Algebra III/Introduction to Algebra III: Scheme Theory

Due: Please upload solutions to NUCT by Tuesday, April 19, 2022.

This week's problem concerns limits and colimits of diagrams of sets. A diagram is the same as a functor. If $p\colon K \to \mathcal{C}$ is a functor, then we also say that it is a diagram in $\mathcal{C}$ indexed by K. Given a category K, its left cone $K^{\triangleleft}$ is obtained from K by adding an object 0 and a single morphism $0 \to k$ to every object k in K. Similarly, its right cone $K^{\triangleright}$ is obtained from K by adding an object 1 and a single morphism $k \to 1$ from every object k in K. We call $0 \in K^{\triangleleft}$ and $1 \in K^{\triangleright}$ the cone points. We define:

(1) A limit of $p\colon K \to \mathcal{C}$ is a functor $\bar{p}\colon K^{\triangleleft} \to \mathcal{C}$ with $\bar{p}|_K = p$ and such that for every functor $f\colon K^{\triangleleft} \to \mathcal{C}$ with $f|_K = p$, there exists a unique natural transformation $\varphi\colon f \to \bar{p}$ with $\varphi|_K = \mathrm{id}_p$.
(2) A colimit of $p\colon K \to \mathcal{C}$ is a functor $\bar{p}\colon K^{\triangleright} \to \mathcal{C}$ with $\bar{p}|_K = p$ and such that for every functor $f\colon K^{\triangleright} \to \mathcal{C}$ with $f|_K = p$, there exists a unique natural transformation $\varphi\colon \bar{p} \to f$ with $\varphi|_K = \mathrm{id}_p$.

If a limit (resp. a colimit) of $p\colon K \to \mathcal{C}$ exists, then it is unique, up to unique isomorphism. Therefore, we also write $\lim p\colon K^{\triangleleft} \to \mathcal{C}$ (resp. $\operatorname{colim}\colon K^{\triangleright} \to \mathcal{C}$) for any choice of limit (resp. colimit) with the understanding that it is well-defined, up to unique isomorphism. It is also common to abuse notation and write $\lim p$ (resp. $\operatorname{colim} p$) for the object $\bar{p}(0)$ (resp. the object $\bar{p}(1)$).

If $p\colon K \to \mathsf{Set}$ is a (small) diagram of sets, then a limit $\bar{p}\colon K^{\triangleleft} \to \mathsf{Set}$ is given as follows. We let $X_i = p(i)$, and let

$$Y \subset X = \textstyle\prod_{i \in \mathrm{ob}(K)} X_i$$

to be the set of solutions $x = (x_i)_{i \in \mathrm{ob}(K)}$ to the system of equations

$$p(a)(x_j) = x_i$$

indexed by the set $\mathrm{mor}(K)$ of morphisms $a\colon j \to i$. We define $\bar{p}\colon K^{\triangleleft} \to \mathsf{Set}$ to the extension of $p\colon K \to \mathsf{Set}$, whose value at the cone point is $\bar{p}(0) = Y$, and whose value at the unique morphism $0 \to k$ with $k \in \mathrm{ob}(K)$ is the composition

$$Y \longrightarrow \textstyle\prod_{i \in \mathrm{ob}(K)} X_i \longrightarrow X_k$$

of the canonical inclusion and the canonical projection.

It is also true that every (small) diagram $p\colon K \to \mathsf{Set}$ of sets admits a colimit, but in general, it is basically impossible to understand.[1] However, the situation improves if K is filtered. A category J is filtered it it satisfies:

(1) The category J is non-empty.
(2) For every pair of objects (i, j) in J, there exists a pair of morphisms

$$\begin{tikzcd} i \arrow[dr, "a"] & \\ & k \\ j \arrow[ur, "b"'] & \end{tikzcd}$$

[1] The value of $\bar{p}\colon K^{\triangleright} \to \mathsf{Set}$ at the cone point is the quotient of $\coprod_{i \in \mathrm{ob}(K)} X_i$ by the equivalence generated by the relation that identifies $x_i \in X_i$ and $x_j \in X_j$ if there exists a morphism $a\colon j \to i$ in K such that $p(a)(x_j) = x_i$. The problem is that the equivalence relation generated by a relation is impossible to understand.

with the given objects as sources and with a common target.

(3) For every pair of parallel morphisms $(a\colon i \to j, b\colon i \to j)$ in J, there exists a morphism $c\colon j \to k$ such that the two composite morphisms

$$\begin{tikzcd} i \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & j \arrow[r, "c"] & k \end{tikzcd}$$

are equal.

If $p\colon J \to \mathsf{Set}$ is a diagram of sets indexed by a (small) filtered category, then its colimit $\bar{p}\colon J^{\triangleright} \to \mathsf{Set}$ is given as follows. Let $X_i = p(i)$, and let $X = \coprod_{i \in \mathrm{ob}(J)} X_i$. The relation $R \subset X \times X$ that consists of the pairs (x_i, x_j) with $x_i \in X_i$ and $x_j \in X_j$ for which there exist $a\colon i \to k$ and $b\colon j \to k$ such that

$$p(a)(x_i) = p(b)(x_j) \in X_k$$

is an equivalence relation, and $\bar{p}\colon J^{\triangleright} \to \mathsf{Set}$ is the extension of $p\colon J \to \mathsf{Set}$, whose value at the cone point is $\bar{p}(1) = X/R$, and whose value at the unique morphism $k \to 1$ with $k \in \mathrm{ob}(J)$ is the composition

$$X_k \longrightarrow \textstyle\coprod_{i \in \mathrm{ob}(J)} X_i \longrightarrow X/R$$

of the canonical inclusion and the canonical projection. Moreover, if $(x_i, x_j) \in R$ with $x_i \in X_i$ and $x_j \in X_j$, then there exists $k \in \mathrm{ob}(J)$ and $x_k \in X_k$ such that both (x_i, x_k) and (x_j, x_k) belong to R. Finally, if $(x_i, x_i') \in R$ with $x_i, x_i' \in X_i$, then there exists $a\colon i \to j$ in J such that $p(a)(x_i) = p(a)(x_i') \in X_j$.

A theorem of Grothendieck (which we will use again and again) states, in the category of sets, filtered colimits and finite limits commute. More precisely, let J and K be small categories with J filtered and with K finite. (A category is finite if it has finitely many objects and finitely many morphisms.) A diagram

$$J \times K \xrightarrow{\ p\ } \mathsf{Set}$$

gives rise to a diagram $p(-,k)\colon J \to \mathsf{Set}$ for every $k \in \mathrm{ob}(K)$ and to a diagram $p(j,-)\colon K \to \mathsf{Set}$ for every $j \in \mathrm{ob}(J)$. Moreover, it follows from the definition of limit and colimit that there is a canonical map

$$\mathrm{colim}_{j \in J} \lim_{k \in K} p(j,k) \longrightarrow \lim_{k \in K} \mathrm{colim}_{j \in J}\, p(j,k),$$

and Grothendieck's theorem states that this map is a bijection. One important consequence of this theorem is that the formula for the colimit of a filtered diagram of sets also is valid for a filtered diagram of rings.

Problem 1. Suppose that the ring $R = \mathrm{colim}_{j \in J} R_j$ is the colimit of a filtered diagram of rings $p\colon J \to \mathrm{CAlg}(\mathrm{Ab})$.

(1) Show that the canonical map

$$|\,\mathrm{Spec}(R)| \longrightarrow \lim_{j \in J} |\,\mathrm{Spec}(R_j)|$$

is a bijection.

(Hint: Let $\varphi_j\colon R_j \to R$ be the ring homomorphism that is part of the colimit. If $\mathfrak{p} \subset R$ is a prime ideal, then show that $\mathfrak{p} = \mathrm{colim}_{j \in J}\, \mathfrak{p}_j$, where $\mathfrak{p}_j = \varphi_j^{-1}(\mathfrak{p}) \subset R_j$. Conclude from the description of the limit given above.)

(2) Show that the map in (1) is a homeomorphism.

(Hint: The limit $\lim_{j \in J} |\operatorname{Spec}(R_j)|$ is a subset of the product $\prod_{j \in \mathrm{ob}(J)} |\operatorname{Spec}(R_j)|$. We give the product the product topology and the limit the subspace topology. The map in question is continuous, but it must be proved that it is also open. It suffices to show that the image of a distinguished open subset $D(f) \subset |\operatorname{Spec}(R)|$ is an open subset of $\lim_{j \in J} |\operatorname{Spec}(R_j)|$.)