(a)\colon p(j) \to p(i)$ and $p(b)\colon p(i) \to p(j)$ in $\mathcal{C}$.*

Proof. Indeed, since $p\colon K \to \mathcal{C}$ is a functor, we have
$$p(a) \circ p(b) = p(a \circ b) = p(\mathrm{id}_i) = \mathrm{id}_{p(i)}$$
$$p(b) \circ p(a) = p(b \circ a) = p(\mathrm{id}_j) = \mathrm{id}_{p(j)}$$
which proves the lemma. □

A category $\mathcal{G}$ is defined to be a *groupoid* if all morphisms in $\mathcal{G}$ are isomorphisms. The categories K and BG in the following example are both groupoids.

Example 2.3. (1) Let K be a set, and let also K denote the static category with object set K. A functor $p\colon K \to \mathcal{C}$ determines and is determined by the map of sets $p_0\colon K \to \mathcal{C}_0$, that is, a K-indexed family of objects in $\mathcal{C}$. Indeed, the map $p_1\colon K \to \mathcal{C}_1$ is necessarily given by $p_1 = e \circ p_0$, and the pair (p_0, p_1) satisfies the axioms in the definition of a functor.

(2) Let G be a group, and let BG be the groupoid that has a single object 1, whose group of automorphism is $\mathrm{Map}(1,1) = G$. A functor $p\colon BG \to \mathrm{Set}$ determines and is determined by the pair (X, ρ) of the set $X = p(1)$ and the group homomorphism $\rho = p\colon G \to \mathrm{Aut}(X)$, that is, by a set with left G-action. Similarly, if k is a field, then a functor $p\colon BG \to \mathrm{Vect}_k$ determines an is determined by the pair (V, π) of the k-vector space $V = p(1)$ and the group homomorphism $\pi = p\colon G \to \mathrm{Aut}_k(V)$, that is, by a k-linear representation of G. Note that the fact that the map π takes values in the $\mathrm{Aut}_k(V) \subset \mathrm{End}_k(V)$ is a consequence of Lemma 2.2.

Definition 2.4. A functor $p\colon K \to \mathcal{C}$ is faithful (resp. full, resp. fully faithful) if for every pair (i, j) of objects in K, the map
$$\mathrm{Map}(j, i) \xrightarrow{p} \mathrm{Map}(y, x),$$
where $x = p(i)$ and $y = p(j)$, is injective (resp. surjective, resp. bijective).

Example 2.5. A subcategory of a category $\mathcal{C}$ is a category $\mathcal{C}'$ such that $\mathcal{C}'_0 \subset \mathcal{C}_0$ and $\mathcal{C}'_1 \subset \mathcal{C}_1$. We write $\mathcal{C}' \subset \mathcal{C}$ to indicate that $\mathcal{C}'$ is a subcategory $\mathcal{C}$. The canonical inclusions of subsets $i_0\colon \mathcal{C}'_0 \to \mathcal{C}_0$ and $i_1\colon \mathcal{C}'_1 \to \mathcal{C}_1$ define a functor $i\colon \mathcal{C}' \to \mathcal{C}$ that we call the canonical inclusion functor. It is always faithful, but it need not be full. If it is full, then we say that $\mathcal{C}' \subset \mathcal{C}$ is a full subcategory. We note that a full subcategory $\mathcal{C}' \subset \mathcal{C}$ is uniquely determined by the subset $\mathcal{C}'_0 \subset \mathcal{C}_0$.

It is tempting to define a category, whose objects are all small categories and whose morphisms are all functors between them, but while it is possible to do so, this misses an important point about functors, which is that we also want to have morphisms, or at least isomorphisms, between functors. These are called natural transformations and natural isomorphisms, respectively.

Definition 2.6. Let $p, q\colon K \to \mathcal{C}$ be functors with a common source and target. A natural transformation from q to p is a family $\varphi = (\varphi_i\colon q(i) \to p(i))_{i \in K_0}$ of morphisms in $\mathcal{C}$ indexed by the set K_0 of objects in K such that for every morphism $a\colon j \to i$ in K, the diagram
$$\begin{tikzcd} q(j) \arrow[r, "\varphi_j"] \arrow[d, "q(a)"] & p(j) \arrow[d, "p(a)"] \\ q(i) \arrow[r, "\varphi_i"] & p(i) \end{tikzcd}$$

in $\mathcal{C}$ commutes. A natural transformation is a natural isomorphism if the morphisms φ_i all are isomorphisms in $\mathcal{C}$. We write $\varphi\colon q \to p$ to indicate that φ is a natural transformation from q to p.

The word "natural" has the precise mathematical meaning expressed by the fact that the diagrams in Definition 2.6 commute. So, in mathematics, we should never use the word "natural," except in this precise meaning. This means that if we feel an urge to say that something is natural, then we should think more and identify the functors that this something is a natural transformation between.

Example 2.7. (1) Continuing Example 2.3, if G is a group, and if $p, q\colon BG \to \mathrm{Set}$ are functors corresponding to two sets with left G-action (X, ρ) and (Y, σ), then a natural transformation $\varphi\colon q \to p$ determines and is determined by the G-equivariant map $f = \varphi_0\colon Y \to X$. Similarly, if $p, q\colon BG \to \mathrm{Vect}_k$ are functors corresponding to k-linear representations (V, π) and (W, τ) of G, then a natural transformation $\varphi\colon q \to p$ determines and is determined by the k-linear map $f = \varphi_0\colon W \to V$ that intertwines between τ and π.
(2) If $p\colon K \to \mathcal{C}$ is a functor, then the family of identity morphisms $(\mathrm{id}_{p(i)})_{i \in K_0}$ is (obviously) a natural isomorphism from p to p. We write id_p and sometimes just p for this natural isomorphism.

If we compare two categories, then it is not resonable to ask if they are equal, just as it is not reasonable to ask if two vector spaces are equal. But it is also not reasonable to ask if they are isomorphic. The reasonable way to compare categories is via the notion of an equivalence, which we now define.

Definition 2.8. A functor $p\colon \mathcal{D} \to \mathcal{C}$ is an equivalence, if there exists a functor $q\colon \mathcal{C} \to \mathcal{D}$ and natural isomorphisms $\epsilon\colon p \circ q \to \mathrm{id}_{\mathcal{C}}$ and $\eta\colon\ \mathrm{id}_{\mathcal{D}} \to q \circ p$.

Example 2.9. Let k be a field, let Vect_k be the category of k-vector spaces, and let

$$\mathrm{Vect}_k^{\omega} \subset \mathrm{Vect}_k$$

be the full subcategory spanned by the finite-dimensional k-vector spaces. We define a new category K as follows. The set of objects is the set ω of finite ordinals, and for every pair (m, n) of objects, the mapping set

$$\mathrm{Map}(n, m) = M_{m,n}(k)$$

is the set of $(m \times n)$-matrices with entries in the field k. The identity morphism of the object n is the $(n \times n)$-identity matrix, and composition

$$\mathrm{Map}(n, m) \times \mathrm{Map}(p, n) \xrightarrow{\circ} \mathrm{Map}(p, m)$$

is the map given by matrix multiplication. We have a functor

$$K \xrightarrow{p} \mathrm{Vect}_k^{\omega}$$

that to the object n assigns the (right) k-vector space $p(n) = M_{n,1}(k)$, and that on mapping sets is given by the map

$$\mathrm{Map}(n, m) = M_{m,n}(k) \xrightarrow{p} \mathrm{Map}(p(n), p(m)) = \mathrm{Hom}_k(M_{n,1}(k), M_{m,1}(k))$$

defined by $p(A)(\boldsymbol{y}) = A\boldsymbol{y}$. As we learn in linear algebra, this map is a bijection, so the functor p is fully faithful. We claim that it is an equivalence. Indeed, this we also learn in linear algebra. In order to define a functor

$$\mathrm{Vect}_k^{\omega} \xrightarrow{q} K,$$

we choose a basis $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$ of every finite-dimensional k-vector space V, and the functor q will depend on this choice.[4] We now define q to be the functor that to a finite-dimensional vector space V assigns its dimension $q(V) = \dim_k(V)$, and that on mapping sets is given by the map

$$\mathrm{Map}(W, V) = \mathrm{Hom}_k(W, V) \xrightarrow{q} \mathrm{Map}(n, m) = M_{m,n}(k)$$

that to a k-linear map $f\colon W \to V$ assigns the matrix $A \in M_{m,n}(k)$ that represents f with respect to the chosen bases of V and W. If $f\colon V \to U$ and $g\colon W \to V$ are composable k-linear maps, then $q(f \circ g) = q(f) \circ q(g)$, because we calculate $q(f)$ and $q(g)$ using the *same* basis for V, and $q(\mathrm{id}_V) = \mathrm{id}_{q(V)}$, because we calculate $q(\mathrm{id}_V)$ using the *same* basis of the domain and target. So q is a functor.

It remains to define $\epsilon\colon p \circ q \to \mathrm{id}_{\mathrm{Vect}_k^{\omega}}$ and $\eta\colon\ \mathrm{id}_K \to q \circ p$. To define ϵ, we again use the choice of bases that we made in order to define q, and let

$$(p \circ q)(V) = M_{m,1}(k) \xrightarrow{\epsilon_V} V$$

be the map given by $\epsilon_V(\boldsymbol{x}) = \boldsymbol{v}_1 x_1 + \dots + \boldsymbol{v}_m x_m$. It is an isomorphism by the definition of a basis. In order that ϵ be a natural transformation, we must show that for every k-linear map $f\colon W \to V$, the diagram

$$\begin{tikzcd}
(p \circ q)(W) = M_{n,1}(k) \arrow[r, "\epsilon_W"] \arrow[d, "(p\circ q)(f)"', shift right=10ex] \arrow[d, "\boldsymbol{y} \mapsto A\boldsymbol{y}"] & W \arrow[d, "f"] \\
(p \circ q)(V) = M_{m,1}(k) \arrow[r, "\epsilon_V"] & V
\end{tikzcd}$$

commutes. But the matrix A that represents f with respect to the chosen bases is exact defined so that this is true. So ϵ is a natural isomorphism. Finally, define

$$m \xrightarrow{\eta_m} m = (q \circ p)(m)$$

to be matrix $Q \in M_{m,m}(k) = \mathrm{Map}(m, m)$ that represents the identity map

$$M_{m,1}(k) \xrightarrow{\mathrm{id}_{M_{m,1}(k)}} M_{m,1}(k)$$

with respect to the standard basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ of the domain and the (possibly different) chosen basis $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$ of the target. The matrix Q is invertible, so η_m is an isomorphism. It is also natural, again by the fact that the matrix that represents the composition of two linear maps with respect to given bases of the three vector spaces involved is equal to the product of the matrices that represent the two linear maps separately with respect to the given bases.

Remark 2.10. It may appear from Example 2.9 that the "inverse" of an equivalence of categories $p\colon \mathcal{D} \to \mathcal{C}$ is not unique in any way. But in fact it is unique, up to contractible ambiguity: isomorphism among objects in a 1-category and equivalence among 1-categories are both special cases the notion of equivalence among objects in an ∞-category, and the "inverse" of an equivalence in an ∞-category is as unique

[4] This uses the axiom of choice.

as it can possibly be, namely, the collection of all inverses of a given equivalence is organized into an anima and that anima is contractible.

Now that we have the proper language to discuss diagrams, it is time to define limits and a colimits of a diagrams. We first define the *join* of two categories. Given categories J and K, their join is a category $J \star K$, whose set of objects is the disjoint union of the sets of objects in J and K, and where, in addition to the morphisms in J and K, there is a unique morphism from every object in J to every object in K, but *not* vice versa. So $J \star K$ and $K \star J$ are typically not equivalent!

Definition 2.11. The join of two categories J and K is the category $J \star K$ defined as follows. The set of objects in $J \star K$ is the disjoint union

$$(J \star K)_0 = J_0 \sqcup K_0$$

of the sets of objects in J and K, respectively, and the set of morphisms in $J \star K$ is the disjoint union

$$(J \star K)_1 = J_1 \sqcup K_1 \sqcup J_0 \times K_0$$

of the sets of morphisms in J and K, respectively, and the product of the sets of objects in J and K. The source map $s\colon (J \star K)_1 \to (J \star K)_0$ is the unique map that makes the diagrams

$$\begin{tikzcd}
J_1 \arrow[r, "i_1"] \arrow[d, "s"] & (J \star K)_1 \arrow[d, "s"] & K_1 \arrow[l, "i_2"'] \arrow[d, "s"] \\
J_0 \arrow[r, "i_1"] & (J \star K)_0 & K_0 \arrow[l, "i_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
J_0 \times K_0 \arrow[r, "i_3"] \arrow[d, "p_1"] & (J \star K)_1 \arrow[d, "s"] \\
J_0 \arrow[r, "i_1"] & (J \star K)_0
\end{tikzcd}$$

commute, and the target map $t\colon (J \star K)_1 \to (J \star K)_0$ is the unique map that makes the diagrams

$$\begin{tikzcd}
J_1 \arrow[r, "i_1"] \arrow[d, "t"] & (J \star K)_1 \arrow[d, "t"] & K_1 \arrow[l, "i_2"'] \arrow[d, "t"] \\
J_0 \arrow[r, "i_1"] & (J \star K)_0 & K_0 \arrow[l, "i_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
J_0 \times K_0 \arrow[r, "i_3"] \arrow[d, "p_2"] & (J \star K)_1 \arrow[d, "t"] \\
K_0 \arrow[r, "i_1"] & (J \star K)_0
\end{tikzcd}$$

commute. Note that, with these definitions, the set of composable morphisms in the join $J \star K$ is given by the disjoint union

$$(J \star K)_2 = J_2 \sqcup K_2 \sqcup J_1 \times K_0 \sqcup J_0 \times K_1.$$

The composition $\circ\colon (J \star K)_2 \to (J \star K)_1$ is the unique map that makes the diagrams

$$\begin{tikzcd}
J_2 \arrow[r, "i_1"] \arrow[d, "\circ"] & (J \star K)_2 \arrow[d, "\circ"] & K_2 \arrow[l, "i_2"'] \arrow[d, "\circ"] \\
J_1 \arrow[r, "i_1"] & (J \star K)_1 & K_1 \arrow[l, "i_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
J_1 \times K_0 \arrow[r, "i_3"] \arrow[d, "s \times K_0"] & (J \star K)_2 \arrow[d, "\circ"] & J_0 \times K_1 \arrow[l, "i_4"'] \arrow[d, "J_0 \times t"] \\
J_0 \times K_0 \arrow[r, "i_3"] & (J \star K)_1 & J_0 \times K_0 \arrow[l, "i_3"']
\end{tikzcd}$$

commute. Finally, the identity map $e\colon (J\star K)_0 \to (J\star K)_1$ is the unique map that makes the diagram

$$\begin{tikzcd} J_1 \arrow[r, "i_1"] & (J\star K)_1 & K_1 \arrow[l, "i_2"'] \\ J_0 \arrow[u, "e"] \arrow[r, "i_1"] & (J\star K)_0 \arrow[u, "e"] & K_0 \arrow[u, "e"] \arrow[l, "i_2"'] \end{tikzcd}$$

commute.

There are functors $i_1\colon J \to J\star K$ and $i_2\colon K \to J\star K$ defined by the canonical inclusions, both of which are fully faithful. Indeed, in $J\star K$, we have added a unique morphism from every object in J to every object in K, but we have not added any new morphisms within either J or K. We now specialize to the case, where one of J and K is the static category 1 with a single object.

Definition 2.12. The left cone on a category K is the join $K^{\triangleleft} = 1\star K$, and the right cone on the category K is the join $K^{\triangleright} = K\star 1$.

Informally, the left cone $K^{\triangleleft}$ is obtained by formally adjoining an initial object to the category K. We write $i\colon K \to K^{\triangleleft}$ for the canonical inclusion, and given a functor $f\colon K^{\triangleleft} \to \mathcal{C}$, we also write $f|_K = f\circ i\colon K \to \mathcal{C}$ and call it the restriction of f to K. Similarly, the right cone $K^{\triangleright}$ is obtained by formally adjoining a final object to K. We again write $i\colon K \to K^{\triangleright}$ for the canonical inclusion, and given a functor $g\colon K^{\triangleright} \to \mathcal{C}$, we write $g|_K = g\circ i\colon K \to \mathcal{C}$ and call it the restriction of g to K.

Definition 2.13. Let $p\colon K \to \mathcal{C}$ be a functor.

(1) A limit of $p\colon K \to \mathcal{C}$ is a functor $\bar{p}\colon K^{\triangleleft} \to \mathcal{C}$ with $\bar{p}|_K = p$ and such that for every functor $f\colon K^{\triangleleft} \to \mathcal{C}$ with $f|_K = p$, there exists a *unique* natural transformation $\varphi\colon f \to \bar{p}$ with $\varphi|_K = \mathrm{id}_p$.
(2) A colimit of $p\colon K \to \mathcal{C}$ is a functor $\bar{p}\colon K^{\triangleright} \to \mathcal{C}$ with $\bar{p}|_K = p$ and with the property that for every functor $g\colon K^{\triangleright} \to \mathcal{C}$ with $g|_K = p$, there exists a *unique* natural transformation $\psi\colon \bar{p} \to g$ with $\psi|_K = \mathrm{id}_p$.

A limit $\bar{p}\colon K^{\triangleleft} \to \mathcal{C}$ of a diagram $p\colon K \to \mathcal{C}$, if it exists, is unique, up to unique isomorphism. Therefore, we also write $\lim p\colon K^{\triangleleft} \to \mathcal{C}$ for any choice of a limit of $p\colon K \to \mathcal{C}$ with the understanding that it is well-defined, up to unique isomorphism only. Similarly, a colimit of $p\colon K \to \mathcal{C}$ is unique, up to unique isomorphism, and we write $\operatorname{colim} p\colon K^{\triangleright} \to \mathcal{C}$ for any choice of a such a colimit.[5]

Example 2.14. (1) Let K be a static category, and let $p\colon K \to \mathcal{C}$ be a functor corresponding to a K-indexed family $(x_i)_{i\in K}$ of objects of $\mathcal{C}$. A diagram $\bar{p}\colon K^{\triangleleft} \to \mathcal{C}$ with $\bar{p}|_K = p$ determines and is determined by a pair $(y, (p_i\colon y \to x_i)_{i\in K})$ of an object y in $\mathcal{C}$ and a family of morphisms as indicated. Moreover, the diagram $\bar{p}\colon K^{\triangleleft} \to \mathcal{C}$ is a limit of $p\colon K \to \mathcal{C}$ if and only if the pair $(y, (p_i\colon y \to x_i)_{i\in K})$ is a product of $(x_i)_{i\in K}$.

Similarly, a diagram $\bar{p}\colon K^{\triangleright} \to \mathcal{C}$ is a pair $(y, (i_s\colon x_s \to y)_{s\in K})$ of an object y in $\mathcal{C}$ and a family of morphisms, and the diagram $\bar{p}\colon K^{\triangleright} \to \mathcal{C}$ is a colimit of $p\colon K \to \mathcal{C}$ if an only if the pair $(y, (i_s\colon x_s \to y)_{s\in K})$ is a coproduct of $(x_s)_{s\in K}$.

[5] Alternative notation for the limit (resp. colimit) is $\varprojlim p$ (resp. $\varinjlim p$), and an alternative name for the limit (resp. colimit) is projective limit (resp. inductive limit).

(2) Let G be a group, let $K = BG$, and let $p\colon K \to \mathrm{Set}$ be a functor corresponding to a pair (X, ρ) of a set X with left G-action $\rho\colon G \to \mathrm{Aut}(X)$ of a group G. A diagram $\bar{p}\colon K^{\triangleleft} \to \mathrm{Set}$ with $\bar{p}|_K = p$ determines and is determined by a pair $(Y, p\colon Y \to X)$ of a set Y and a map $p\colon Y \to X$ such that $\rho(g) \circ p = p$ for all $g \in G$. Moreover, the diagram $\bar{p}\colon K^{\triangleleft} \to \mathrm{Set}$ is a limit of $p\colon K \to \mathrm{Set}$ if and only if the map $p\colon Y \to X$ induces a bijection of Y onto the subset

$$X^G = \{x \in X \mid \rho(g)(x) = x \text{ for all } g \in G\} \subset X$$

of fixed points for the left G-action. Similarly, a diagram $\bar{p}\colon K^{\triangleright} \to \mathrm{Set}$ with $\bar{p}|_K = p$ is a pair $(Y, i\colon X \to Y)$ of a set Y and a map $i\colon X \to Y$ such that $i \circ \rho(g) = i$ for all $g \in G$. The diagram $\bar{p}\colon K^{\triangleright} \to \mathrm{Set}$ is a colimit of $p\colon K \to \mathrm{Set}$ if and only if the map $i\colon X \to Y$ induces a bijection from the quotient set

$$X \longrightarrow X_G = X/R$$

of orbits for the left G-action onto Y. Here $R \subset X \times X$ is the relation

$$R = \{(\rho(g)(x), x) \in X \times X \mid g \in G, x \in X\},$$

which is an equivalence relation, because G is a group, as opposed to a monoid.

The description of the limit and colimit of a diagram $p\colon BG \to \mathrm{Set}$ is typical for diagrams of sets. Let us show that every diagram $p\colon K \to \mathrm{Set}$ with K a small category admits a limit and a colimit.

Proposition 2.15. *The category* Set *admits all small limits and colimits.*

Proof. Let $p\colon K \to \mathrm{Set}$ be diagram with K small, and let $X_i = p(i)$. We first show that this diagram admits a limit. To this end, we let

$$Y \subset X = \textstyle\prod_{i \in K_0} X_i$$

be the set of solutions $x = (x_i)_{i \in K_0}$ to the system of equations

$$p(a)(x_j) = x_i$$

indexed by the set K_1 of morphisms $a\colon j \to i$ in K. Define $\bar{p}\colon K^{\triangleleft} \to \mathrm{Set}$ to be the extension of $p\colon K \to \mathrm{Set}$, whose value at the cone point is Y, and whose value at the unique morphism from the cone point to the object $i \in K_0$ is the composition

$$Y \longrightarrow \textstyle\prod_{s \in K_0} X_s \xrightarrow{p_i} X_i$$

of the canonical inclusion and the projection on the ith factor. It is clear that the diagram $\bar{p}\colon K^{\triangleleft} \to \mathrm{Set}$ is a limit of $p\colon K \to \mathrm{Set}$.

We next show that the diagram $p\colon K \to \mathrm{Set}$ admits a colimit. We let

$$X = \textstyle\coprod_{i \in K_0} X_i$$

be the coproduct of the family $(X_i)_{i \in K_0}$. Last time, we proved that coproducts in Set are universal. This implies, in particular, that the map

$$\textstyle\coprod_{(i,j) \in K_0 \times K_0} X_i \times X_j \longrightarrow X \times X$$

induced by the canonical inclusions is a bijection. We now define

$$S = \textstyle\bigcup_{(a\colon j \to i) \in K_1} S_a \subset X \times X$$

to be the relation, where for $a\colon j \to i$ a morphism in K,

$$S_a = \{(p(a)(x), x) \in X_i \times X_j \mid x \in X_i\} \subset X_i \times X_j \subset X \times X.$$

The relation $S \subset X \times X$ is reflexive and transitive, but it is generally not symmetric. So $S \subset X \times X$ is generally *not* an equivalence relation. However, there exists a smallest equivalence relation $R \subset X \times X$ with $S \subset R$, and we let

$$X = \textstyle\coprod_{i \in K_0} X_i \longrightarrow Y = X/R$$

be the quotient of X by this equivalence relation. We now define $\bar{p}\colon K^{\triangleright} \to \mathrm{Set}$ to be extension of $p\colon K \to \mathrm{Set}$, whose value at the cone point is Y, and whose value at the unique morphism from $s \in K_0$ to the cone point is the composition

$$X_s \xrightarrow{i_s} \textstyle\coprod_{i \in K_0} X_i \longrightarrow Y$$

of the inclusion of the sth summand and the canonical projection. We claim that the diagram $\bar{p}\colon K^{\triangleright} \to \mathrm{Set}$ is a colimit of $p\colon K \to \mathrm{Set}$. Indeed, let $f\colon K^{\triangleright} \to \mathrm{Set}$ be any diagram with $f|_K = p$, let Z be the value of f at the cone point, and for all $i \in K_0$, let $h_i\colon X_i \to Z$ be the map induced by the unique maps from i to the cone point, and let $h = \sum_{i \in K_0} h_i\colon X \to Z$. The map h defines the equivalence relation

$$Q = \{(x, y) \in X \times X \mid h(x) = h(y)\} \subset X \times X,$$

and $S \subset Q$, since $h_i = p(a) \circ h_j\colon X_j \to Z$ for every morphism $a\colon j \to i$ in K. But then $R \subset Q$, so $h\colon X \to Z$ factors through the canonical projection $X \to Y$. $\square$

Remark 2.16. There is a stark asymmetry between limits and colimits of diagrams of sets. If $\bar{p}\colon K^{\triangleleft} \to \mathrm{Set}$ is a limit of the diagram $p\colon K \to \mathrm{Set}$, then the set $\bar{p}(0)$ is quite explicitly given as the set of solutions to a system of equations. By contrast, if $\bar{p}\colon K^{\triangleright} \to \mathrm{Set}$ is a colimit of $p\colon K \to \mathrm{Set}$, then the set $\bar{p}(0)$ is all but unknowable in general. Indeed, while the diagram $p\colon K \to \mathrm{Set}$ gives a "concrete" description of the relation $S \subset X \times X$ in the proof of Proposition 2.15, it is in general very difficult to understand the equivalence relation $R \subset X \times X$ that it generates.

Exercise 2.17. First, let $p, q, r\colon K \to \mathcal{C}$ be functors, and let $\varphi\colon q \to p$ and $\psi\colon r \to q$ be natural transformations.

(a) Show that the family of morphisms

$$(\varphi_k \circ \psi_k)_{k \in K}$$

constitute a natural transformation $\varphi \cdot \psi\colon r \to p$.

The natural transformation $\varphi \cdot \psi\colon r \to p$ is called the vertical composition of the natural transformations $\varphi\colon q \to p$ and $\psi\colon r \to q$.

Next, let $p, q\colon J \to \mathcal{C}$ and $r, s\colon K \to J$ be functors, and let $\varphi\colon q \to p$ and $\psi\colon s \to r$ be natural transformations. (So these are different from the functors and natural transformations considered above.)

(b) Show that for all $k \in K$, the diagram

$$\begin{tikzcd}
(q \circ s)(k) \arrow[r, "q(\psi_k)"] \arrow[d, "\varphi_{s(k)}"] & (q \circ r)(k) \arrow[d, "\varphi_{r(k)}"] \\
(p \circ s)(k) \arrow[r, "p(\psi_k)"] & (p \circ r)(k)
\end{tikzcd}$$

commutes.

We let $(\varphi \circ \psi)_k \colon (q \circ s)(k) \to (p \circ r)(k)$ denote the common composition of the morphisms in diagram in (b).

(c) Show that the family of morphisms

$$((\varphi \circ \psi)_k)_{k \in K}$$

constitutes a natural transformation $\varphi \circ \psi \colon q \circ s \to p \circ r$.

The natural transformation $\varphi \circ \psi \colon q \circ s \to p \circ r$ is called the horizontal composition of $\varphi \colon q \to p$ and $\psi \colon s \to r$.

Remark 2.18. The following diagram

$$\begin{tikzcd} K \arrow[r, bend left=50, "r"{name=R}] \arrow[r, "q"{name=Q}] \arrow[r, bend right=50, "p"'{name=P}] & \mathcal{C} \arrow[from=R, to=Q, Rightarrow, "\psi"] \arrow[from=Q, to=P, Rightarrow, "\phi"] \end{tikzcd}$$

depicts the "vertical composition" in (a), whereas the diagram

$$\begin{tikzcd} K \arrow[r, bend left=40, "s"{name=S}] \arrow[r, bend right=40, "r"'{name=R}] & J \arrow[r, bend left=40, "q"{name=Q}] \arrow[r, bend right=40, "p"'{name=P}] & \mathcal{C} \arrow[from=S, to=R, Rightarrow, "\psi"] \arrow[from=Q, to=P, Rightarrow, "\phi"] \end{tikzcd}$$

depicts the "horizontal composition" in (b) and (c).

3. Adjoint functors

Let us recall the fundamental difference between set theory and category theory. In a set, it is a *property* of two elements x and y that equality $x = y$ holds, whereas in a category, we must provide the *structure* of a morphism $f\colon y \to x$ to compare two objects x and y. We are familiar with this phenomenon from linear algebra, where we must provide a k-linear map $f\colon W \to V$ in order to compare two k-vector spaces V and W. The structure encoded in the morphisms in a category gives rise to new phenomena that are very general and yet very powerful, and that have applications in all parts of mathematics. Adjunctions, which is the subject of this lecture, is such a phenomenon.

Definition 3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An adjunction from $\mathcal{D}$ to $\mathcal{C}$ is a quadruple (f, g, ϵ, η) of two functors $f\colon \mathcal{D} \to \mathcal{C}$ and $g\colon \mathcal{C} \to \mathcal{D}$ and two natural transformations $\epsilon\colon f \circ g \to \mathrm{id}_{\mathcal{C}}$ and $\eta\colon \mathrm{id}_{\mathcal{D}} \to g \circ f$ such that the diagrams of natural transformations

$$\begin{tikzcd} & f \circ g \circ f \arrow[dr, "\epsilon \circ f"] & \\ f \arrow[ur, "f \circ \eta"] \arrow[rr, equal] & & f \end{tikzcd} \qquad \begin{tikzcd} & g \circ f \circ g \arrow[dr, "g \circ \epsilon"] & \\ g \arrow[ur, "\eta \circ g"] \arrow[rr, equal] & & g \end{tikzcd}$$

commute.

Suppose that (f, g, ϵ, η) is an adjunction. We say that f is the left adjoint functor of the adjunction, that g is the right adjoint functor of the adjunction, that ϵ is the counit of the adjunction, and that is η the unit of the adjunction. We also express that the two diagrams in the definition of an adjunction commute by saying that the triangle identities hold.

Remark 3.2. Let us spell out what it means for the triangle identities to hold. The natural transformation $f \circ \eta$ is the horizontal composition of $\eta\colon \mathrm{id}_{\mathcal{D}} \to g \circ f$ and the identity natural transformation $\mathrm{id}_f\colon f \to f$, which, by abuse of notation, we simply denote by f. So, by definition, we have $(f \circ \eta)_y = f(\eta_y)$. Similarly, the natural transformation $\epsilon \circ f$ is the horizontal composition of the identity natural transformation $\mathrm{id}_f\colon f \to f$ and $\epsilon\colon f \circ g \to \mathrm{id}_{\mathcal{C}}$. So $(\epsilon \circ f)_y = \epsilon_{f(y)}$. Now, for the left-hand diagram to commute means that the vertical composition of $\epsilon \circ f$ and $f \circ \eta$ equal to id_f, which, in turn, means that for all $y \in \mathcal{D}$, the diagram

$$\begin{tikzcd} & f(g(f(y))) \arrow[dr, "\epsilon_{f(y)}"] & \\ f(y) \arrow[ur, "f(\eta_y)"] \arrow[rr, "\mathrm{id}_{f(y)}"] & & f(y) \end{tikzcd}$$

of objects and morphisms in $\mathcal{C}$ commutes. Similarly, for the right-hand diagram in Definition 3.1 to commute means that for all $x \in \mathcal{C}$, the diagram

$$\begin{tikzcd} & g(f(g(x))) \arrow[dr, "g(\epsilon_x)"] & \\ g(x) \arrow[ur, "\eta_{g(x)}"] \arrow[rr, "\mathrm{id}_{g(x)}"] & & g(x) \end{tikzcd}$$

of objects and morphisms in $\mathcal{D}$ commutes.

Example 3.3. Let Set be the category of small sets, and let Vect_k be the category of small k-vector spaces, where k is a field. We define an adjunction (f, g, ϵ, η) from Set to Vect_k as follows. The functor $f\colon \mathrm{Set} \to \mathrm{Vect}_k$ assigns to a set S the k-vector space $f(S) = (f(S), +, \cdot\,)$ with $f(S)$ the set consisting of maps $x\colon S \to k$, whose support[6] is finite, and with vector sum and scalar multiplication defined by

$$\begin{aligned}(x+y)(s) &= x(s) + y(s)\\ (x \cdot a)(s) &= x(s) \cdot a.\end{aligned}$$

Moreover, the functor $f\colon \mathrm{Set} \to \mathrm{Vect}_k$ assigns to a map $p\colon T \to S$ of sets the k-linear map $f(p)\colon f(T) \to f(S)$ of k-vector spaces defined by

$$f(p)(y)(s) = \sum_{t \in p^{-1}(s)} y(t),$$

where the sum is well-defined, because y has finite support. It is straightforward to check that the map $f(p)$ is k-linear. The functor $g\colon \mathrm{Vect}_k \to \mathrm{Set}$ assigns to a k-vector space $(V, +, \cdot\,)$ its "underlying" set $g(V, +, \cdot\,) = V$ and to a k-linear map $h\colon (W, +, \cdot\,) \to (V, +, \cdot\,)$ the same map $g(h) = h\colon W \to V$. The counit

$$(f \circ g)(V, +, \cdot) \xrightarrow{\epsilon_{(V,+,\,\cdot\,)}} (V, +, \cdot)$$

is the k-linear map that to a map $x\colon V \to k$ of finite support assigns the sum

$$\epsilon_{(V,+,\cdot\,)}(x) = \sum_{\boldsymbol{v} \in V} \boldsymbol{v} \cdot x(\boldsymbol{v}) \in V,$$

which is well-defined, because x has finite support, and the unit

$$S \xrightarrow{\eta_S} g(f(S))$$

is the map that to $s \in S$ assigns the δ-function $\eta_S(s) = \delta_s\colon S \to k$ defined by

$$\delta_s(t) = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{if } s \neq t. \end{cases}$$

Let us check that the triangle identities hold. We note that $(\delta_s)_{s \in S}$ is a basis of the k-vector space $f(S)$. Indeed, every $x \in f(S)$ can be written uniquely as

$$x = \sum_{s \in S} \delta_s \cdot x(s).$$

To verify the first triangle identity, we must prove that the composite map

$$f(S) \xrightarrow{f(\eta_S)} f(g(f(S))) \xrightarrow{\epsilon_{f(S)}} f(S)$$

is the identity map. We find that

$$\begin{aligned}(\epsilon_{f(S)} \circ f(\eta_S))(x) &= \sum_{y \in f(S)} y \cdot f(\eta_S)(x)(y) = \sum_{y \in f(S)} y \cdot \sum_{s \in \eta_S^{-1}(y)} x(s)\\ &= \sum_{s \in S} \eta_S(s) \cdot x(s) = \sum_{s \in S} \delta_s \cdot x(s) = x,\end{aligned}$$

as desired. Here the first two identities consist in spelling out the definitions, the third identity holds, because $\eta_S\colon S \to g(f(S))$ is injective, and the final identity

[6] The support of $x\colon S \to k$ is the subset $\mathrm{supp}(x) = \{s \in S \mid x(s) \neq 0\} \subset S$.

holds, because $(\delta_s)_{s\in S}$ is a basis of $f(S)$. Similarly, to verify the second triangle identity, we must prove that the composite map

$$g(V) \xrightarrow{\eta_{g(V)}} g(f(g(V))) \xrightarrow{g(\epsilon_V)} g(V),$$

where we abbreviate $V = (V, +, \cdot)$, is the identity map. So we calculate

$$(g(\epsilon_V) \circ \eta_{g(V)})(\boldsymbol{v}) = \sum_{\boldsymbol{w}\in V} \boldsymbol{w} \cdot \eta_{g(V)}(\boldsymbol{v})(\boldsymbol{w}) = \sum_{\boldsymbol{w}\in V} \boldsymbol{w} \cdot \delta_{\boldsymbol{v}}(\boldsymbol{w}) = \boldsymbol{v},$$

which shows what we wanted.

The example above illustrates one source of adjunctions (f, g, ϵ, η). We have a category $\mathcal{C}$, whose objects are pairs consisting of a set and some structure on that set, and whose morphisms are maps of sets that preserve the given structure. In the example, the structure is a structure of k-vector space. The right adjoint functor $g\colon \mathcal{C} \to \mathrm{Set}$ is given by forgetting the structure, so we call it a forgetful functor. It is faithful, because for maps to preserve structure is a property. The corresponding left adjoint functor $f\colon \mathrm{Set} \to \mathcal{C}$ assigns to a set S the *free* example of a set with the structure in question generated by the set S.

Proposition 3.4. *If (f, g, ϵ, η) is an adjunction from $\mathcal{D}$ to $\mathcal{C}$, then for all pairs (y, x) of an object in $\mathcal{D}$ and an object in $\mathcal{C}$, the maps*

$$\begin{tikzcd} \mathrm{Map}(f(y), x) \arrow[r, shift left, "\alpha_{(y,x)}"] & \mathrm{Map}(y, g(x)) \arrow[l, shift left, "\beta_{(y,x)}"] \end{tikzcd}$$

defined by $\alpha_{(y,x)}(a) = g(a) \circ \eta_y$ and $\beta_{(y,x)}(b) = \epsilon_x \circ f(b)$ are each other's inverses.

Proof. By definition, the morphism $\alpha_{(y,x)}(a)$ is the composite morphism

$$y \xrightarrow{\eta_y} g(f(y)) \xrightarrow{g(a)} g(x)$$

so the morphism $(\beta_{(y,x)} \circ \alpha_{(y,x)})(a)$ is the composition of the upper horizontal morphisms and right-hand vertical morphism in the following diagram.

$$\begin{tikzcd} f(y) \arrow[r, "(f\circ\eta)_y"] \arrow[dr, equal] & f(g(f(y))) \arrow[r, "f(g(a))"] \arrow[d, "\epsilon_{f(y)}"] & f(g(x)) \arrow[d, "\epsilon_x"] \\ & f(y) \arrow[r, "a"] & x \end{tikzcd}$$

But the left-hand triangle commutes by the triangle identities, and the right-hand square commutes by the naturality of ϵ. So we find that $(\beta_{(y,x)} \circ \alpha_{(y,x)})(a) = a$, as desired. Similarly, the morphism $\beta_{(y,x)}(b)$ is the composite morphism

$$f(y) \xrightarrow{f(b)} f(g(x)) \xrightarrow{\epsilon_x} x$$

so $(\alpha_{(y,x)} \circ \beta_{(y,x)})(b)$ is the composition of the left-hand vertical morphism and the lower horizontal morphisms in the following diagram.

$$\begin{tikzcd} y \arrow[r, "b"] \arrow[d, "\eta_y"] & g(x) \arrow[d, "\eta_{g(x)}"] \arrow[dr, equal] & \\ g(f(y)) \arrow[r, "g(f(b))"] & g(f(g(x))) \arrow[r, "g(\epsilon_x)"] & g(x) \end{tikzcd}$$

Again, the left-hand square commutes by the naturality of η, and the right-hand triangle commutes by the triangle identities, so $(\alpha_{(y,x)} \circ \beta_{(y,x)})(b) = b$. $\square$

Example 3.5. For the adjunction in Example 3.3, Proposition 3.4 gives a bijection

$$\mathrm{Map}(S, g(V, +, \cdot)) \xrightarrow{\beta} \mathrm{Map}(f(S), (V, +, \cdot))$$

that to a map $\boldsymbol{v}\colon S \to V$ assigns the k-linear map $\beta(\boldsymbol{v})\colon f(S) \to (V, +, \cdot)$ given by

$$\beta(\boldsymbol{v})(x) = \sum_{s \in S} \boldsymbol{v}(s) \cdot x(s).$$

We also say that a map $\boldsymbol{v}\colon S \to V$ is a family of vectors in V indexed by S, and we sometimes write $(\boldsymbol{v}_s)_{s \in S}$ instead of $\boldsymbol{v}\colon S \to V$. In linear algebra, we define three important properties of such families, namely, the property of being linearly independent, the property of being a generating family, and the property of being a basis. We can now state these definitions in a way that it is easy both to understand and to remember: A family $\boldsymbol{v}\colon S \to V$ is

(i) linearly independent, if the map $\beta(\boldsymbol{v})\colon f(S) \to (V, +, \cdot)$ is injective,
(ii) a generating family, if the map $\beta(\boldsymbol{v})\colon f(S) \to (V, +, \cdot)$ is surjective, and
(iii) a basis, if the map $\beta(\boldsymbol{v})\colon f(S) \to (V, +, \cdot)$ is bijective.

This also makes it clear why the correct definition of a basis of $(V, +, \cdot)$ is that it is a family of vectors in V that satisfies (iii) and not e.g. a subset of V with some property. Indeed, it is for families of vectors that we have the bijection β.

We proceed to discuss an extremely useful interaction between adjoints functors and limits and colimits. In preparation, we need a generalization of Proposition 3.4. If $f, g\colon K \to \mathcal{C}$ are functors with a common domain and a common target, then we write $\mathrm{Map}(f, g)$ for the set of natural transformations $\varphi\colon f \to g$.

Addendum 3.6. *Let (f, g, ϵ, η) be an adjunction from $\mathcal{D}$ to $\mathcal{C}$, and let $p\colon K \to \mathcal{D}$ and $q\colon K \to \mathcal{C}$ be functors. There are mutally inverse maps*

$$\begin{tikzcd} \mathrm{Map}(f \circ p, q) \arrow[r, shift left, "\alpha_{(p,q)}"] & \mathrm{Map}(p, g \circ q) \arrow[l, shift left, "\beta_{(p,q)}"] \end{tikzcd}$$

defined by $\alpha_{(p,q)}(\varphi) = (g \circ \varphi) \cdot (\eta \circ p)$ and $\beta_{(p,q)}(\psi) = (\epsilon \circ q) \cdot (f \circ \psi)$.

Proof. The diagram of functors and natural transformations

$$\begin{tikzcd} f \circ p \arrow[r, "f \circ \eta \circ p"] \arrow[dr, equal] & f \circ g \circ f \circ p \arrow[r, "f \circ g \circ \varphi"] \arrow[d, "\epsilon \circ f"] & f \circ g \circ q \arrow[d, "\epsilon"] \\ & f \circ p \arrow[r, "\varphi"] & q \end{tikzcd}$$

commutes and shows that $(\beta_{(p,q)} \circ \alpha_{(p,q)})(\varphi) = \varphi$. Similarly, the diagram

$$\begin{tikzcd} p \arrow[r, "\psi"] \arrow[d, "\eta \circ p"] & g \circ q \arrow[d, "\eta \circ g \circ q"] \arrow[dr, equal] & \\ g \circ f \circ p \arrow[r, "g \circ f \circ \psi"] & g \circ f \circ g \circ q \arrow[r, "g \circ \epsilon \circ q"] & g \circ q \end{tikzcd}$$

commutes and shows that $(\alpha_{(p,q)} \circ \beta_{(p,q)})(\psi) = \psi$. $\square$

We now prove that every left adjoint functor preserves colimits and that every right adjoint functor preserves limits.

Proposition 3.7. *Let (f, g, ϵ, η) be an adjunction from $\mathcal{D} \to \mathcal{C}$.*

(1) *Suppose that $\bar{p}\colon K^{\triangleright} \to \mathcal{D}$ is a colimit of $p\colon K \to \mathcal{D}$. In this situation, the composite functor $f \circ \bar{p}\colon K^{\triangleright} \to \mathcal{C}$ is a colimit of $f \circ p\colon K \to \mathcal{C}$.*
(2) *Suppose that $\bar{p}\colon K^{\triangleleft} \to \mathcal{C}$ is a limit of $p\colon K \to \mathcal{C}$. In this situation, the composite functor $g \circ \bar{p}\colon K^{\triangleleft} \to \mathcal{D}$ is a limit of $g \circ p\colon K \to \mathcal{D}$.*

Proof. We prove (1); the proof of (2) is analogous. Given a diagram $h\colon K^{\triangleright} \to \mathcal{C}$ such that $h|_K = f \circ p\colon K \to \mathcal{C}$, we must show that there exists a unique natural transformation $\varphi\colon f \circ \bar{p} \to h$ such that $\varphi|_K = f \circ p$. By Addendum 3.6, we may instead show that there exists a unique natural transformation $\psi\colon \bar{p} \to g \circ h$ such that $\psi|_K = \eta \circ p$. Now, since $K^{\triangleright} = K \star 1$ is obtained from K by adjoining the final object $0 \in 1$, there exists a unique exists a unique natural transformation

$$q \xrightarrow{\zeta} g \circ h$$

such that $q|_K = p$, $q|_1 = g \circ h|_1$, $\zeta|_K = \eta \circ p$, and $\zeta|_1 = g \circ h|_1$. Indeed, the only part that remains undefined is the value of the functor q on the unique morphism $a\colon i \to 0$ from an object $i \in K$ to the object $0 \in 1$. But in order that ζ be a natural transformation, the diagram

$$\begin{tikzcd}
q(i) = p(i) \arrow[r, "\zeta_i = (\eta \circ p)_i"] \arrow[d, "q(a)"] & (g \circ h)(i) = (g \circ f \circ p)(i) \arrow[d, "(g \circ h)(a)"] \\
q(0) = (g \circ h)(0) \arrow[r, "\zeta_0 = \mathrm{id}_{(g \circ h)(0)}"] & (g \circ h)(0)
\end{tikzcd}$$

must commute, so this determines $q(a)\colon q(i) \to q(0)$ uniquely. Since $\bar{p}\colon K^{\triangleright} \to \mathcal{D}$ is a colimit of $p\colon K \to \mathcal{D}$, there exists a unique natural transformation $\xi\colon \bar{p} \to q$ such that $\xi|_K = p$. But then the composition of the natural transformations

$$\bar{p} \xrightarrow{\xi} q \xrightarrow{\zeta} g \circ h$$

is the desired unique natural transformation $\psi\colon \bar{p} \to g \circ h$ such that $\psi|_K = \eta \circ p$. $\square$

Example 3.8. Let (f, g, ϵ, η) be the adjunction from Set to Vect_k that we defined in Example 3.3. Given a small diagram $p\colon K \to \mathrm{Set}$, we proved in Proposition 2.15 that it admits a colimit $\bar{p}\colon K^{\triangleright} \to \mathrm{Set}$. So it follows from Proposition 3.7 that $f \circ \bar{p}\colon K^{\triangleright} \to \mathrm{Vect}_k$ is a colimit of $f \circ p\colon K \to \mathrm{Vect}_k$.

Now, suppose that K is static. We claim that for every $q\colon K \to \mathrm{Vect}_k$, there exists a diagram of sets $p\colon K \to \mathrm{Set}$ and natural isomorphism

$$f \circ p \xrightarrow{\varphi} q.$$

Indeed, if we choose, for every $i \in K$, a basis $\psi_i\colon p(i) \to g(q(i))$ of the k-vector space $q(i)$, then, since K is static, the family of sets $(p(i))_{i \in K}$ determine a unique functor $p\colon K \to \mathrm{Set}$ and the family of maps $(\psi_i\colon p(i) \to g(q(i)))_{i \in K}$ determine a unique natural transformation $\psi\colon p \to g \circ q$. Moreover, by the definition of what it means to be a basis, which we recalled in Example 3.5, the adjunct natural transformation $\varphi = \alpha(\psi)\colon f \circ p \to q$ is a natural isomorphism. This proves the claim.

Let $\bar{p}\colon K^{\triangleright} \to \mathrm{Set}$ be a coproduct of $p\colon K \to \mathrm{Set}$, so that $f \circ \bar{p}\colon K^{\triangleright} \to \mathrm{Vect}_k$ is a coproduct of $f \circ p\colon K \to \mathrm{Vect}_k$. If we choose a natural isomorphism

$$f \circ \bar{p} \xrightarrow{\bar{\varphi}} \bar{q}$$

such that $\bar{\varphi}|_K = \varphi$, then $\bar{q}\colon K^{\triangleright} \to \mathrm{Vect}_k$ is a coproduct of $q\colon K \to \mathrm{Vect}_k$. In less precise terms, we may restate this conclusion by saying that the canonical map

$$\textstyle\bigoplus_{i \in K} f(S_i) \longrightarrow f(\coprod_{i \in K} S_i)$$

is an isomorphism. Note also that the forgetful functor $g\colon \mathrm{Vect}_k \to \mathrm{Set}$ does not preserve colimits, since it does not preserve (binary) coproducts. Therefore, we conclude from Proposition 3.7 that g does not admit a right adjoint.

If K is not static, then not every diagram $q\colon K \to \mathrm{Vect}_k$ is naturally isomorphic to a diagram of the form $f \circ p\colon K \to \mathrm{Vect}_k$ for some $p\colon K \to \mathrm{Vect}_k$. For example, if G is a group, then a diagram $q\colon BG \to \mathrm{Vect}_k$ determines and is determined by a k-linear representation (V, π) of G, and $q \simeq f \circ p$ if and only if the k-linear representation (V, π) is a permutation representation.

The data of an adjunction (f, g, ϵ, η) is redundant. In particular, as the following result shows, it suffices to specify the value of one of the two functors f and g on objects, only. The requirement that it be part of an adjunction then automatically determines its value on morphisms.[7]

Proposition 3.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories.*

(1) *Let $f\colon \mathcal{D} \to \mathcal{C}$ be a functor and suppose given a map $g_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ and a family of morphisms $(\epsilon_x\colon f(g_0(x)) \to x)_{x \in \mathcal{C}_0}$ such that for every $x \in \mathcal{C}_0$ and $y \in \mathcal{D}_0$, the composite map*

$$\mathrm{Map}(y, g_0(x)) \xrightarrow{f} \mathrm{Map}(f(y), f(g_0(x))) \xrightarrow{\epsilon_x \circ (-)} \mathrm{Map}(f(y), x)$$

is a bijection. In this case, there exists a unique adjunction (f, g, ϵ, η) from $\mathcal{D}$ to $\mathcal{C}$ such that $g(x) = g_0(x)$ for all $x \in \mathcal{C}_0$ and such that $\epsilon = (\epsilon_x)_{x \in \mathcal{C}_0}$.

(2) *Let $g\colon \mathcal{C} \to \mathcal{D}$ be a functor and suppose given a map $f_0\colon \mathcal{D}_0 \to \mathcal{C}_0$ and a family of morphisms $(\eta_y\colon y \to g(f_0(y)))_{y \in \mathcal{D}_0}$ such that for every $y \in \mathcal{D}_0$ and $x \in \mathcal{C}_0$, the composite map*

$$\mathrm{Map}(f_0(y), x) \xrightarrow{g} \mathrm{Map}(g(f_0(y)), g(x)) \xrightarrow{(-) \circ \eta_y} \mathrm{Map}(y, g(x))$$

is a bijection. In this case, there exists a unique adjunction (f, g, ϵ, η) from $\mathcal{D}$ to $\mathcal{C}$ such that $f(y) = f_0(y)$ for all $y \in \mathcal{D}_0$ and such that $\eta = (\eta_y)_{y \in \mathcal{D}_0}$.

Proof. We sketch a proof of (1); the proof of (2) is analogous. We claim that there is a unique functor $g\colon \mathcal{C} \to \mathcal{D}$ such that the map of object sets is the given map $g_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ and such that $\epsilon\colon f \circ g \to \mathrm{id}_{\mathcal{C}}$ is a natural transformation. Indeed, if $a\colon y \to x$ is a morphism in $\mathcal{C}$, then for $\epsilon\colon f \circ g \to \mathrm{id}_{\mathcal{C}}$ to be a natural transformation,

[7] The analogous result for adjunctions between ∞-categories also holds, and it is one of the main tools available in ∞-category theory for defining functors.

the desired morphism $g(a)\colon g(y) \to g(x)$ must make the diagram

$$\begin{tikzcd}
\mathrm{Map}(g(y), g(y)) \arrow[r, "f"] \arrow[d, "g(a)\circ(-)"] & \mathrm{Map}(f(g(y)), f(g(y))) \arrow[r, "\epsilon_y\circ(-)"] \arrow[d, "f(g(a))\circ(-)"] & \mathrm{Map}(f(g(y)), y) \arrow[d, "a\circ(-)"] \\
\mathrm{Map}(g(y), g(x)) \arrow[r, "f"] & \mathrm{Map}(f(g(y)), f(g(x))) \arrow[r, "\epsilon_x\circ(-)"] & \mathrm{Map}(f(g(y)), x)
\end{tikzcd}$$

commute. But the composition of the maps in the top and bottom rows are assumed to be bijections, so there is a unique morphism $g(a)\colon g(y) \to g(x)$ with this property, namely, the unique solution to the equation

$$\epsilon_x(f(g(a))) = a \circ \epsilon_y(f(\mathrm{id}_{g(y)})).$$

We leave it as an exercise to show that, with this definition, $g\colon \mathcal{C} \to \mathcal{D}$ is a functor and $\epsilon\colon f \circ g \to \mathrm{id}_{\mathcal{C}}$ is a natural transformation. It remains to define the unit natural transformation $\eta\colon \mathrm{id}_{\mathcal{D}} \to g \circ f$. But in order to ensure that the triangle identities hold, we are forced to define $\eta_y\colon y \to g(f(y))$ to be the unique morphism that is mapped to $\mathrm{id}_{f(y)}\colon f(y) \to f(y)$ by the composite map

$$\begin{tikzcd}
\mathrm{Map}(y, g(f(y))) \arrow[r, "f"] & \mathrm{Map}(f(y), f(g(f(y)))) \arrow[r, "\epsilon_{f(y)}\circ(-)"] & \mathrm{Map}(f(y), f(y)).
\end{tikzcd}$$

We leave it as an exercise to check that η is a natural transformation and that the triangle identities are hold. $\square$

Exercise 3.10. We consider categories and functors

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift left, "f"] \arrow[d, shift right, "h"'] & \mathcal{B} \arrow[l, shift left, "f'"] \arrow[d, shift right, "g"'] \\
\mathcal{C} \arrow[r, shift left, "k"] \arrow[u, shift right, "h'"'] & \mathcal{D} \arrow[l, shift left, "k'"] \arrow[u, shift right, "g'"']
\end{tikzcd}$$

and assume that f (resp. g, resp. h, resp, k) is left adjoint to f' (resp. g', resp. h', resp. k') and that counits and units for these adjunctions have been chosen. Prove the following statements:

(a) A natural transformation $\varphi\colon g \circ f \to k \circ h$ determines and is determined by a natural transformation $\varphi'\colon h' \circ k' \to f' \circ g'$.

(b) A natural transformation $\varphi\colon g \circ f \to k \circ h$ is a natural isomorphism if and only if the corresponding natural transformation $\varphi'\colon h' \circ k' \to f' \circ g'$ is a natural isomorphism.

Exercise 3.11. Let K and $\mathcal{C}$ be categories with K small, and let $\mathrm{Fun}(K, \mathcal{C})$ be the category, whose objects are the functors $f\colon K \to \mathcal{C}$, and whose morphisms are the natural transformations between such functors. Let

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta"] & \mathrm{Fun}(K, \mathcal{C})
\end{tikzcd}$$

be the "diagonal" functor that to an object x of $\mathcal{C}$ assigns the constant functor $\Delta(x)\colon K \to \mathcal{C}$ with value x, and that to a morphism $f\colon y \to x$ in $\mathcal{C}$ assigns the natural transformation $\Delta(f)\colon \Delta(y) \to \Delta(x)$ with $\Delta(f)_i = f$ for all $i \in K_0$.

(a) Suppose that every $p\colon K \to \mathcal{C}$ admits a limit $\bar{p}\colon K^{\triangleleft} \to \mathcal{C}$. Show that there is an adjunction $(\Delta, g, \epsilon, \eta)$ from $\mathcal{C}$ to $\mathrm{Fun}(K, \mathcal{C})$ such that

$$g(p) = \bar{p}(0)$$

and such that $\epsilon_p\colon (\Delta \circ g)(p) \to p$ is the natural transformation, whose value $\epsilon_{p,i}\colon (\Delta \circ g)(p)(i) = \bar{p}(0) \to p(i)$ at $i \in K_0$ is the image by $\bar{p}$ of the unique morphism $0 \to i$ in $K^{\triangleleft}$.

(b) Suppose that every $p\colon K \to \mathcal{C}$ admits a colimit $\bar{p}\colon K^{\triangleright} \to \mathcal{C}$. Show that there is an adjunction $(f, \Delta, \epsilon, \eta)$ from $\mathrm{Fun}(K, \mathcal{C})$ to $\mathcal{C}$ such that

$$f(p) = \bar{p}(0)$$

and such that $\eta_p\colon p \to (\Delta \circ f)(p)$ is the natural transformation, whose value $\eta_{p,i}\colon p(i) \to p(0) = (\Delta \circ f)(p)(i)$ at $i \in K_0$ is the image by $\bar{p}$ of the unique morphism $i \to 0$ in $K^{\triangleright}$.

[Hint: Use Proposition 3.9.]

We have defined limits and colimits "pointwise" which is to say one diagram at a time. Exercise 3.11 promotes these definitions to functors.

Exercise 3.12. Given a category K, we define the category $K^{\triangleleft} = 1 \star K$ by formally adjoining an initial object 0 to K. Suppose that K already has an initial object i.

(a) Show that the initial object $i \in K_0$ determines and is determined by a limit

$$K^{\triangleleft} \xrightarrow{\overline{\mathrm{id}}_K} K$$

of the identity functor $\mathrm{id}_K\colon K \to K$.

(b) Show that the initial object $i \in K_0$ determines and is determined by a limit

$$K^{\triangleleft} \xrightarrow{\bar{p}} \mathcal{C}$$

for any K-indexed diagram $p\colon K \to \mathcal{C}$ with $\mathcal{C}$ any category.

If K admits a final object i, then the dual statements for colimits hold, as follows from the canonical equivalence $(K^{\triangleright})^{\mathrm{op}} \simeq (K^{\mathrm{op}})^{\triangleleft}$.

4. The Yoneda embedding

The Yoneda lemma is a very powerful result in category theory that, at the same time, is very easy to prove. For instance, it allows us to transfer definitions made in the category of sets to a general category.

Given categories K and $\mathcal{C}$ with K small, we define the category of presheaves

$$\mathcal{P}_{\mathcal{C}}(K) = \mathrm{Fun}(K^{\mathrm{op}}, \mathcal{C})$$

as follows. The set of objects $\mathcal{P}_{\mathcal{C}}(K)_0$ is the set of all functors $\mathcal{F}\colon K^{\mathrm{op}} \to \mathcal{C}$ and the set of morphisms $\mathrm{Map}(\mathcal{G}, \mathcal{F})$ for $\mathcal{F}, \mathcal{G} \in \mathcal{P}_{\mathcal{C}}(K)$ is the set of natural transformations $\varphi\colon \mathcal{G} \to \mathcal{F}$. Given $\mathcal{F} \in \mathcal{P}_{\mathcal{C}}(K)_0$, the identity morphism $\mathrm{id}_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{F}$ is the identity natural transformation, and given $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathcal{P}_{\mathcal{C}}(K)_0$, the composition

$$\mathrm{Map}(\mathcal{G}, \mathcal{F}) \times \mathrm{Map}(\mathcal{H}, \mathcal{G}) \xrightarrow{\circ} \mathrm{Map}(\mathcal{H}, \mathcal{G})$$

is the vertical composition of natural transformations. A functor $\mathcal{F}\colon K^{\mathrm{op}} \to \mathcal{C}$ is said to be a $\mathcal{C}$-valued presheaf on K and a natural transformation $\varphi\colon \mathcal{G} \to \mathcal{F}$ between such functors is said to be a map of $\mathcal{C}$-valued presheaves on K.

We will only consider $\mathcal{C} = \mathrm{Set}$ the category of small set, and write $\mathcal{P}(K)$ instead of $\mathcal{P}_{\mathrm{Set}}(K)$. We define the Yoneda embedding to be the functor

$$K \xrightarrow{h} \mathcal{P}(K)$$

given as follows. If $S \in K_0$, then $h(S)\colon K^{\mathrm{op}} \to \mathrm{Set}$ is the functor that to $T \in K_0^{\mathrm{op}}$ assigns the small set $h(S)(T) = \mathrm{Map}(T, S)$ of morphisms in K from T to S, and if $g\colon U \to T$ is a morphism in K, then $h(S)(g)\colon h(S)(T) \to h(S)(U)$ is the map that to $f\colon T \to S$ assigns $f \circ g\colon U \to S$. Finally, if $f\colon T \to S$ is a morphism in K, then $h(f)\colon h(T) \to h(S)$ is the natural transformation, whose value at $U \in K_0$ is the map $h(f)_U\colon h(T)(U) \to h(S)(U)$ that to $g\colon U \to T$ assigns $f \circ g\colon U \to S$.

Theorem 4.1 (Yoneda lemma)**.** *Let K be a small category. For every object S in K and every functor $\mathcal{F} \in K^{\mathrm{op}} \to \mathrm{Set}$, the map*

$$\mathrm{Map}(h(S), \mathcal{F}) \xrightarrow{\epsilon_{(\mathcal{F},S)}} \mathcal{F}(S)$$

that to $\varphi\colon h(S) \to \mathcal{F}$ assigns $\epsilon_{(\mathcal{F},S)}(\varphi) = \varphi_S(\mathrm{id}_S)$ is a bijection.

Proof. Suppose that $\varphi\colon h(S) \to \mathcal{F}$ is a natural transformation and that $f\colon T \to S$ is an element of $h(S)(T)$. On the one hand, the diagram

$$\begin{tikzcd} h(S)(S) \arrow[r, "\varphi_S"] \arrow[d, "h(S)(f)"] & \mathcal{F}(S) \arrow[d, "\mathcal{F}(f)"] \\ h(S)(T) \arrow[r, "\varphi_T"] & \mathcal{F}(T) \end{tikzcd}$$

commutes, by the naturality of φ, and shows that

$$\varphi_T(f) = (\varphi_T \circ h(f))(\mathrm{id}_S) = (\mathcal{F}(f) \circ \varphi_S)(\mathrm{id}_S) = \mathcal{F}(f)(\epsilon_{(\mathcal{F},S)}(\varphi)),$$

and on the other hand, given $a \in \mathcal{F}(S)$, the same formula $\varphi_T(f) = \mathcal{F}(f)(a)$ defines a natural transformation $\varphi\colon h(S) \to \mathcal{F}$. $\square$

Theorem 4.1 has the following corollary, which explains the "embedding" part of the name for the functor $h\colon K \to \mathcal{P}(K)$.

Corollary 4.2. *For every small category K, the Yoneda embedding*

$$K \xrightarrow{h} \mathcal{P}(K)$$

is fully faithful.

Proof. Indeed, by Theorem 4.1, for all $S, T \in K_0$, the maps

$$h(S)(T) = \mathrm{Map}(T, S) \underset{\epsilon_{(h(S),T)}}{\overset{h_{(S,T)}}{\rightleftarrows}} \mathrm{Map}(h(T), h(S))$$

are each other's inverses. □

Example 4.3. Let G be a group, and let BG be the category with a single object 0 and $\mathrm{Map}(0,0) = G$. A functor $\mathcal{F} \colon BG^{\mathrm{op}} \to \mathrm{Set}$ determines and is determined by the right G-set (X, ρ), where $X = \mathcal{F}(0)$, and where $\rho \colon G^{\mathrm{op}} \to \mathrm{Aut}(X)$ is the group homomorphism given by $\rho(g)(x) = \mathcal{F}(g) \colon X \to X$. In this situation, the functor

$$BG \xrightarrow{h} \mathcal{P}(BG)$$

takes the unique object 0 in BG to the right G-set (G, ρ) with $\rho(g)(f) = fg$, and it is given on morphism sets by the map

$$G = \mathrm{Map}(0,0) \xrightarrow{h_{(0,0)}} \mathrm{End}(G, \rho) = \mathrm{Map}(h(0), h(0))$$

defined by $h_{(0,0)}(f)(g) = fg$. By Corollary 4.2, the latter map is a bijection. From this, we learn two things. First, we have $\mathrm{End}(G, \rho) \subset \mathrm{End}(G)$, by definition, but we learn that, in fact, $\mathrm{End}(G, \rho) \subset \mathrm{Aut}(G)$. Second, we learn that

$$G \xrightarrow{\lambda} \mathrm{Aut}(G)$$

given by $\lambda(f)(g) = fg$ is injective. This shows that the group G is canonically isomorphic to the subgroup $\lambda(G) \subset \mathrm{Aut}(G)$ of the group $\mathrm{Aut}(G)$ of permutations of the set G, which is a classical theorem attributed to Cayley.

To state and prove the next result, it is convenient to introduce a bit of notation. Given an object S in K, we define

$$\mathcal{P}(K) \xrightarrow{\Gamma(S,-)} \mathrm{Set}$$

to be the functor that takes a presheaf $\mathcal{F}$ to the set $\Gamma(S, \mathcal{F}) = \mathcal{F}(S)$ and that takes a map of presheaves $\varphi \colon \mathcal{G} \to \mathcal{F}$ to the map $\Gamma(S, \varphi) = \varphi_S \colon \mathcal{G}(S) \to \mathcal{F}(S)$, and given a morphism $f \colon T \to S$ in K, we define a natural transformation

$$\Gamma(S, -) \xrightarrow{\Gamma(f,-)} \Gamma(T, -),$$

whose value at $\mathcal{F}$ is the map $\Gamma(f, \mathcal{F}) = \mathcal{F}(f) \colon \mathcal{F}(S) \to \mathcal{F}(T)$. Note that the natural transformation $\Gamma(f, -)$ goes in the opposite direction of the morphism f. We say that $\Gamma(S, \mathcal{F})$ is the set of sections of the presheaf $\mathcal{F}$ over S and that $\Gamma(f, \mathcal{F})$ is the restriction along f.

Proposition 4.4. *Let K be a small category.*

(1) *A small diagram $\bar{p} \colon J^{\triangleright} \to \mathcal{P}(K)$ is a colimit of $p = \bar{p} \circ i \colon J \to \mathcal{P}(K)$ if and only if $\Gamma(S, \bar{p}) \colon J^{\triangleright} \to \mathrm{Set}$ is a colimit of $\Gamma(S, p) \colon J \to \mathrm{Set}$ for all $S \in K_0$.*

(2) *A small diagram* $\bar{p}\colon J^{\triangleleft} \to \mathcal{P}(K)$ *is a limit of* $p = \bar{p} \circ i\colon J \to \mathcal{P}(K)$ *if and only if* $\Gamma(S, \bar{p})\colon J^{\triangleleft} \to \mathrm{Set}$ *is a limit of* $\Gamma(S, p)\colon J \to \mathrm{Set}$ *for all* $S \in K_0$.
(3) *The Yoneda embedding* $h\colon K \to \mathcal{P}(K)$ *preserves all small limits that exist in* K.

In particular, the category $\mathcal{P}(K)$ *admits all small colimits and limits.*

Proof. We first prove (1); the proof of (2) is analogous. So let $p\colon J \to \mathcal{P}(K)$ be a small diagram. For every $S \in K_0$, the composite functor $\Gamma(S, p)\colon J \to \mathrm{Set}$ is a small diagram of sets, so it admits a colimit $\overline{\Gamma(S, p)}\colon J^{\triangleright} \to \mathrm{Set}$. We wish to show that there exists a unique functor $\bar{p}\colon J^{\triangleright} \to \mathcal{P}(K)$ such that $\Gamma(S, \bar{p}) = \overline{\Gamma(S, p)}$ for all $S \in K_0$, and that $\bar{p}$ is a colimit of p. Since $\overline{\Gamma(S, p)}$ is a colimit of $\Gamma(S, p)$, there is, for every morphism $f\colon T \to S$ in K, a unique natural transformation

$$\overline{\Gamma(S, p)} \xrightarrow{\overline{\Gamma(f,p)}} \overline{\Gamma(T, p)}$$

whose restriction along $i\colon J \to J^{\triangleright}$ is $\Gamma(f, p)\colon \Gamma(S, p) \to \Gamma(T, p)$. The uniqueness of these natural transformations implies that for every $S \in K_0$,

$$\overline{\Gamma(\mathrm{id}_S, p)} = \mathrm{id}_{\overline{\Gamma(S,p)}},$$

and that for every pair $(f\colon T \to S, g\colon U \to T)$ of composable morphism in K,

$$\overline{\Gamma(f \circ g, p)} = \overline{\Gamma(g, p)} \circ \overline{\Gamma(f, p)}.$$

So there is a unique functor $\bar{p}\colon J^{\triangleright} \to \mathcal{P}(K)$ such that $\Gamma(S, \bar{p}) = \overline{\Gamma(S, p)}$ for every object S in K and such that $\Gamma(f, \bar{p}) = \overline{\Gamma(f, p)}$ for every morphism $f\colon T \to S$ in K. We must verify that $\bar{p}$ is indeed a colimit of p. By definition, the restriction of $\bar{p}\colon J^{\triangleright} \to \mathcal{P}(K)$ along $i\colon J \to J^{\triangleright}$ is equal to $p\colon J \to \mathcal{P}(K)$, and if $q\colon J^{\triangleright} \to \mathcal{P}(K)$ is another functor with this property, then for every $S \in K_0$, there is a unique natural transformation $\varphi_S\colon \Gamma(S, \bar{p}) \to \Gamma(S, q)$ such that $\varphi_S \circ i = \Gamma(S, p)$, and by the uniqueness of these natural transformations, we conclude that there is a unique natural transformation $\varphi\colon \bar{p} \to q$ such that $\Gamma(S, \varphi) = \varphi_S$ for all objects S in K. But this unique natural transformation satisfies $\varphi \circ i = p$, which shows that $\bar{p}\colon J^{\triangleright} \to \mathcal{P}(K)$ is a colimit of $p\colon J \to \mathcal{P}(K)$. This proves (1).

To prove (3), let $\bar{p}\colon J^{\triangleleft} \to K$ be a limit of $p\colon J \to K$. By definition of the Yoneda embedding, we have $\Gamma(S, h \circ p) = \mathrm{Map}(S, p)$ and $\Gamma(S, h \circ \bar{p}) = \mathrm{Map}(S, \bar{p})$ as functor from J and $J^{\triangleleft}$, respectively, to Set. Since $\bar{p}$ is a limit of p, it follows from the definition of limit that $\mathrm{Map}(S, \bar{p})$ is a limit of $\mathrm{Map}(S, p)$ for all $S \in K_0$. So by (2), we conclude that $h \circ \bar{p}\colon J^{\triangleleft} \to \mathcal{P}(K)$ is a limit of $h \circ p\colon J \to \mathcal{P}(K)$ as stated. □

Example 4.5. Let K be a small category and suppose that K admits finite products. We define a group object in K to be a pair (G, μ) of an object G and a morphism $\mu\colon G \times G \to G$ such that the pair $(\mathrm{Map}(S, G), \mathrm{Map}(S, \mu))$ is a group for every $S \in K_0$. For example, the category Mfd^{ω} of compact smooth manifolds and smooth maps is essentially small[8] and admits finite products. A group object in Mfd^{ω} is called a compact Lie group.

[8] A category is defined to be essentially small if it is equivalent to a small category.

Example 4.6. Let K be a small category and suppose that K admits finite products. A morphism in K of the form $(s,t)\colon R \to X \times X$ is defined to be an equivalence relation if for every $S \in K_0$, the induced map of sets

$$\mathrm{Map}(S,R) \xrightarrow{(\mathrm{Map}(S,s),\mathrm{Map}(S,t))} \mathrm{Map}(S,X) \times \mathrm{Map}(S,X)$$

is an equivalence relation. This means that it is injective, and that its image is symmetric, reflexive, and transitive.

The Yoneda embedding $h\colon K \to \mathcal{P}(K)$ does not preserve colimits that might exist in K. Instead, we will now prove that $h\colon K \to \mathcal{P}(K)$ exhibits $\mathcal{P}(K)$ as the category freely generated by K under small colimits. We first prove two lemmas.

Lemma 4.7. *Let $\mathcal{C}$ be a category. A morphism $f\colon y \to x$ in $\mathcal{C}$ is an isomorphism if and only if for all object z in $\mathcal{C}$, the induced map of sets*

$$\mathrm{Map}(x,z) \xrightarrow{\mathrm{Map}(f,z)} \mathrm{Map}(y,z)$$

is a bijection.

Proof. If $\mathcal{C}$ is small, then this follows from the Yoneda lemma applied to $\mathcal{C}^{\mathrm{op}}$. But let us give a direct proof. The "only if" part of the statement is clear, and to prove the "if" part, we let $z = y$, and define $g\colon x \to y$ to be the unique morphism in $\mathcal{C}$ such that $\mathrm{Map}(f,y) = \mathrm{id}_y$, or equivalently, such that $g \circ f = \mathrm{id}_y$. It remains to show that also $f \circ g = \mathrm{id}_x$. This is an equality in $\mathrm{Map}(x,x)$, which, by assumption, is equivalent to the equality $f \circ g \circ f = f$ in $\mathrm{Map}(y,x)$. But this equality holds, since $g \circ f = \mathrm{id}_y$, so we conclude that $f \circ g = \mathrm{id}_x$, as desired. $\square$

To state the next lemma, let K be a small category, and let $h\colon K \to \mathcal{P}(K)$ be the Yoneda embedding. Given an object $\mathcal{F}$ of $\mathcal{P}(K)$, we define the slice category $K_{/\mathcal{F}}$ as follows. The object set is the set of pairs (S,φ), where S is an object of K, and where $\varphi\colon h(S) \to \mathcal{F}$ is a morphism in $\mathcal{P}(K)$. A morphism from (T,ψ) to (S,φ) is a morphism $f\colon T \to S$ in K with the property that $\psi = \varphi \circ h(f)$. We claim that $K_{/\mathcal{F}}$ is essentially small. Indeed, let $K'_{/\mathcal{F}}$ be the category, where an object is a pair (S,x) of an object S in K and an element $x \in \mathcal{F}(S)$, and where a morphism $f\colon (T,y) \to (S,x)$ is a morphism $f\colon T \to S$ in K such that $\mathcal{F}(f)(x) = y$. It is a small category, since K is small and since the sets $\mathcal{F}(S)$ are all small. Moreover, the Yoneda lemma shows that the functor

$$K_{/\mathcal{F}} \xrightarrow{e} K'_{/\mathcal{F}}$$

that to (S,φ) assigns $(S,\varphi_S(\mathrm{id}_S))$ and that to $f\colon (T,\psi) \to (S,\varphi)$ assigns the same map $f\colon (T,\psi_T(\mathrm{id}_T)) \to (S,\varphi_S(\mathrm{id}_S))$ is an equivalence. So $K_{/\mathcal{F}}$ is essentially small, as claimed. The category $K'_{/\mathcal{F}}$ is called the category of elements in $\mathcal{F}$.

We define $q\colon K_{/\mathcal{F}} \to K$ to be the functor that, on object sets, takes (S,φ) to S, and that, on morphism sets, takes $f\colon (T,\psi) \to (S,\varphi)$ to $f\colon T \to S$. It is a faithful functor that, typically, is not full.

Lemma 4.8. *In the situation above, a colimit of $p = h \circ q\colon K_{/\mathcal{F}} \to \mathcal{P}(K)$ is given by the functor $\bar{p}\colon (K_{/\mathcal{F}})^{\triangleright} \to \mathcal{P}(K)$, whose restriction to $K_{/\mathcal{F}}$ is p, whose value at the cone point 0 is $\mathcal{F}$, and whose value at the unique morphism from (S,φ) to 0 is the morphism $\varphi\colon h(S) \to \mathcal{F}$.*

Proof. By lemma 4.7, it suffices to show that for every object $\mathcal{G}$ of $\mathcal{P}(K)$,

$$((K_{/\mathcal{F}})^{\mathrm{op}})^{\triangleleft} = ((K_{/\mathcal{F}})^{\triangleright})^{\mathrm{op}} \xrightarrow{\mathrm{Map}(\bar{p},\mathcal{G})} \mathrm{Set}$$

is a limit of $\mathrm{Map}(p, \mathcal{G})\colon (K_{/\mathcal{F}})^{\mathrm{op}} \to \mathrm{Set}$. As we now explain, this is a consequence of the Yoneda lemma and the definitions. We replace $K_{/\mathcal{F}}$ with the equivalent small category $K'_{/\mathcal{F}}$ of elements in $\mathcal{F}$. There is a natural transformation

$$\mathrm{Map}(p, \mathcal{G}) \xrightarrow{\alpha} \mathcal{G} \circ q,$$

where $\alpha_{(S,x)}\colon \mathrm{Map}(p, \mathcal{G})(S, x) \to \mathcal{G}(S)$ is the map that to $\psi\colon p(S, x) = h(S) \to \mathcal{G}$ assigns $\psi_S(\mathrm{id}_S) \in \mathcal{G}(S)$. The Yoneda lemma shows that this natural transformation is a natural isomorphism. Moreover, by definition,

$$\mathrm{Map}(\bar{p}, \mathcal{G})(0) = \mathrm{Map}(\mathcal{F}, \mathcal{G})$$

is the set of natural transformations from $\xi\colon \mathcal{F} \to \mathcal{G}$, and the composition

$$\mathrm{Map}(\bar{p}, \mathcal{G})(0) \longrightarrow \mathrm{Map}(\bar{p}, \mathcal{G})(S, x) \xrightarrow{\alpha_{(S,x)}} \mathcal{G}(S)$$

of the map induced by the unique morphism to the cone point and the bijection $\alpha_{(S,x)}$ takes ξ to $\xi_S(x) \in \mathcal{G}(S)$. In order that the diagram $\mathrm{Map}(\bar{p}, \mathcal{G})$ of sets be a limit of the diagram $\mathrm{Map}(p, \mathcal{G})$ of sets, the family $(\xi_S(x))_{(S,x)}$ is required to have the property that for every morphism $f\colon (T, y) \to (S, x)$ in $K'_{/\mathcal{F}}$, the map

$$\mathcal{G}(S) \xrightarrow{\mathcal{G}(f)} \mathcal{G}(T)$$

takes $\xi_S(x)$ to $\xi_T(y) = \xi_T(f(x))$. But this is precisely the definition of what it means for $\xi\colon \mathcal{F} \to \mathcal{G}$ to be a natural transformation. $\square$

A presheaf $\mathcal{F}$ on a small category K is defined to be representable if it belongs to the essential image of the Yoneda embedding $h\colon K \to \mathcal{P}(K)$, that is, if there exists and object S in K and an isomorphism $\varphi\colon h(S) \to \mathcal{F}$ in $\mathcal{P}(K)$. Thus, stated in less precise terms, Lemma 4.8 shows that every presheaf on K is a small colimit of representable presheaves on K.

Proposition 4.9. *If K is a small category, and if $\mathcal{C}$ is a category that admits small colimits, then the Yoneda embedding induces an equivalence*

$$\mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(K), \mathcal{C}) \subset \mathrm{Fun}(\mathcal{P}(K), \mathcal{C}) \xrightarrow{\mathrm{Fun}(h,\mathcal{C})} \mathrm{Fun}(K, \mathcal{C})$$

from the full subcategory spanned by the functors $F\colon \mathcal{P}(K) \to \mathcal{C}$ that preserve small colimits to the category of functors $f\colon K \to \mathcal{C}$.

Proof. We first use Proposition 3.9 to produce a functor

$$\mathrm{Fun}(K, \mathcal{C}) \xrightarrow{h_!} \mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(K), \mathcal{C}).$$

that is left adjoint to the composite functor h^* in the statement. Given a functor $f\colon K \to \mathcal{C}$, we must define a colimit-preserving functor $h_!(f)\colon \mathcal{P}(K) \to \mathcal{C}$ and a natural transformation $\eta_f\colon f \to h^*h_!(f)$ such that the map $\alpha_{(f,G)}$ given by

$$\mathrm{Map}(h_!(f), G) \xrightarrow{h^*} \mathrm{Map}(h^*h_!(f), h^*(G)) \xrightarrow{-\circ\eta_f} \mathrm{Map}(f, h^*(G))$$

is a bijection for every colimit-preserving functor $G\colon \mathcal{P}(K) \to \mathcal{C}$. Let $q_{\mathcal{F}}\colon K_{/\mathcal{F}} \to K$ be the canonical projection. We proved in Lemma 4.8 that the composite diagram $p_{\mathcal{F}} = h \circ q_{\mathcal{F}}\colon K_{/\mathcal{F}} \to \mathcal{P}(K)$ admits the colimit $\bar{p}_{\mathcal{F}}\colon K^{\triangleright}_{/\mathcal{F}} \to \mathcal{P}(K)$ with $\bar{p}_{\mathcal{F}}(0) = \mathcal{F}$. By the assumption that the category $\mathcal{C}$ admits small colimits, we may choose a colimit $\bar{r}_{\mathcal{F}}\colon K^{\triangleright}_{/\mathcal{F}} \to \mathcal{C}$ of the composite diagram $r_{\mathcal{F}} = f \circ q_{\mathcal{F}}\colon K_{/\mathcal{F}} \to \mathcal{C}$ and define

$$h_!(f)(\mathcal{F}) = \bar{r}_{\mathcal{F}}(0).$$

Moreover, if $\mathcal{F} = h(S)$, then we define the morphism

$$f(S) \xrightarrow{\eta_{f,S}} h_!(f)(h(S)) = h^*h_!(f)(S)$$

to be the image by $\bar{r}_{h(S)}\colon (K_{/h(S)})^{\triangleright} \to \mathcal{C}$ of the unique morphism from $(S, \mathrm{id}_{h(S)})$ to the cone point 0. We note that $\eta_{f,S}$ is in fact an isomorphism, because $(S, \mathrm{id}_{h(S)})$ is a final object in $K_{/h(S)}$. The defining universal property of the colimit implies that the family $(\eta_{f,S})_{S \in K_0}$ is a natural transformation $\eta_f\colon f \to h^*h_!(f)$, which, in fact, is a natural isomorphism. To prove that the map $\alpha_{(f,G)}$ is a bijection, we define an inverse map $\beta_{(f,G)}$. By definition, if $\bar{\varphi}\colon h_!(f) \to G$ is a natural transformation, then, at S in K, $\varphi = \alpha_{(f,G)}(\bar{\varphi})\colon f \to h^*(G)$ is given by the composite morphism

$$f(S) \xrightarrow{\eta_{f,S}} h_!(f)(h(S)) \xrightarrow{\bar{\varphi}_{h(S)}} G(h(S)).$$

Conversely, a natural transformation $\varphi\colon f \to h^*(G)$ induces

$$r_{\mathcal{F}} = f \circ q_{\mathcal{F}} \xrightarrow{\varphi \circ q_{\mathcal{F}}} G \circ h \circ q_{\mathcal{F}} = G \circ p_{\mathcal{F}}$$

which extends uniquely to a natural transformation between their colimits

$$\bar{r}_{\mathcal{F}} \xrightarrow{\overline{\varphi \circ q}_{\mathcal{F}}} G \circ \bar{p}_{\mathcal{F}}.$$

Its value at the cone point 0 is a morphism

$$h_!(f)(\mathcal{F}) \xrightarrow{\bar{\varphi}_{\mathcal{F}}} G(\mathcal{F})$$

that we take as our definition $\bar{\varphi}_{\mathcal{F}} = \beta_{(f,G)}(\varphi)_{\mathcal{F}}$. The universal property of colimits implies that the family $(\bar{\varphi}_{\mathcal{F}})$ is a natural transformation $\bar{\varphi} = \beta_{(f,G)}(\varphi)$. So we conclude from Proposition 3.9 that there is an adjunction $(h_!, h^*, \epsilon, \eta)$ from $\mathrm{Fun}(K, \mathcal{C})$ to $\mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(K), \mathcal{C})$ with $h_!(f)$ and $\eta_f\colon f \to h^*h_!(f)$ as above.

We have already seen that η is a natural isomorphism, so it remains only to prove that the same is true for ϵ. By definition, $\epsilon_G\colon h_!h^*(G) \to G$ is the image by the map $\beta_{(h^*(G),G)}$ of $\mathrm{id}_{h^*(G)}\colon h^*(G) \to h^*(G)$, and its value at $\mathcal{F}$ is calculated as follows. The identity natural transformation

$$r_{\mathcal{F}} = h^*(G) \circ q_{\mathcal{F}} \xrightarrow{h^*(G) \circ q_{\mathcal{F}}} G \circ h \circ q_{\mathcal{F}} = G \circ p_{\mathcal{F}}$$

extends uniquely to a natural transformation between the chosen colimits

$$\bar{r}_{\mathcal{F}} \xrightarrow{\overline{h^*(G) \circ q}_{\mathcal{F}}} G \circ \bar{p}_{\mathcal{F}},$$

and the its value at the cone point 0 is the morphism

$$h_!h^*(G)(\mathcal{F}) \xrightarrow{\epsilon_{G,\mathcal{F}}} G(\mathcal{F}).$$

It is an isomorphism, since colimits are unique, up to unique isomorphism. □

The category $\mathcal{P}(K)$ is not small, unless K is the empty category, but we learn from Proposition 4.9 that it nevertheless controlled by the small category K.

Example 4.10. The simplest non-trivial case of Proposition 4.9 is, where $K = 1$ a static category with a single object 0, and $\Gamma(0, -)\colon \mathcal{P}(1) \to \mathrm{Set}$ is an equivalence. In this case, we learn that the functor

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Set}, \mathcal{C}) \longrightarrow \mathcal{C}.$$

given by evaluation at a singleton set $1 = \{0\}$ is an equivalence. Informally, this statement says that the category Set of small sets is freely generated under small colimits by the singleton set $1 = \{0\}$.

The category $\mathcal{P}(K)$ of presheaves on a small category K is an example of what is called a topos. Here is a general definition.

Definition 4.11 (Grothendieck)**.** A category $\mathcal{X}$ is a topos if there exists a small category K and a fully faithful embedding $\iota\colon \mathcal{X} \to \mathcal{P}(K)$ that admits a left adjoint functor $L\colon \mathcal{P}(K) \to \mathcal{X}$ that preserves finite limits.

We stress that to be a topos is a property of a category $\mathcal{X}$. In Grothendieck's philosophy, a topos is similar to the category of sets in all respects, except for one: The axiom of choice does not hold in a general topos. The axiom of choice is the statement that every epimorphism $p\colon Y \to X$ admits a section $s\colon X \to Y$. We use the axiom of choice to prove that free modules are projective. In a general topos, the corresponding statement fails and leads to the notion of cohomology.[9]

Example 4.12. Let us see that the axiom of choice fails in the topos

$$\mathcal{X} = \mathcal{P}(BG),$$

where G is any non-trivial group. We have earlier identified $\mathcal{X}$ with the category of right G-sets (X, ρ) and G-equivariant maps. Such a map $p\colon (Y, \sigma) \to (X, \rho)$ is an epimorphism if and only if the map $p\colon Y \to X$ is surjective. Let $(Y, \sigma) = (G, \sigma)$, where $\sigma\colon G^{\mathrm{op}} \to \mathrm{Aut}(G)$ is action by right multiplication, and let $(X, \rho) = (1, \rho)$, where $1 = \{0\}$ and $\rho\colon G^{\mathrm{op}} \to \mathrm{Aut}(1)$ is the unique map. The unique map

$$(Y, \sigma) \xrightarrow{\;p\;} (X, \rho)$$

is G-equivariant, but it does not admit a G-equivariant section. Indeed, such a map $s\colon (X, \rho) \to (Y, \sigma)$ would map $0 \in X$ to a point $s(0) \in Y$ that is fixed by the G-action. But the G-action on Y is free, so in particular, it there are no points in Y that are fixed by the G-action.

Exercise 4.13. Let $j\colon K' \to K$ be a functor between small categories, and let

$$\mathcal{P}(K) \xrightarrow{\;j^*\;} \mathcal{P}(K')$$

be the functor defined by $j^*(\mathcal{F})(S') = \mathcal{F}(j(S'))$ and $j^*(\varphi)_{S'} = \varphi_{j(S')}$.

(1) Show that j^* preserves small colimits.
(2) Show that j^* preserves small limits.

[9] Coherent cohomology on a scheme $(X, \mathcal{O}_X)$ is given by $H^i(X, \mathcal{F}) = \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$, so the rank 1 free $\mathcal{O}_X$-module $\mathcal{O}_X$ is not generally not projective!

Exercise 4.14. Let Grp^{ω} be the category of finitely generated[10] groups and group homomorphisms, and let $\mathcal{F}\colon \mathrm{Grp}^{\omega} \to \mathrm{Set}$ be the forgetful functor that to a group (G, μ) assigns the set G, and that to a group homomorphism $f\colon (G', \mu') \to (G, \mu)$ assigns the same map $f\colon G' \to G$. We view $\mathcal{F}$ as a presheaf on $K = (\mathrm{Grp}^{\omega})^{\mathrm{op}}$.

(1) Show that the forgetful functor $\mathcal{F}\colon \mathrm{Grp}^{\omega} \to \mathrm{Set}$ is representable.

Let $\mathcal{C}$ be a category, and let x be an object of $\mathcal{C}$. The composition of morphisms in $\mathcal{C}$ gives the set $\mathrm{End}(x) = \mathrm{Map}(x, x)$ the structure of a monoid.

(2) Describe the monoid $\mathrm{End}(\mathcal{F})$, where $\mathcal{F}\colon \mathrm{Grp}^{\omega} \to \mathrm{Set}$ is the forgetful functor.

[10] The only reason that we consider finitely generated groups instead of all groups is to not have to deal with set-theoretic issues.

5. Grothendieck's theorem

In this last lecture, we introduce two very important tools in category theory, namely, Kan extensions and Grothendieck's theorem that, in the category of sets, filtered colimits and finite limits commute. For example, every theorem concerning sheaves (in algebraic geometry and elsewhere) is obtained by an application of some (often clever) combination of these two tools.

Let K be a small category, and let $f\colon K \to 1$ be the unique functor to the final category 1, which is static and has a single object 0. We have seen that for any category $\mathcal{C}$, the functor f induces a "diagonal" functor

$$\mathcal{C} \simeq \mathcal{P}_{\mathcal{C}}(1) \xrightarrow{f^*} \mathcal{P}_{\mathcal{C}}(K) \simeq \mathrm{Fun}(K^{\mathrm{op}}, \mathcal{C}).$$

We have further seen that an adjunction $(f_!, f^*, \epsilon, \eta)$ determines and is determined by a colimit $\bar{\mathcal{F}}\colon (K^{\mathrm{op}})^{\triangleright} \to \mathcal{C}$ of every diagram $\mathcal{F}\colon K^{\mathrm{op}} \to \mathcal{C}$, and similarly, that an adjunction $(f^*, f_*, \epsilon, \eta)$ determines and is determined by a limit $\bar{\mathcal{F}}\colon (K^{\mathrm{op}})^{\triangleleft} \to \mathcal{C}$ for every diagram $\mathcal{F}\colon K^{\mathrm{op}} \to \mathcal{C}$.

The Kan extensions concern the "in families" generalization of the discussion above. We replace $f\colon K \to 1$ by an arbitrary functor $f\colon L \to K$ between small categories and replace the diagonal functor by the functor

$$\mathcal{P}_{\mathcal{C}}(K) \xrightarrow{f^*} \mathcal{P}_{\mathcal{C}}(L)$$

defined by $f^*(-) = (-) \circ f^{\mathrm{op}}$ as before. The "in families" generalization of colimit is the left Kan extension $f_!$ and the "in families" generalization of limit is the right Kan extension f_*. We now show that if these exist in the absolute case of $K \to 1$, then they also exist in the relative case of $L \to K$.

Theorem 5.1. *Let $f\colon L \to K$ be a functor between small categories.*

(1) *If $\mathcal{C}$ is a category that admits small colimits, then f^* admits a left adjoint*

$$\begin{tikzcd} \mathcal{P}_{\mathcal{C}}(L) \arrow[r, shift left, "f_!"] & \mathcal{P}_{\mathcal{C}}(K). \arrow[l, shift left, "f^*"] \end{tikzcd}$$

(2) *If $\mathcal{C}$ is a category that admits small limits, then f^* admits a right adjoint*

$$\begin{tikzcd} \mathcal{P}_{\mathcal{C}}(K) \arrow[r, shift left, "f^*"] & \mathcal{P}_{\mathcal{C}}(L). \arrow[l, shift left, "f_*"] \end{tikzcd}$$

Proof. It suffices to prove (1), since (2) is (1) for $\mathcal{C}^{\mathrm{op}}$. Given $\mathcal{G}\colon L^{\mathrm{op}} \to \mathcal{C}$, we define $f_!(\mathcal{G})\colon K^{\mathrm{op}} \to \mathcal{C}$ as follows. Let S be an object of K, and let $L_{S/}$ be the slice category. We recall that the objects of $L_{S/}$ are all pairs (T, a) of an object T of L and a morphism $a\colon S \to f(T)$ in K and that a morphism in $L_{/S}$ from (U, b) to (T, a) is a morphism $h\colon U \to T$ in L with the property that $a = f(h) \circ b$. We also recall the functor $q_S\colon L_{S/} \to L$ that to $h\colon (U, b) \to (T, a)$ assigns $h\colon U \to T$. A morphism $k\colon S \to R$ in K gives rise to a functor $L_{k/}\colon L_{R/} \to L_{S/}$ in the opposite direction that to $h\colon (U, b) \to (T, a)$ assigns $h\colon (U, b \circ k) \to (T, a \circ k)$. Moreover, the diagram

$$\begin{tikzcd} L_{R/} \arrow[rr, "L_{k/}"] \arrow[dr, "q_R"'] & & L_{S/} \arrow[dl, "q_S"] \\ & L & \end{tikzcd}$$

commutes. Now, given $\mathcal{G}\colon L^{\mathrm{op}} \to \mathcal{C}$, we define $f_!(\mathcal{G})\colon K^{\mathrm{op}} \to \mathcal{C}$ as follows. If S is an object of K, then we let $p_S\colon (L_{S/})^{\mathrm{op}} \to \mathcal{C}$ be the composite functor

$$(L_{S/})^{\mathrm{op}} \xrightarrow{q_S^{\mathrm{op}}} L^{\mathrm{op}} \xrightarrow{\mathcal{G}} \mathcal{C}$$

and choose a colimit $\bar{p}_S\colon ((L_{S/})^{\mathrm{op}})^{\triangleright} \to \mathcal{C}$ thereof and define

$$f_!(\mathcal{G})(S) = \bar{p}_S(0)$$

to be the value at the cone point 0. Moreover, if $k\colon S \to R$ is a morphism in K, then there is a unique natural transformation $\varphi\colon \bar{p}_R \to \bar{p}_S \circ ((L_{k/})^{\mathrm{op}})^{\triangleright}$ that restricts to the identity natural transformation of $p_R = p_s \circ (L_{k/})^{\mathrm{op}}$ on $(L_{R/})^{\mathrm{op}}$, and we define

$$f_!(\mathcal{G})(R) \xrightarrow{f_!(\mathcal{G})(k)=\varphi_0} f_!(\mathcal{G})(S)$$

to be the value of φ at the cone point. This defines the presheaf $f_!(\mathcal{G})$. We define

$$\mathcal{G}(T) \xrightarrow{\eta_{\mathcal{G},T}} f^*f_!(\mathcal{G})(T) = f_!(\mathcal{F})(f(T))$$

to be the image by $\bar{p}_{f(T)}$ of the unique morphism $(T, \mathrm{id}_{f(T)}) \to 0$. It remains to show that the composite map

$$\mathrm{Map}(f_!(\mathcal{G}), \mathcal{F}) \xrightarrow{f^*} \mathrm{Map}(f^*f_!(\mathcal{G}), f^*(\mathcal{F})) \xrightarrow{-\circ\eta_{\mathcal{G}}} \mathrm{Map}(\mathcal{G}, f^*(\mathcal{F}))$$

is a bijection. We define the inverse map as follows. Given $\psi\colon \mathcal{G} \to f^*(\mathcal{F})$, we extend the functor $p_S\colon (L_{S/})^{\mathrm{op}} \to \mathcal{C}$ to a functor $r_S\colon ((L_{S/})^{\mathrm{op}})^{\triangleright} \to \mathcal{C}$ that to the unique morphism $(T, a\colon S \to f(T)) \to 0$ assigns the composite morphism

$$\mathcal{G}(T) \xrightarrow{\psi_T} \mathcal{F}(f(T)) \xrightarrow{\mathcal{F}(a)} \mathcal{F}(S).$$

There is a unique natural transformation $\varphi\colon \bar{p}_S \to r_S$ with $\varphi|_{(L_{S/})^{\mathrm{op}}} = p_S$, and its value at the cone point is a morphism

$$f_!(\mathcal{G})(S) \xrightarrow{\varphi_0} \mathcal{F}(S).$$

The uniqueness property of the colimit implies that the family consisting of these morphisms is a natural transformation $f_!(\mathcal{G}) \to \mathcal{F}$. One checks that this is indeed an inverse to the map in question. □

Remark 5.2. We record the formulas for $f_!(\mathcal{G})(S)$ and $f_*(\mathcal{G})(S)$, which we derived during the proof of Proposition 5.1. Here $\mathcal{G}\colon L^{\mathrm{op}} \to \mathcal{C}$ is a presheaf on L with values in $\mathcal{C}$, and S is an object of K. If we let $p_S = \mathcal{G} \circ q_S^{\mathrm{op}}\colon (L_{S/})^{\mathrm{op}} \to \mathcal{C}$ and choose a colimit $\bar{p}_S$ of p, then $f_!(\mathcal{G})(S) \simeq \bar{p}_S(0)$. Similarly, if $p_S\colon \mathcal{G} \circ q_S^{\mathrm{op}}\colon (L_{/S})^{\mathrm{op}} \to \mathcal{C}$ and if $\bar{p}_S$ is a limit of p_S, then $f_*(\mathcal{G})(S) \simeq \bar{p}_S(0)$. So informally, we have

$$f_!(\mathcal{G})(S) = \varinjlim_{(T, a\colon S \to f(T))} \mathcal{G}(T),$$
$$f_*(\mathcal{G})(S) = \varprojlim_{(T, a\colon f(T) \to S)} \mathcal{G}(T).$$

We now specialize to the case, where $\mathcal{C} = \text{Set}$ is the category of small sets, where we have the following more useful characterization of the Kan extensions.

Proposition 5.3. *Let $f\colon L \to K$ be a functor between small categories. Up to unique natural isomorphism, the right Kan extension along f is the unique functor that preserves small colimits and makes the diagram*

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "h_L"] & K \arrow[d, "h_K"] \\ \mathcal{P}(L) \arrow[r, "f_!"] & \mathcal{P}(K) \end{tikzcd}$$

commute, up to unique natural isomorphism.

Proof. By definition, the right Kan extension has this property, and the uniqueness statement follows from Proposition 4.9. □

Proposition 5.3 further characterizes f^* as a right adjoint of $f_!$ and f_* as a right adjoint of f^*, both of which are unique, up to unique natural isomorphism. This has the following useful consequence.

Corollary 5.4. *Let $\mathcal{C}$ be a category that admits small limits and colimits, and let*

$$\begin{tikzcd} L \arrow[r, shift left, "f"] & K \arrow[l, shift left, "g"] \end{tikzcd}$$

be an adjoint pair of functors between small categories. In this situation, there are unique natural isomorphisms between adjoint functors

$$\begin{tikzcd} \mathcal{P}_{\mathcal{C}}(L) \arrow[r, bend left=40, "f_!"] \arrow[r, shift right, "f_* \simeq g^*"'] & \mathcal{P}_{\mathcal{C}}(K) \arrow[l, shift right, "f^* \simeq g_!"'] \arrow[l, bend left=40, "g_*"] \end{tikzcd}$$

Proof. If $\mathcal{C} = \text{Set}$, then Proposition 5.3 shows that $g_!$ is a right adjoint to $f_!$. But so is f^*, and therefore, we conclude, by the uniqueness of right adjoints, up to unique natural isomorphism, that f^* and $g_!$ are uniquely naturally isomorphic as stated. By the same argument, we further conclude that also f_* and g^* are uniquely naturally isomorphic. For general $\mathcal{C}$, we use the fact, which we proved in Lemma 4.8, that limits and colimits in categories of $\mathcal{C}$-valued presheaves are calculated pointwise to reduce to the case $\mathcal{C} = \text{Set}$. □

Example 5.5. A small category K admits an initial object if and only if the unique functor $p\colon K \to 1$ admits a left adjoint $s\colon 1 \to K$. In this case, Corollary 5.4 shows that $p_! \simeq s^*$, which is the familiar fact that the a colimit indexed by a small category that admits a final object is given by evaluation at said final object.

Similarly, a small category K admits a final object if and only if the unique functor $p\colon K \to 1$ admits a right adjoint $s\colon 1 \to K$. So we conclude that $p_* \simeq s^*$, or equivalently, that limits indexed by a small category that admits an initial object are given by evaluation at said initial object.

Let K and L be small categories. We define $K \times L$ to be the category, whose set of objects is $K_0 \times L_0$, whose set of morphisms is $K_1 \times L_1$, and whose structure maps are the products of the structure maps for K and L. We would like to say that $K \times L$ is the "product" of K and L and that the diagram

$$\begin{tikzcd} K \times L \arrow[r, "q'"] \arrow[d, "p'"] & K \arrow[d, "p"] \\ L \arrow[r, "q"] & 1, \end{tikzcd}$$

where 1 is a final category, where p and q are the unique functors, and where p' and q' are the canonical projection functors, is a "cartesian" diagram of categories. These statements are both true, but to make sense of them, we need to pass to ∞-categories.[11] The induced diagram of restriction functors

$$\begin{tikzcd} \mathcal{P}_{\mathcal{C}}(K \times L) & \mathcal{P}_{\mathcal{C}}(K) \arrow[l, "q'^*"'] \\ \mathcal{P}_{\mathcal{C}}(L) \arrow[u, "p'^*"] & \mathcal{P}_{\mathcal{C}}(1) \arrow[l, "q^*"'] \arrow[u, "p^*"'] \end{tikzcd}$$

commutes, but it does not make sense to ask if the "base-change" diagram

$$\begin{tikzcd} \mathcal{P}_{\mathcal{C}}(K \times L) \arrow[d, "p'_*"] & \mathcal{P}_{\mathcal{C}}(K) \arrow[l, "q'^*"'] \arrow[d, "p_*"] \\ \mathcal{P}_{\mathcal{C}}(L) & \mathcal{P}_{\mathcal{C}}(1) \arrow[l, "q^*"'] \end{tikzcd}$$

does so, because, the functors p_* and p'_* are well-defined, up to unique natural isomorphism only. However, by the commutativity of the diagram of restriction functors, we may consider the composite natural transformation

$$q^*p_* \xrightarrow{\eta q^* p_*} p'_*p'^*q^*p_* = p'_*q'^*p^*p_* \xrightarrow{p'_*q'^*\epsilon} p'_*q'^*$$

between the two composite functors in the "base-change" diagram, and we can ask whether or not it is a natural isomorphism. We call this natural transformation the base-change map, and we say that base-change holds, if it is a natural isomorphism. For cartesian diagrams of categories in general, this is not true, but it is so in the case that we consider, because the functor $p\colon K \to 1$ is "proper." The following is a special case of proper base-change for presheaves on categories.

Proposition 5.6. *Let $\mathcal{C}$ be a category that admits small limits, and let K and L be small categories. The base-change map $q^*p_* \to p'_*q'^*$ for the diagram*

$$\begin{tikzcd} \mathcal{P}_{\mathcal{C}}(K \times L) \arrow[d, "p'_*"] & \mathcal{P}_{\mathcal{C}}(K) \arrow[l, "q'^*"'] \arrow[d, "p_*"] \\ \mathcal{P}_{\mathcal{C}}(L) & \mathcal{P}_{\mathcal{C}}(1), \arrow[l, "q^*"'] \end{tikzcd}$$

is a natural isomorphism.

[11] It suffices to pass to $(2,1)$-categories, but it is easier to pass to $(\infty,1)$-categories right away.

Proof. We wish to prove that for every $\mathcal{F} \in \mathcal{P}_{\mathcal{C}}(K)$, the base-change map

$$q^* p_*(\mathcal{F}) \longrightarrow p'_* q'^*(\mathcal{F})$$

in $\mathcal{P}_{\mathcal{C}}(L)$ is an isomorphism. This map, in turn, is an isomorphism if and only if for every object T in L, the induced map

$$q^* p_*(\mathcal{F})(T) \longrightarrow p'_* q'^*(\mathcal{F})(T)$$

in $\mathcal{C}$ is an isomorphism. To prove that this is so, we consider the diagram

$$\begin{tikzcd} K \arrow[r, "T'"] \arrow[d, "p"] & K \times L \arrow[r, "q'"] \arrow[d, "p'"] & K \arrow[d, "p"] \\ 1 \arrow[r, "T"] & L \arrow[r, "q"] & 1, \end{tikzcd}$$

where T' is the functor that to S in K assigns (S, T) in $K \times L$. Now, we can rewrite the map in $\mathcal{C}$, which we wish to prove is an isomorphism, as the map

$$T^* q^* p_*(\mathcal{F}) \longrightarrow T^* p'_* q'^*(\mathcal{F})$$

induced by the base-change map for the right-hand square. But the composition

$$T^* q^* p_*(\mathcal{F}) \longrightarrow T^* p'_* q'^*(\mathcal{F}) \longrightarrow p_* T'^* q'^*(\mathcal{F})$$

of this map and the map induced by the base-change map for the left-hand square is the base-change map for the outer square, which is the identity map of $p_*(\mathcal{F})$, so we may instead show that the right-hand map is an isomorphism. In fact, we will show for all $\mathcal{G} \in \mathcal{P}(K \times L)$, the base-change map

$$T^* p'_*(\mathcal{G}) \longrightarrow p_* T'^*(\mathcal{G})$$

for the left-hand square is an isomorphism. To this end, we consider the diagram

$$\begin{tikzcd} (K \times L)_{/T} \arrow[dr, shift left, "f"] \arrow[drr, bend left, "q_T"] \arrow[ddr, bend right, "r"'] & & \\ & K \arrow[ul, shift left, "g"] \arrow[r, "T'"] \arrow[d, "p"] & K \times L \arrow[d, "p'"] \\ & 1 \arrow[r, "T"] & L, \end{tikzcd}$$

where $f((S, T'), a \colon T' \to T) = S$ and $g(S) = ((S, T), \mathrm{id}_T \colon T \to T)$. (The outer square and the top triangular diagram do not commute.) We now have

$$T^* p'_*(\mathcal{G}) \simeq r_* q_T^*(\mathcal{G}) \simeq p_* f_* q_T^*(\mathcal{G}) \simeq p_* g^* q_T^*(\mathcal{G}) \simeq p_* T'^*(\mathcal{G}),$$

where the first isomorphism is the formula for the right Kan extension that we gave in Remark 5.2, and where the next to last isomorphism follows from Corollary 5.4. We leave it to the reader to check that the composite isomorphism is equal to the base-change map. This completes the proof. □

We may now repeat this process. So we consider the diagram

$$\begin{tikzcd} \mathcal{P}_{\mathcal{C}}(K \times L) \arrow[r, "q'_!"] \arrow[d, "p'_*"] & \mathcal{P}_{\mathcal{C}}(K) \arrow[d, "p_*"] \\ \mathcal{P}_{\mathcal{C}}(L) \arrow[r, "q_!"] & \mathcal{P}_{\mathcal{C}}(1) \end{tikzcd}$$

and use the fact that Proposition 5.6 gives rise to a natural transformation

$$q_! p'_* \xrightarrow{q_! p'_* \eta} q_! p'_* q'^* q'_! \xleftarrow{\sim} q_! q^* p_* q'_! \xrightarrow{\epsilon p_* q'_!} p_* q'_!,$$

where the "wrong-way" arrow is the base-change map. Indeed, by Proposition 5.6, the base-change morphism is a natural isomorphism, so it has a unique inverse natural transformation, and it is this unique inverse natural transformation that we use to form the composite natural transformation $q_! p'_* \to p_* q'_!$. Informally, a presheaf on $K \times L$ is a functor $\mathcal{H}\colon K^{\mathrm{op}} \times L^{\mathrm{op}} \to \mathcal{C}$ of two variables, and $q_! p'_*$ takes $\mathcal{H}$ to the object of $\mathcal{C}$ obtained by first taking the limit in the K^{op}-variable and then the colimit in the L^{op}-variable, whereas $p_* q'_!$ instead takes $\mathcal{H}$ to the object of $\mathcal{C}$ obtained by first taking the colimit in the L^{op}-variable and then the limit in the K^{op}-variable. For this reason, we call $q_! p'_* \to p_* q'_!$ the limit-colimit-interchange map, and we also write it informally as

$$\varinjlim_{L^{\mathrm{op}}} \varprojlim_{K^{\mathrm{op}}} \mathcal{H} \longrightarrow \varprojlim_{K^{\mathrm{op}}} \varinjlim_{L^{\mathrm{op}}} \mathcal{H}$$

even though the functors p'_* and $q'_!$ are not really limits and colimits. In general, it is not a natural isomorphism. Let us give a counterexample.

Example 5.7. In order to produce a counterexample, it is always good to think about extreme cases. So let us suppose that K and L are both the empty category. In this case, also $K \times L$ is the empty category, and $\mathcal{H}\colon K^{\mathrm{op}} \times L^{\mathrm{op}} \to \mathcal{C}$ is necessarily the unique functor. Similarly, $p'_*(\mathcal{H})\colon L^{\mathrm{op}} \to \mathcal{C}$ and $q'_!(\mathcal{H})\colon K^{\mathrm{op}} \to \mathcal{C}$ are both the unique functors, since this is the only possibility. So $q_! p'_*(\mathcal{H})\colon 1^{\mathrm{op}} \to \mathcal{C}$ is an initial object of $\mathcal{C}$, whereas $p_* q'_!(\mathcal{H})\colon 1^{\mathrm{op}} \to \mathcal{C}$ is a final object of $\mathcal{C}$, and therefore, the limit-colimit-interchange map is the unique morphism $q_! p'_*(\mathcal{H}) \to p_* q'_!(\mathcal{H})$ from an initial object to a final object in $\mathcal{C}$. If $\mathcal{C} = \mathrm{Set}$, then this is not an isomorphism, since the unique initial object in Set is the empty set $\emptyset$, whereas a final object in Set is any set with exactly one element.

In the case $\mathcal{C} = \mathrm{Set}$, Grothendieck's theorem gives sufficient[12] conditions for the limit-colimit-interchange map to be a natural isomorphism.

Definition 5.8. A category J is filtered the following conditions are satisfied.[13]

(1) The category J is non-empty.
(2) For every pair of objects (i, j) in J, there exists a pair of morphisms

$$\begin{tikzcd} i \arrow[dr, "a"] & \\ & k \\ j \arrow[ur, "b"'] & \end{tikzcd}$$

[12] It turns out that, for $\mathcal{C} = \mathrm{Set}$, these conditions are also necessary.

[13] Equivalently, a category J is filtered, if every functor $p\colon K \to J$ from a finite category K extends to a functor $\tilde{p}\colon K^{\triangleright} \to J$.

with the given objects as sources and with a common target.

(3) For every pair of parallel morphisms $(a\colon i \to j, b\colon i \to j)$ in J, there exists a morphism $c\colon j \to k$ such that the two composite morphisms

$$\begin{tikzcd} i \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & j \arrow[r, "c"] & k \end{tikzcd}$$

are equal.

Example 5.9. Here is a typical example of a filtered category. Let X be a topological space, and let $x \in X$ be a point. Let K be category, whose objects are open subsets $x \in U \subset X$ that contain x, and where there is a unique morphism $i_U^V\colon U \to V$ if $U \subset V$. If $U \not\subset V$, then there are no morphisms from U to V. In this case, the opposite category $J = K^{\mathrm{op}}$ is filtered. We also say that K is cofiltered.

We proved in Proposition 2.15 that Set admits all small colimits, but we also remarked in Remark 2.16 that if $\bar{p}\colon J \to \mathrm{Set}$ is a colimit of $p\colon J \to \mathrm{Set}$, then the set $\bar{p}(0) =: \varinjlim_J p$ is all but unknowable, in general. However, for J filtered, the situation turns out to be much better.

Proposition 5.10. *Let J be a small filtered category, let $p\colon J \to \mathrm{Set}$ be a diagram, and let $\bar{p}\colon J \to \mathrm{Set}$ be a colimit. Let $X = \coprod_{j \in J_0} X_j$, and let $R \subset X \times X$ be the relation that consists of the pairs (x_i, x_j) with $x_i \in p(i)$ and $x_j \in p(j)$ for which there exists $a\colon i \to k$ and $b\colon j \to k$ such that $p(a)(x_i) = p(b)(x_j) \in p(k)$.*

(1) *The relation $R \subset X \times X$ is an equivalence relation, and the map $X \to \bar{p}(0)$ induced by the unique maps $j \to 0$ in $J^{\triangleright}$ factors through a bijection*

$$X/R \longrightarrow \bar{p}(0).$$

(2) *Given $(x_i, x_j) \in R$ with $x_i \in p(i)$ and $x_j \in p(j)$, there exists $k \in J_0$ and $x_k \in p(k)$ such that both $(x_i, x_k), (x_j, x_k) \in R$.*

(3) *Given $(x_i, x_i') \in R$ with $x_i, x_i' \in p(i)$, there exists $a\colon i \to j$ in J such that $p(a)(x_i) = p(a)(x_i') \in p(j)$.*

Proof. The statement (1) is clear, once we prove that R is an equivalence relation. Moreover, it is clear that R is both reflexive and symmetric, so only transitivity needs proof. So we assume that $(x_i, x_j) \in R$ and $(x_j, x_k) \in R$ with $x_i \in p(i)$, $x_j \in p(j)$, and $x_k \in p(k)$ and must prove that $(x_i, x_k) \in R$. We use this the assumption and the fact that J is filtered to choose morphisms

$$\begin{tikzcd}[column sep=small] i \arrow[dr] & & & \\ & l \arrow[dr] & & \\ j \arrow[ur] \arrow[dr] & & n \arrow[r] & o \\ & m \arrow[ur] & & \\ k \arrow[ur] & & & \end{tikzcd}$$

such that the two composite morphisms $j \to o$ are equal and such that, in the induced diagram of sets

$$\begin{tikzcd}
p(i) \arrow[dr] & & & \\
& p(l) \arrow[dr] & & \\
p(j) \arrow[ur] \arrow[dr] & & p(n) \arrow[r] & p(o), \\
& p(m) \arrow[ur] & & \\
p(k) \arrow[ur] & & &
\end{tikzcd}$$

the elements $x_i \in p(i)$ and $x_j \in p(j)$ have the same image $x_m \in p(l)$ and the elements $x_j \in p(j)$ and $x_k \in p(k)$ have the same image $x_n \in p(m)$. Since p is a functor, we conclude that $x_i \in p(i)$, $x_j \in p(j)$, and $x_k \in p(k)$ all have the same image $x_o \in p(o)$, which shows that $(x_i, x_k) \in R$, as desired.

The statement (2) follows immediately from the fact that J is filtered, so it remains only to prove (3). If $(x_i, x_i') \in R$ with $x_i, x_i' \in p(i)$, then, by definition, there exists $f, g\colon i \to i'$ such that $p(f)(x_i) = p(g)(x_i') \in p(i')$. Since J is filtered, we can choose $h\colon i' \to j$ such that $h \circ f = h \circ g\colon i \to j$. Hence, if $a\colon i \to j$ is the common morphism, then we find that

$$p(a)(x_i) = (p(h) \circ p(f))(x_i)) = (p(h) \circ p(g))(x_i') = p(a)(x_i'),$$

as desired. □

We define a category K to be finite if both the set of object K_0 and the set of morphisms K_1 are finite. For example, the empty category is finite. We can now state and prove Grothendieck's theorem.

Theorem 5.11. *Let K and L be small categories. If K is finite and if L is cofiltered, then, in the diagram of categories of presheaves of small sets*

$$\begin{tikzcd}
\mathcal{P}(K \times L) \arrow[r, "q_!'"] \arrow[d, "p_*'"] & \mathcal{P}(K) \arrow[d, "p_*"] \\
\mathcal{P}(L) \arrow[r, "q_!"] & \mathcal{P}(1),
\end{tikzcd}$$

the limit-colimit-interchange map

$$q_! p_*' \longrightarrow p_* q_!'$$

is a natural isomorphism.

Proof. We wish to show that the limit-colimit-interchange map

$$q_! p_*'(\mathcal{H}) \longrightarrow p_* q_!'(\mathcal{H})$$

is a bijection for every presheaf $\mathcal{H}\colon K^{\mathrm{op}} \times L^{\mathrm{op}} \to \mathrm{Set}$. In order to do so, we first determine the source and target of this map by using the description of limits given in Proposition 2.15 and the description of filtered colimits given in Proposition 5.10 above. An element of $q_! p_*'(\mathcal{H})$ is an equivalence class of families of the form $(x_{i,j})_{i \in K_0}$, where $x_{i,j} \in \mathcal{H}(i,j)$ and $\mathcal{H}(a,j)(x_{i,j}) = x_{i',j}$ for all $a\colon i' \to i$ in

K, and two such families $(x_{i,j})_{i\in K_0}$ and $(x'_{i,k})_{i\in K_0}$ are equivalent if there exists morphisms $b\colon l\to j$ and $c\colon l\to k$ in L such that $\mathcal{H}(i,b)(x_{i,j}) = \mathcal{H}(i,c)(x'_{i,k})$ for all $i\in K_0$. Similarly, an element of $p_*q'_!(\mathcal{H})$ is a family $(\mathrm{class}(x_{i,j}))_{i\in K_0}$ of equivalence classes of elements $x_{i,j}\in\mathcal{H}(i,j)$, where $x_{i,j}$ and $x_{i,k}$ are equivalent if there exists morphisms $b\colon l\to j$ and $c\colon l\to k$ such that $\mathcal{H}(i,b)(x_{i,j}) = \mathcal{H}(i,c)(x_{i,k})$, and this family is required to satisfy that $\mathrm{class}(\mathcal{H}(a,j)(x_{i,j})) = \mathrm{class}(x_{i',j})$ for all $a\colon i'\to i$ in K. Finally, the limit-colimit-interchange map is given by

$$\mathrm{class}((x_{i,j})_{i\in K_0}) \longmapsto (\mathrm{class}(x_{i,j}))_{i\in K_0}.$$

Since K is finite, it follows immediately from Proposition 5.10 that it is a bijection. Indeed, given an element $(\mathrm{class}(x_{i,j}))_{i\in K_0}$ of $p_*q'_!(\mathcal{H})$, we can find a common $m\in L_0$ and a family $(x_{i,m})_{i\in K_0}$ of representatives of the given classes such that every morphism $a\colon i'\to i$ in K, we have $\mathcal{H}(a,m)(x_{i,m}) = x_{i',m}$. Hence, the map

$$\mathrm{class}((x_{i,m})_{i\in K_0}) \longleftarrow\!\shortmid (\mathrm{class}(x_{i,j}))_{i\in K_0}$$

is inverse to the limit-colimit-interchange map. $\square$

Exercise 5.12. Use Proposition 5.10 to check that the limit-colimit-interchange map is an isomorphism in the case, where K is the empty category and L is a small cofiltered category.

Remark 5.13. Theorem 5.11 is valid for presheaves with values in a topos $\mathcal{X}$, but it does not hold for presheaves with values in a general category $\mathcal{C}$. For example, it fails for the category $\mathcal{C} = \mathrm{Set}^{\mathrm{op}}$.

Exercise 5.14. Let $\mathcal{C}$ be a category that admits small limits, and let K and L be small categories. We consider the diagram of categories of $\mathcal{C}$-valued presheaves

$$\begin{tikzcd} \mathcal{P}_{\mathcal{C}}(K\times L) \arrow[r, "q'_!"] & \mathcal{P}_{\mathcal{C}}(K) \\ \mathcal{P}_{\mathcal{C}}(L) \arrow[u, "p'^*"] \arrow[r, "q_!"] & \mathcal{P}_{\mathcal{C}}(1). \arrow[u, "p^*"'] \end{tikzcd}$$

Show that the composite natural transformation

$$q'_!p'^* \xrightarrow{q'_!p'^*\eta} q'_!p'^*q^*q_! \simeq q'_!q'^*p^*q_! \xrightarrow{\epsilon p^* q_!} p^*q_!$$

is a natural isomorphism.
[Hint: Show that this statement is equivalent to Proposition 4 for $\mathcal{C}^{\mathrm{op}}$.]

Exercise 5.15. Let K be a small category that admits finite coproducts, and let

$$\mathcal{P}^{\Sigma}(K) \subset \mathcal{P}(K)$$

be the full subcategory spanned by the functors $\mathcal{F}\colon K^{\mathrm{op}}\to\mathrm{Set}$ that preserve finite products. Since finite products in K^{op} are given by finite coproducts in K, the requirement that $\mathcal{F}$ preserve finite products amounts to the requirement that for every finite family $(x_i)_{i\in I}$ of objects in K, the canonical map

$$\mathcal{F}(\textstyle\coprod_{i\in I} x_i) \longrightarrow \prod_{i\in I}\mathcal{F}(x_i)$$

is a bijection.

(1) Show that the $\mathcal{P}^{\Sigma}(K) \subset \mathcal{P}(K)$ is closed under small filtered colimits. More precisely, given a functor $p\colon J \to \mathcal{P}^{\Sigma}(K)$ with J small, we know that its composition $q = i \circ p\colon J \to \mathcal{P}(K)$ with the inclusion $i\colon \mathcal{P}^{\Sigma}(K) \to \mathcal{P}(K)$ admits a colimit $\bar{q}\colon J^{\triangleright} \to \mathcal{P}(K)$. Show that if J is filtered, then there exists a (unique) functor $\bar{p}\colon J^{\triangleright} \to \mathcal{P}^{\Sigma}(K)$ such that $\bar{q} = i \circ \bar{p}$.
(2) Conclude that small filtered colimits in $\mathcal{P}^{\Sigma}(K)$ are calculated pointwise.[14]

Remark 5.16. As an application of Exercise 5.15, we let CAlg(Ab) be the category of commutative rings and ring homomorphisms, and let $K \subset \mathrm{CAlg}(\mathrm{Ab})$ to be the full subcategory spanned by the polynomial rings $\mathbb{Z}[x_1, \dots, x_n]$ in finitely many variables, including $n = 0$. It admits finite coproducts. There is a functor

$$\mathrm{CAlg}(\mathrm{Ab}) \xrightarrow{\ h\ } \mathcal{P}^{\Sigma}(K)$$

defined by $h(R)(-) = \mathrm{Map}(-, R)$, and this functor is an equivalence of categories. Thus, the fact that small filtered colimits in $\mathcal{P}^{\Sigma}(K)$ are calculated pointwise implies that a diagram $\bar{p}\colon J^{\triangleright} \to \mathrm{CAlg}(\mathrm{Ab})$ with J small and filtered is a colimit diagram of commutative rings if and only its composition $\bar{q}\colon J^{\triangleright} \to \mathrm{Set}$ with the forgetful functor $\mathrm{CAlg}(\mathrm{Ab}) \to \mathrm{Set}$ is a colimit diagram of sets.[15]

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[14] More generally, small sifted colimits in $\mathcal{P}^{\Sigma}(K)$ are calculated pointwise.
[15] Again, this statement is true more generally for small sifted diagrams.