Algebra III/Introduction to Algebra III: Scheme Theory

Due: Please upload solutions to NUCT by Tuesday, June 14, 2022.

Let $|X|$ be a topological space, and let $x, \eta \in |X|$ be two points. If x is contained in the closure of $\{\eta\} \subset |X|$, then we say that x is a specialization of η and that η is a generalization of x. If $p\colon |Y| \to |X|$ is a continuous map and if $y \in |Y|$ is a specialization of $\xi \in |Y|$, then $x = f(y) \in |X|$ is a specialization of $\eta = f(\xi) \in |X|$.

Problem 1. Let X be a scheme.

(1) Let R be a local ring with prime spectrum Y, and let $y \in Y$ be the (unique) closed point corresponding to the (unique) maximal ideal $\mathfrak{m} \subset R$. A map of schemes $f = (p, \phi)\colon Y \to X$ determines a point $x = p(y) \in |X|$ and a local ring homomorphism $\phi_x\colon \mathcal{O}_{X,x} \to R$. Show that, conversely, for every pair (x, ψ) of a point $x \in |X|$ and a local ring homomorphism $\psi\colon \mathcal{O}_{X,x} \to R$, there is a unique map of schemes $f = (p, \phi)\colon Y \to X$ such that $(x, \psi) = (f(y), \phi_x)$.
[Hint: First consider the case, where X is affine. In the general case, show that if $f = (p, \phi)\colon Y \to X$ is a map of schemes with Y as above, and if $U \subset |X|$ is an affine open subset such that $x = p(y) \subset U$, then $p(|Y|) \subset U$.]

Given $x \in |X|$, we conclude from (1) that there is a unique map of schemes

$$\operatorname{Spec}(\mathcal{O}_{X,x}) \xrightarrow{f=(p,\phi)} X$$

such that $p(x) = x$ and $\phi_x\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is the identity map. Let us write $f_{X,x}$ for this map.

(2) Let $g = (q, \psi)\colon X \to S$ be a map of schemes, let $x \in |X|$, and let $s = q(x) \in |S|$. Show that the diagram

$$\begin{array}{ccc} \operatorname{Spec}(\mathcal{O}_{X,x}) & \xrightarrow{f_{X,x}} & X \\ \downarrow{\scriptstyle \operatorname{Spec}(\psi_x)} & & \downarrow{\scriptstyle g} \\ \operatorname{Spec}(\mathcal{O}_{S,s}) & \xrightarrow{f_{S,s}} & S \end{array}$$

commutes.

(3) Given a point $x \in |X|$, show that the image of the underlying map ofspaces

$$|\operatorname{Spec}(\mathcal{O}_{X,x})| \xrightarrow{|f_{X,x}|} |X|$$

is equal to the set of points $\eta \in |X|$ that specialize to $x \in |X|$.