Algebra III/Introduction to Algebra III: Scheme Theory

Due: Please upload solutions to NUCT by Tuesday, June 28, 2022.

Another intuitive description of the projective space $\mathbb{P}^n_{\mathbb{Z}}$ is as the quotient of the punctured affine space $\mathbb{A}^{n+1}_{\mathbb{Z}} \smallsetminus \{0\}$ by nonzero scalar multiplication. Here you will prove a precise version of this.

Problem 1. Recall that $\mathbb{A}^{n+1}_{\mathbb{Z}} \smallsetminus \{0\}$ represents the functor

$$\mathsf{Sch}^{\mathrm{op}} \xrightarrow{H} \mathsf{Set},$$

which a scheme T assigns the set $H(T)$ of $(n+1)$-tuples of global functions

$$(f_0, \dots, f_n) \in \mathcal{O}_T(T)^{\oplus(n+1)}$$

such that for all $x \in |T|$, the element $(f_0(x), \dots, f_n(x))$ of $k(x)^{\oplus(n+1)}$ is nonzero, and that the group scheme $\mathbb{G}_m$ represents the functor

$$\mathsf{Sch}^{\mathrm{op}} \xrightarrow{G} \mathsf{Grp},$$

which to a scheme T assigns the group of global units $G(T) = \mathcal{O}_T(T)^{\times}$.

(1) Define an action of G on H, and show that the Zariski sheafification of the functor $T \mapsto H(T)/G(T)$ identifies with the functor

$$\mathsf{Sch}^{\mathrm{op}} \xrightarrow{F} \mathsf{Set}$$

represented by $\mathbb{P}^n_{\mathbb{Z}}$.
[Hint: Construct a map of presheaves $H(T)/G(T) \to F(T)$ and show that this map is a Zariski sheafification.]

(2) Give an example to show that, in (1), Zariski sheafification is necessary.