Algebra III/Introduction to Algebra III: Scheme Theory

Due: Please upload solutions to NUCT by Tuesday, June 7, 2022.

Problem 1. Let X be a scheme, and let

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

be an exact sequence of quasi-coherent $\mathcal{O}_X$-modules.

(1) Show that if $\mathcal{E}$ and $\mathcal{E}''$ both are vector bundles on X, then so is $\mathcal{E}'$.
(2) Show by example that if $\mathcal{E}'$ and $\mathcal{E}$ both are vector bundles on X, then $\mathcal{E}''$ need not be so.
(3) Show by example that if a map $\mathcal{E} \to \mathcal{E}''$ between two vector bundles on X is not surjective, then its kernel (in $\mathrm{QCoh}(X)$) need not be a vector bundle.

[Hints: In (1), use Theorem 17. In (3), think syzygy.]