Algebra III/Introduction to Algebra III: Scheme Theory

Due: Please upload solutions to NUCT by Tuesday, May 31, 2022.

Problem 1. Let $f = (p, \phi)\colon (|Y|, \mathcal{O}_Y) \to (|X|, \mathcal{O}_X)$ be a map of schemes and recall that the continuous map $p\colon |Y| \to |X|$ gives rise to an adjunction

$$\mathrm{Sh}(|X|) \underset{p_*}{\overset{p^*}{\rightleftarrows}} \mathrm{Sh}(|Y|)$$

between the respective categories of sheaves of sets. We call p^* the inverse image functor and p_* the direct image functor. It induces an adjunction

$$\mathrm{Mod}_{\mathcal{O}_X}(\mathrm{Sh}(|X|)) \underset{p_*}{\overset{p^*}{\rightleftarrows}} \mathrm{Mod}_{p^*\mathcal{O}_X}(\mathrm{Sh}(|Y|))$$

between the indicated categories of modules. Moreover, the map $\phi\colon p^*\mathcal{O}_X \to \mathcal{O}_Y$ ring objects in $\mathrm{Sh}(|Y|)$ gives rise to an additional adjunction

$$\mathrm{Mod}_{p^*\mathcal{O}_X}(\mathrm{Sh}(|Y|)) \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{O}_Y}(\mathrm{Sh}(|Y|))$$

with ϕ^* given by extension of scalars along ϕ and with ϕ_* given by restriction of scalars along ϕ. We denote the composite adjunction by

$$\mathrm{Mod}_{\mathcal{O}_X}(\mathrm{Sh}(|X|)) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{O}_Y}(\mathrm{Sh}(|Y|))$$

and, by abuse of language, we also call f^* the inverse image functor and f_* the direct image functor.

(1) Show that the inverse image functor f^* restricts to a functor

$$\mathrm{QCoh}(X) \xrightarrow{f^*} \mathrm{QCoh}(Y).$$

We define the map $f\colon Y \to X$ to be affine if every affine open subset $U \subset |X|$ has affine open preimage $p^{-1}(U) \subset |Y|$.[1]

(2) Show that if $f\colon Y \to X$ is affine, then the direct image functor f_* restricts to

$$\mathrm{QCoh}(Y) \xrightarrow{f_*} \mathrm{QCoh}(X).$$

We define $f\colon Y \to X$ to be quasi-compact if for every affine open $U \subset |X|$, the preimage $p^{-1}(U) \subset |Y|$ is quasi-compact open, and we define $f\colon Y \to X$ to be quasi-separated if for every affine open $U \subset |X|$ and every pair of affine open $V, V' \subset p^{-1}(U) \subset |Y|$, their intersection $V \cap V' \subset |Y|$ is a finite union of affine open subsets.[2]

(3) Show that if $f\colon Y \to X$ is quasi-compact and quasi-separated, then the direct image functor f_* restricts to a functor

$$\mathrm{QCoh}(Y) \xrightarrow{f_*} \mathrm{QCoh}(X).$$

[Hint: Prove the case, where X is affine, first, and then prove the general case.]

[1] There are many equivalent definitions affine maps between schemes.

[2] There are many equivalent definitions of quasi-compact and quasi-separated maps.