SCHEME THEORY

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Contents

1. The Zariski space of a ring

In this course, every ring will be commutative, unless explicitly stated otherwise. The basic idea, due to Grothendieck, is that every such ring R deserves to be viewed as the ring of "functions" on some geometric object. Concretely, this geometric object will be defined to be a pair

$$\operatorname{Spec}(R) = (|\operatorname{Spec}(R)|, \mathcal{O}_{\operatorname{Spec}(R)})$$

of a topological space and a sheaf of rings on this topological space. In this lecture, we define the topological space $|\operatorname{Spec}(R)|$, which we call the Zariski space of R. For every $f \in R$, it should have a closed subset

$$Z(f) = V(f) \subset |\operatorname{Spec}(R)|,$$

given by the locus where "$f = 0$."

Definition 1.1. Let R be a ring. Its Zariski space is the topological space

$$|\operatorname{Spec}(R)|$$

given by the set of prime ideals $\mathfrak{p} \subset R$ with the topology with closed subsets

$$V(S) = \{\mathfrak{p} \in |\operatorname{Spec}(R)| \mid S \subset \mathfrak{p}\},$$

where S ranges over all subsets $S \subset R$.

Here we interpret $V(S)$ as the locus, where "$f = 0$" for all $f \in S$.

We will check that this is a topological space later, but let us first give some motivation for this definition. How did we decide that the points of $|\operatorname{Spec}(R)|$ should be the prime ideals of R? Two principles:

(1) A ring homomorphism $\varphi\colon R \to R'$ should give rise to a continuous map

$$|\operatorname{Spec}(R')| \xrightarrow{\;p\;} |\operatorname{Spec}(R)|,$$

and this assignment should be functorial.

(2) For a field k, $|\operatorname{Spec}(k)|$ is a point. Indeed, every $f \in k$ is either a unit or 0, so the locus "$f = 0$" is either empty or the whole space.

According to these two principles, every ring homomorphism $\varphi\colon R \to k$ to a field should determine a point $x \in |\operatorname{Spec}(R)|$. But every such ring homomorphsm factors canonically as the composition

$$R \longrightarrow R/\mathfrak{p} \longrightarrow \operatorname{Frac}(R/\mathfrak{p}) \xrightarrow{\;\bar{\varphi}\;} k,$$

where $\mathfrak{p} \subset R$ is the kernel of φ. Here $R/\mathfrak{p}$ is an integral domain, and $\operatorname{Frac}(R/\mathfrak{p})$ is its quotient field. We write $k(x)$ for this field and call it the residue field at $x \in |\operatorname{Spec}(R)|$. Now, according to the functoriality of principle (1), the commutative diagram of rings and ring homomorphisms

$$\begin{tikzcd} k(x) \arrow[rr, "\bar{\varphi}"] & & k \\ & R \arrow[ul, "\psi"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

gives rise to a commutative diagram of topological spaces and continuous maps

$$
\begin{tikzcd}
{|\operatorname{Spec}(k(x))|} \arrow[dr, "q"'] & & {|\operatorname{Spec}(k)|} \arrow[ll, "\bar{p}"'] \arrow[dl, "p"] \\
& {|\operatorname{Spec}(R)|} &
\end{tikzcd}
$$

and according to the principle (2), the topological spaces in the top row are both one-point spaces, so the map $\bar{p}$ is necessarily a homeomorphism. Thus, the maps p and q define the same point $x \in |\operatorname{Spec}(R)|$. But the map q only depends on the prime ideal $\mathfrak{p} \subset R$. Whence the definition of the Zariski space.

We now fix a ring R. We will also write

$$|X| = |\operatorname{Spec}(R)|$$

for its Zariski space and $x \in |X|$ for a point therein. So $x \in |X|$ determines and is determined by a prime ideal $\mathfrak{p} \subset R$. In fact, we literally have $x = \mathfrak{p}$, but it is useful to have the separate notation. We have already defined the residue field

$$k(x) = \operatorname{Frac}(R/\mathfrak{p})$$

at $x \in |X|$ and the canonical map $\psi\colon R \to k(x)$. We write

$$f(x) \in k(x)$$

for the image of $f \in R$ by this map and call it the value of $f \in R$ at $x \in |X|$. This is the beginning of the definition of the sheaf of functions $\mathcal{O}_X$, which we will give in Lecture 4. Its stalk at $x \in |X|$ will be the localization

$$\mathcal{O}_{X,x} = R_{\mathfrak{p}}$$

of R with respect to the multiplicative subset $R \smallsetminus \mathfrak{p} \subset R$. We call $\mathcal{O}_{X,x}$ the local ring at $x \in |X|$, and we think of elements of $\mathcal{O}_{X,x}$ as "germs of functions defined in a neighborhood of $x \in |X|$." We will write

$$\mathfrak{m}_x \subset \mathcal{O}_{X,x}$$

for the unique maximal ideal $\mathfrak{p}R_{\mathfrak{p}} \subset R_{\mathfrak{p}}$, and we note that $k(x)$ is also the residue field $\mathcal{O}_{X,x}/\mathfrak{m}_x$ of the local ring $\mathcal{O}_{X,x}$.

Example 1.2. The ring of integers $R = \mathbb{Z}$ has a prime ideal $p\mathbb{Z} \subset \mathbb{Z}$ for every prime number p, and, in addition, the zero ideal $\{0\} \subset \mathbb{Z}$ is a prime ideal. So

$$|X| = |\operatorname{Spec}(\mathbb{Z})| = \{p\mathbb{Z} \subset \mathbb{Z} \mid p \text{ is a prime number}\} \cup \{\{0\}\},$$

and the closed subsets are $V = |X|$ and the finite subsets $V \subset |X| \smallsetminus \{\{0\}\}$. So this is not a Hausdorff space. In fact, it even has a non-closed point. Indeed, the only closed subset that contains $\eta = \{0\} \in |X|$ is all of $|X|$. So not only is the subset

$$A = \{\eta\} \subset |X|$$

not closed, but its closure is all of $|X|$. We say that such a point is a generic point. The residue fields at the various points in $|X|$ are $k(p\mathbb{Z}) = \mathbb{F}_p$ and $k(\eta) = \mathbb{Q}$, and if $n \in \mathbb{Z}$, then its value at $x = p\mathbb{Z} \in |X|$ is

$$n(x) = n + p\mathbb{Z} \in \mathbb{Z}/p\mathbb{Z} = k(x),$$

whereas its value at $\eta = \{0\} \in |X|$ is

$$n(\eta) = n \in \mathbb{Q} = k(\eta).$$

Example 1.3. Let $k = \bar{k}$ be an algebraically closed field, and let $R = k[x_1, \dots, x_n]$. In this case, Hilbert's Nullstellensatz shows that the map

$$k^n \longrightarrow |X| = |\operatorname{Spec}(R)|$$

that to $(a_1, \dots, a_n)$ assigns the maximal ideal $(x_1 - a_1, \dots, x_n - a_n)$ is injective that its image precisely is the subset

$$|\operatorname{max-Spec}(R)| \subset |\operatorname{Spec}(R)|$$

consisting of the maximal ideals $\mathfrak{m} \subset R$. Moreover, if

$$x = (x_1 - a_1, \dots, x_n - a_n) \in |\operatorname{max-Spec}(R)|,$$

then $k(x) = k$ and $f(x) \in k(x)$ is literally the value of f at $(a_1, \dots, a_n)$. However,

$$|\operatorname{max-Spec}(R)| \subsetneq |\operatorname{Spec}(R)|,$$

unless $n = 0$. For example, the generic point

$$\eta = \{0\} \in |\operatorname{Spec}(R)|$$

is not in $|\operatorname{max-Spec}(R)|$.

Remark 1.4. Why not use $|\operatorname{max-Spec}(R)|$ instead of $|\operatorname{Spec}(R)|$? One reason is that it is not functorial. For example, for $\mathbb{Z} \to \mathbb{Q}$, which point in $|\operatorname{max-Spec}(\mathbb{Z})|$ should the unique point in $|\operatorname{max-Spec}(\mathbb{Q})|$ map to? Another reason is that the extra points in $|\operatorname{Spec}(R)|$ are useful. E.g. "spreading out":

(1) For many properties of points on a variety over a algebraically closed field, if the generic point has the property, then so do all points in an open neighborhood of the generic point. In dimension 1, for example, this means that if the property holds for the generic point, then it holds for all but finitely many "classical" points, that is, points that correspond to maximal ideals.
(2) Similar techniques let you reduce statements about varieties over fields of characteristic 0 to varieties over fields of positive characteristic p. This is surprisingly useful! (Mori's bend-and-break technique is of this kind.) For example, given a variety over $\mathbb{Q}$, one first spreads it out to a scheme over $\mathbb{Z}[1/N]$ for some N, and then specializes it to a scheme over $\mathbb{F}_p$ for $p \nmid N$.

Let R be a ring, and let $|X| = |\operatorname{Spec}(R)|$ be its Zariski space. Is $f \in R$ determined by its values $f(x) \in k(x)$ for all $x \in |X|$? Equivalently, if $f(x) = 0$ for all $x \in |X|$, then is $f = 0$? Well, that $f(x) = 0$ for all $x \in |X|$ is equivalent to saying that $f \in \mathfrak{p}$ for all prime ideals $\mathfrak{p} \subset R$. So the answer to the question is "yes" if and only if

$$\textstyle\bigcap_{\mathfrak{p} \subset R} \mathfrak{p} = \{0\}.$$

However, as the following result shows, this is not always the case.

Lemma 1.5. *If R is a ring, then $f \in \bigcap_{\mathfrak{p} \subset R} \mathfrak{p}$ if and only if $f \in R$ is nilpotent.*

Proof. If $f \in R$ is nilpotent, then $f^N = 0$ for some $N \geq 0$. In particular, f^N belong to every ideal in R. If $\mathfrak{p} \subset R$ is a prime ideal, then $f^N \in \mathfrak{p}$ implies that $f \in \mathfrak{p}$. So we conclude that if $f \in R$ is nilpotent, then $f \in \bigcap_{\mathfrak{p} \subset R} \mathfrak{p}$.

Conversely, suppose that $f \in \bigcap_{\mathfrak{p} \subset R} \mathfrak{p}$. We consider the localization

$$R \xrightarrow{\gamma} R_f = R[1/f]$$

at the multiplicative subset $\{1, f, f^2, \dots\} \subset R$. It has the universal property that if $\varphi \colon R \to R'$ is a ring homomorphism such that $\varphi(f) \in R'$ is a unit, then there exists a unique ring homomorphism $\bar{\varphi} \colon R_f \to R'$ that makes the diagram

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr, "\gamma"'] & & R' \\ & R_f \arrow[ur, "\bar{\varphi}"'] & \end{tikzcd}$$

commute. So if $R' = k$ be a field, then a ring homomorphism $\bar{\varphi} \colon R_f \to k$ determines and is determined by a ring homomorphism $\varphi \colon R \to k$ such that $\varphi(f) \neq 0$, or equivalently, such that $f \notin \ker(\varphi)$. But $\ker(\varphi) \subset R$ is a prime ideal, so we conclude from our hypothesis that the ring R_f does not admit a ring homomorphism to a field. This implies that R_f does not have any maximal ideals, which, by Zorn's lemma, implies that $R_f = \{0\}$. So $1 = 0$ in R_f, which by the usual construction of R_f, means that there exists some $N \geq 0$ such that $f^N \cdot 1 = f^N \cdot 0$ in R. So we conclude that f is nilpotent. $\square$

Therefore, if R contains nonzero nilpotent elements, then $f \in R$ is not determined by its values $f(x) \in k(x)$ for all $x \in |X|$. So we cannot think of $f \in R$ as a function in the usual sense. Why not just restrict ourselves to reduced rings? (A ring R reduced if $0 \in R$ is the only nilpotent element.)

First, nilpotent elements make some statements much clearer. To wit, if $k = \bar{k}$ is an algebraically closed field, then two curves $C, D \subset \mathbb{P}^2_k$ of degree m and n intersect in $m \cdot n$ points, generically, but not always. For example, the curves $y = x^2$ and $y = 0$ intersect only in the point $(x, y) = (0, 0)$. However, in this example, the "scheme-theoretic intersection" of the two curves is

$$\operatorname{Spec}(k[x]/(x^2)),$$

which, as a topological space, is a single point, but since

$$\dim_k(k[x]/(x^2)) = 2,$$

it gives the correct multiplicity $2 \cdot 1$. Moreover, the scheme-theoretic intersection is also easier to determine:

$$k[x, y]/(y - x^2) \otimes_{k[x,y]} k[x, y]/(y) \simeq k[x, y]/(y, x^2) \simeq k[x]/(x^2).$$

Second, nilpotent elements are useful for studying reduced rings. Consider

$$R \longrightarrow \mathcal{O}_{X,x} \longrightarrow \widehat{\mathcal{O}}_{X,x} = \lim_n \mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1}.$$

To understand the ring R, we begin with information at $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ and lift successively to the "nilpotent thickenings" $\mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1}$ for all $n \geq 0$, and hence, by passing to the limit, to the complete local ring

$$\widehat{\mathcal{O}}_{X,x} = \lim_n \mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1}.$$

If R is noetherian, then $\mathcal{O}_{X,x} \to \widehat{\mathcal{O}}_{X,x}$ is faithfully flat, so we can pass to $\mathcal{O}_{X,x}$ by faithfully flat descent. Finally, we pass to R by varying $x \in |X|$. This is a powerful strategy in which the nilpotent elements play a key role.

Third, as we will prove next week, $f \in R$ *is* determined by its germs $f_x \in \mathcal{O}_{X,x}$ for all $x \in |X|$. This provides a workable salvage for the failure of $f \in R$ to be determined by its values $f(x) \in k(x)$ for all $x \in |X|$.

After this motivation, we now return to study the Zariski space

$$|X| = |\operatorname{Spec}(R)|.$$

We wish to show that, as S varies over all subsets of R, the subsets

$$V(S) = \{x \in |X| \mid f(x) = 0 \text{ for all } f \in S\} \subset |X|$$

form the closed subsets of a topology on the set of prime ideals in R. We prove the following more detailed result.

Proposition 1.6. *Let R be a ring, and let $|X| = |\operatorname{Spec}(R)|$ be its Zariski space.*

(1) $V(R) = \emptyset$
(2) $V(\{0\}) = |X|$
(3) $V(\bigcup_i S_i) = \bigcap_i V(S_i)$
(4) $V(S \cdot T) = V(S) \cup V(T)$
(5) $V(S) \subset V(T)$ *if and only if* $T \subset \sqrt{(S)}$
(6) $V(S) = V(T)$ *if and only if* $\sqrt{(S)} = \sqrt{(T)}$

Here $(S) \subset R$ is the ideal generated by $S \subset R$, and $\sqrt{(S)} \subset R$ is its radical.

Proof. Parts (1)–(4) are straightforward, as is the fact that (5) implies (6), so we only prove (5).

Suppose that $T \subset \sqrt{(S)}$. If $x \in V(S)$, then $f(x) = 0$ for all $f \in S$. This implies that $f(x) = 0$ for all $f \in (S)$, which, in turn, implies that $f(x) = 0$ for all $f \in \sqrt{(S)}$. So $f(x) = 0$ for all $f \in T$, which shows that $V(S) \subset V(T)$.

Conversely, suppose that $V(S) \subset V(T)$, or equivalent, suppose that for every ring homomorphism $\varphi \colon R \to k$ to a field, $S \subset \ker(\varphi)$ implies that $T \subset \ker(\varphi)$. This, in turn, implies that every ring homomorphism $\bar{\varphi} \colon R/(S) \to k$ to a field annihilates all $f \in T$. By Lemma 1.5, this implies that every $f \in T$ has nilpotent image in $R/(S)$, or equivalently, that every $f \in T$ belongs to $\sqrt{(S)} \subset R$. So $T \subset \sqrt{(S)}$. $\square$

Corollary 1.7. *Let R be a ring.*

(1) *There exists a unique topology on the set of prime ideals in R, the closed subsets of which are the subsets of the form $V(S)$ for some subset $S \subset R$.*
(2) *Every closed subset in this topology is equal to $V(I)$ for a unique radical ideal $I \subset R$.*

The topology on the Zariski space defined in Definition 1.1 is called the Zariski topology. We note that since $V(S) = \cap_{f \in S} V(\{f\})$, we only really need to remember the closed subsets $V(f) = V(\{f\})$.

Definition 1.8. Let R be a ring, and let $|X| = |\operatorname{Spec}(R)|$ be its Zariski space. The distinguished open subset corresponding to $f \in R$ is the subset

$$|X_f| = D(f) = |X| \smallsetminus V(f) = \{x \in |X| \mid f(x) \neq 0\} \subset |X|.$$

Corollary 1.9. *Let R be a ring, and let $|X| = |\operatorname{Spec}(R)|$ be its Zariski space. The family of distinguished open subsets $|X_f| \subset |X|$ indexed by $f \in R$ forms a basis for the Zariski topology on $|X|$. It is closed under finite intersections, since*

$$|X_f| \cap |X_g| = |X_{fg}|$$

for all $f, g \in R$.

Remark 1.10. For many purposes, the points of $|X| = |\operatorname{Spec}(R)|$ are irrelevant! It is only the underlying locale of the topological space $|X|$ that is relevant. This, in turn, is uniquely determined by the family of distinguished open subsets together with the following information:

(1) $|X_f| \subset |X_g|$ if and only if $f \in \sqrt{(g)}$.
(2) The family $(|X_{f_i}| \subset |X_g|)_{i \in I}$ is a cover if and only if $g \in \sqrt{(f_i)_{i \in I}}$.

Remark 1.11. Let R be a ring. As we have already remarked, its Zariski space $|X|$ is typically not Hausdorff or even T_1. If $x, y \in |X|$ are points corresponding to prime ideals $\mathfrak{p}, \mathfrak{q} \subset R$, then the following statements are equivalent:

(1) $y \in \overline{\{x\}}$
(2) If $f \in R$ and $f(x) = 0$, then $f(y) = 0$.
(3) $\mathfrak{p} \subset \mathfrak{q}$

If these equivalent statements hold, then we say that x specializes to y and write $x \rightsquigarrow y$. In particular, a point $x \in |X|$ is closed if and only if the corresponding prime ideal $\mathfrak{p} \subset R$ is a maximal ideal.

Example 1.12. A ring R is an integral domain if and only if the zero ideal $\{0\} \subset R$ is a prime ideal. Suppose that this is the case. Since every prime ideal $\mathfrak{p} \subset R$ contains the zero ideal, the point $\eta \in |X|$ corresponding to the zero ideal is a generic point: its closure is all of $|X|$. So every point $y \in |X|$ is a specialization of the generic point $\eta \in |X|$.

Remark 1.13. Specialization is a partial order on $|X|$. Therefore, it gives rise to a topology on $|X|$ called the specialization topology or the Alexandroff topology, in which a subset $V \subset |X|$ is defined to be closed if it is specialization closed in the sense that if $x \in V$ and $x \rightsquigarrow y$, then $y \in V$. A Zariski closed subset is specialization closed, but the converse is generally not true. For instance, the complement $V = |\operatorname{Spec}(\mathbb{Z})| \smallsetminus \{\eta\} \subset |\operatorname{Spec}(\mathbb{Z})|$ of the generic point is specialization closed but not Zariski closed.

Finally, we establish the functoriality of the Zariski space.

Proposition 1.14. *If $\varphi \colon R \to R'$ is a ring homomorphism, then the map*

$$|\operatorname{Spec}(R')| \xrightarrow{p} |\operatorname{Spec}(R)|$$

defined by $p(\mathfrak{p}') = \varphi^{-1}(\mathfrak{p})$ is well-defined and continuous.

Proof. To show that p is well-defined we must prove that if $\mathfrak{p}' \subset R'$ is a prime ideal, then so is $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}') \subset R$. This is elementary, e.g. if we write $\mathfrak{p}'$ as the kernel of a homomorphism $R' \to k$ to a field (the residue field of $\mathfrak{p}'$), then $\mathfrak{p}$ is the kernel of the composite $R \to R' \to k$, hence also prime.

To prove that $p \colon |X'| \to |X|$ is continuous, it suffices to show that for every distinguished open subset $|X_f| \subset |X|$, the inverse image $p^{-1}(|X_f|) \subset |X|$ is open. In fact, the inverse image is itself a distinguished open subset, namely,

$$p^{-1}(|X_f|) = |X_{\varphi(f)}| \subset |X'|.$$

Indeed, by the definition of p, the left-hand side of this equality is the subset

$$\{\mathfrak{p} \subset R \mid f \notin \varphi^{-1}(\mathfrak{p})\} \subset |\operatorname{Spec}(R)|,$$

whereas the right-hand side of the equality is the subset

$$\{\mathfrak{p} \subset R \mid \varphi(f) \notin \mathfrak{p}\} \subset |\operatorname{Spec}(R)|.$$

So equality holds by the definition of inverse image of a map. $\square$

We will occasionally (but rarely) write $|\operatorname{Spec}(\varphi)|$ for the map p in Proposition 1.14 to indicate its dependence on the map φ. With this notation, we now prove the promised functoriality of the Zariski space construction.

Addendum 1.15. *The following statements hold:*

(1) *For every ring R, the map*

$$|\operatorname{Spec}(R)| \xrightarrow{|\operatorname{Spec}(\mathrm{id}_R)|} |\operatorname{Spec}(R)|$$

is the identify map.

(2) *For every commutative diagram*

$$\begin{tikzcd} R'' & & R' \arrow[ll, "\varphi'"'] \\ & R \arrow[ul, "\psi"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

of rings and ring homomorphisms, the diagram

$$\begin{tikzcd} |\operatorname{Spec}(R'')| \arrow[rr, "|\operatorname{Spec}(\varphi')|"] \arrow[dr, "|\operatorname{Spec}(\psi)|"'] & & |\operatorname{Spec}(R')| \arrow[dl, "|\operatorname{Spec}(\varphi)|"] \\ & |\operatorname{Spec}(R)| & \end{tikzcd}$$

of topological spaces and continuous maps commutes.

Proof. Both properties are immediate consequences of the definition of the map $p = |\operatorname{Spec}(\varphi)|$ given in Proposition 1.14. $\square$

Remark 1.16. Addendum 1.15 is the statement that the Zariski space is a functor

$$\mathrm{CAlg}(\mathrm{Ab})^{\mathrm{op}} \xrightarrow{|\operatorname{Spec}(-)|} \mathrm{Top}$$

from the opposite of the category rings and ring homomorphisms to the category of topological spaces and continuous maps. Here "opposite" refers to the fact that $|\operatorname{Spec}(-)|$ reverses the direction of maps. This functor looses *a lot* of information. For example, it maps every field to a one-point space. We will remedy this disaffect in Lecture 4, when we upgrade the Zariski space to an object of geometry, as opposed to an object of topology.

Last time, we assigned to a ring R a topological space $|X|$, the Zariski space. Later, we will define a sheaf of rings $\mathcal{O}_X$ on $|X|$, and it is the pair

$$X = (|X|, \mathcal{O}_X)$$

of the Zariski space $|X|$ and the (structure) sheaf $\mathcal{O}_X$ that is the scheme associated with R. In this kind of situation, a sheaf of "functions" is what separates geometry from topology. For example, a complex manifold is similarly a pair $X = (|X|, \mathcal{O}_X)$ of a topological space $|X|$ and a sheaf $\mathcal{O}_X$ of "holomorphic functions" on $|X|$ such that, locally on $|X|$, the pair $(|X|, \mathcal{O}_X)$ is isomorphic to a pair $U = (|U|, \mathcal{O}_U^{\mathrm{hol}})$, where $U \subset \mathbb{C}^d$ is an open subset, and where $\mathcal{O}_U^{\mathrm{hol}}$ is the sheaf of holomorphic functions on $|U|$. But before we can define the structure sheaf $\mathcal{O}_X$ on the Zariski space $|X|$, me first need to make ourselves familiar with sheaves. So in this lecture, we will concern ourselves entirely with sheaves on a topological space $|X|$.

Since there will be no geometry this time, we will write X instead of $|X|$ for a topological space. A sheaf on X is an abstraction that encodes the way in which we expect "functions on X" to behave. In fact, functions and sheaves are both precise modern manifestations of the less precise concept of function that existed before set theory was introduced.

Now, given a topological space X, we define X_{Zar} to be the category with objects the open subsets of X and with a single morphism

$$U \xrightarrow{i_U^V} V$$

whenever $U \subset V$. So if $U \not\subset V$, then there are no morphisms from U to V in X_{Zar}.

Definition 2.1. Let X be a topological space. A presheaf on X is a functor

$$X_{\mathrm{Zar}}^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathsf{Set}$$

and a map between presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ on X is a natural transformation. We write $\mathcal{P}(X)$ for the category of presheaves on X and maps between these.

If we think of $\mathcal{F}(V)$ as a set of "functions defined on V," then we should think of the map $\mathcal{F}(i_U^V)\colon \mathcal{F}(V) \to \mathcal{F}(U)$ as the operator that restricts a "function defined on V" to a "function defined on U." To enforce this understanding, we also write

$$\mathcal{F}(V) \xrightarrow{\mathrm{res}_U^V} \mathcal{F}(U)$$

for this operator and call it restriction from V to U. Moreover, if $s \in \mathcal{F}(V)$, then we will also write $s|_U \in \mathcal{F}(U)$ for $\mathrm{res}_U^V(s) \in \mathcal{F}(U)$.

Remark 2.2. More generally, if X is a topological space and $\mathcal{C}$ a category, then we define a $\mathcal{C}$-valued presheaf on X to be a functor

$$X_{\mathrm{Zar}}^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathcal{C},$$

and we again define a map between $\mathcal{C}$-valued presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ to be a natural transformation. We write $\mathcal{P}(X, \mathcal{C})$ for the category of $\mathcal{C}$-values presheaves on X and maps between these. So $\mathcal{P}(X) = \mathcal{P}(X, \mathsf{Set})$.

If $\mathcal{C} = \mathsf{Ab}$ is the category of abelian groups, then we also say that a $\mathcal{C}$-valued presheaf is a presheaf of abelian groups. The category $\mathcal{P}(X, \mathsf{Ab})$ of presheaves of abelian groups on X behaves very similar to the category Ab of abelian groups.

Lemma 2.3. *Let X be a topological space. The category $\mathcal{P}(X, \mathsf{Ab})$ of presheaves of abelian groups on X is abelian, and it admits all (small) products and coproducts. Moreover, products, coproducts, kernels, and cokernels are calculated pointwise.*

The statement that e.g. cokernels in $\mathcal{P}(X, \mathsf{Ab})$ are computed pointwise means that if $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism in this category, then its cokernel is given by

$$\operatorname{coker}(\mathcal{F} \xrightarrow{\varphi} \mathcal{G})(U) = \operatorname{coker}(\mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U)).$$

Proof. The proof is a series of straightforward checks. □

Remark 2.4. Let K be any (small) category. Lemma 2.3 holds more generally for the category $\operatorname{Fun}(K, \mathsf{Ab})$, whose objects are the functors $\mathcal{F}\colon K \to \mathsf{Ab}$, and whose morphisms are the natural transformations between such functors. Moreover, the category $\operatorname{Fun}(K, \mathsf{Ab})$ admits all (small) limits and colimits, and these are calculated pointwise in the following sense. Let $\operatorname{ev}_k\colon \operatorname{Fun}(K, \mathsf{Ab}) \to \mathsf{Ab}$ be the functor given by evaluation at $k \in \operatorname{ob}(K)$. So $\operatorname{ev}_k(\mathcal{F}) = \mathcal{F}(k)$, and $\operatorname{ev}_k(\varphi) = \varphi_k$. Now, suppose that $\bar{p}\colon J^{\triangleleft} \to \operatorname{Fun}(K, \mathsf{Ab})$ is a limit of $p\colon J \to \operatorname{Fun}(K, \mathsf{Ab})$. That this limit is calculated pointwise means that for every $k \in \operatorname{ob}(K)$, the composite functor

$$J^{\triangleleft} \xrightarrow{\bar{p}} \operatorname{Fun}(K, \mathsf{Ab}) \xrightarrow{\operatorname{ev}_k} \mathsf{Ab}$$

is the limit of the composite functor

$$J \xrightarrow{p} \operatorname{Fun}(K, \mathsf{Ab}) \xrightarrow{\operatorname{ev}_k} \mathsf{Ab}.$$

So the value of a limit in $\operatorname{Fun}(K, \mathsf{Ab})$ at the object $k \in \operatorname{ob}(K)$ is given by the corresponding limit in Ab. The similar statement is true for colimits.

We now define sheaves. The definition encodes the property of a "function" that if we know it locally, then we know it globally. A sheaf is defined to be a presheaf that has this property. Before we state the definition, we recall that by a family of elements in a set A indexed by a set I, we mean a map $a\colon I \to A$. We also use the notation $(a_i)_{i\in I}$ instead of $a\colon I \to A$, where $a_i = a(i)$. The definition of a sheaf involves families of open subsets of a topological space X. So here A is the set $\operatorname{ob}(X_{\mathrm{Zar}})$ of open subsets of X.

Definition 2.5. Let X be a topological space. A presheaf $\mathcal{F}$ on X is a sheaf if it satisfies the following condition: For every open subset $U \subset X$ and every open covering $(U_i)_{i\in I}$ of U, the map

$$\mathcal{F}(U) \xrightarrow{(\operatorname{res}^U_{U_i})} \textstyle\prod_{i\in I} \mathcal{F}(U_i)$$

induces a bijection between $\mathcal{F}(U)$ and the subset of $\prod_{i\in I} \mathcal{F}(U_i)$ consisting of the tuples $(s_i)_{i\in I}$ with $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \in \mathcal{F}(U_i \cap U_j)$$

for all $(i, j) \in I \times I$.

We refer to the condition in Definition 2.5 as the "sheaf condition."

Remark 2.6. The sheaf condition applied to the case, where $U = \emptyset \subset X$, and where $(U_i)_{i \in I}$ is the empty covering (meaning that $I = \emptyset$), shows that $\mathcal{F}(\emptyset)$ is a singleton set (meaning a set with exactly one element). Indeed, the product of the empty family of sets is a singleton set.

Example 2.7. Let X be a topological space. If A is any set, then we may consider the presheaf $\mathcal{F} \in \mathcal{P}(X)$ defined by

$$\mathcal{F}(U) = \{\text{constant functions } s \colon U \to A\}$$

and $\operatorname{res}^V_U(s)(u) = s(u)$. It is basically never a sheaf. Instead, its "sheafification" (more on this later) is the sheaf $\operatorname{ass}(\mathcal{F}) \in \mathcal{P}(X)$ defined by

$$\operatorname{ass}(\mathcal{F}) = \{\text{locally constant functions } s \colon U \to A\}$$

and where $\operatorname{res}^V_U(s)(u) = s(u)$. We note that there is a canonical map of presheaves

$$\mathcal{F} \xrightarrow{\eta} \operatorname{ass}(\mathcal{F}),$$

since every constant function is locally constant. We say that $\operatorname{ass}(\mathcal{F})$ is the constant sheaf associated with A. It is often denoted by $\underline{A}$. We note that if A has at least two elements, then for all $\emptyset \neq U \subset X$, the map

$$A = \mathcal{F}(U) \xrightarrow{\eta_U} \operatorname{ass}(\mathcal{F})(U) = \underline{A}(U)$$

is a bijection if and only if $U \subset X$ is connected. This gives a first indication indicattion that sheaf theory can be a useful tool for understanding the topology of X. This observation deepens significantly in the context of sheaf *cohomology*.

Let us state the sheaf condition for $\mathcal{C}$-valued presheaves.

Definition 2.8. Let X be a topological space, and let $\mathcal{C}$ be a category that admits (small) limits. A $\mathcal{C}$-valued presheaf $\mathcal{F}$ on X is a sheaf if for all open subsets $U \subset X$ and all open coverings $(U_i)_{i \in I}$ of U, the diagram

$$\mathcal{F}(U) \xrightarrow{(\operatorname{res}^U_{U_i})} \textstyle\prod_{i \in I} \mathcal{F}(U_i) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \textstyle\prod_{(i,j) \in I \times I} \mathcal{F}(U_i \cap U_j)$$

is a limit diagram (= an equalizer). Here α and β are the unique maps such that

$$\operatorname{pr}_{(i,j)} \circ \alpha = \operatorname{res}^{U_i}_{U_i \cap U_j} \circ \operatorname{pr}_i$$
$$\operatorname{pr}_{(i,j)} \circ \beta = \operatorname{res}^{U_j}_{U_i \cap U_j} \circ \operatorname{pr}_j$$

for all $(i,j) \in I \times I$.

Remark 2.9. A diagram of abelian groups (resp. rings) is a limit diagram if and only if the underlying diagram of sets is a limit diagram. Hence, a presheaf of abelian groups (resp. rings) is a sheaf if and only if the underlying presheaf of sets is a sheaf. For example, if A is an abelian group, then the presheaf $\mathcal{F}$ of constant functions and the sheaf $\underline{A} = \operatorname{ass}(\mathcal{F})$ of locally constant functions that we considered in Example 2.7 both acquire the structure of a presheaf (resp. sheaf) of abelian groups on X with addition given by the pointwise addition of functions.

Example 2.10. Let X be a topological space. If A is a topological abelian group (such as $A = (\mathbb{R}, +)$), then we obtain a presheaf of abelian groups $C(-, A)$, where

$$C(U, A) = \{\text{continuous functions } \varphi \colon U \to A\}$$

with pointwise addition of functions, and where $\mathrm{res}^V_U(\varphi)(u) = \varphi(u)$. It is a sheaf, since the condition for a function to be continuous is a local condition. In particular, if A^δ is the topological abelian group obtained from an abelian group A by giving it the discrete topology, then we have

$$C(-, A^\delta) = \underline{A}.$$

So the sheaf of continuous functions generalizes the constant sheaf.

Example 2.11. Let X be a topological space. The presheaf of abelian groups $\mathcal{F}$, where

$$\mathcal{F}(U) = \{\text{bounded continous functions } \varphi\colon U \to \mathbb{R}\}$$

with pointwise addition, and where $\mathrm{res}^V_U(\varphi)(u) = \varphi(u)$, is generally not a sheaf. Indeed, the condition for a function to be bounded is not a local condition. In fact, every continuous function $\varphi\colon U \to \mathbb{R}$ is locally bounded, but not every continuous function $\varphi\colon U \to \mathbb{R}$ is globally bounded.

Example 2.12. The presheaf $C^\infty(-, \mathbb{R})$ of abelian groups on $X = \mathbb{R}^d$, where

$$C^\infty(U, \mathbb{R}) = \{\text{smooth functions } \varphi\colon U \to \mathbb{R}\}$$

with pointwise addition, and where $\mathrm{res}^V_U(\varphi)(u) = \varphi(u)$, is a sheaf. Indeed, the condition of a map $\varphi\colon U \to \mathbb{R}$ to be smooth is a local condition.

Abstract context for these examples: We say that $\mathcal{F}' \in \mathcal{P}(X)$ is a sub-presheaf of $\mathcal{F} \in \mathcal{P}(X)$ and write $\mathcal{F}' \subset \mathcal{F}$ if $\mathcal{F}'(U) \subset \mathcal{F}(U)$ for every open subset $U \subset X$.

Lemma 2.13. *Let X be a topological space, and let $\mathcal{F}$ be a sheaf on X. Suppose that $\mathcal{F}' \subset \mathcal{F}$ is a sub-presheaf, which satisfies the following condition: For every open subset $U \subset X$ and every open covering $(U_i)_{i \in I}$ of U, a section $s \in \mathcal{F}(U)$ belongs to $\mathcal{F}'(U) \subset \mathcal{F}(U)$ if and only if the local sections $s|_{U_i} \in \mathcal{F}(U_i)$ belongs to $\mathcal{F}'(U_i) \subset \mathcal{F}(U_i)$ for all $i \in I$. In this case, the presheaf $\mathcal{F}'$ is a sheaf.*

Proof. This follows immediately from the definition of a sheaf. $\square$

Let X be a topological space, and let A be a set. The presheaf $\mathcal{F} \in \mathcal{P}(X)$, where

$$\mathcal{F}(U) = \{\text{all maps } \varphi\colon U \to A\},$$

and where $\mathrm{res}^V_U(\varphi)(u) = \varphi(u)$, is a sheaf. The examples that we have considered above are all sub-presheaves of this sheaf, and in each examples, we have indicated whether or not the condition in Lemma 2.13 is satisfied.

Example 2.14. Let $f\colon Y \to X$ be a continuous map between topological spaces, and let $C(-, Y)$ be on X that we considered in Example 2.10. We define

$$\mathrm{sec}_f \subset C(-, Y)$$

to be the sub-presheaf of those $s\colon U \to Y$ that satisfy $f \circ s = i^X_U\colon U \to X$. We say that continuous map $s\colon U \to Y$, which satisfies this condition, is a section of $f\colon Y \to X$ over $U \subset X$. It follows from Lemma 2.13 that sec_f is a sheaf. We may view a section $s \in \mathrm{sec}_f(U)$ as a generalization of a function on U, where the value of s at $x \in U$ is a point $s(x) \in f^{-1}(x) \subset Y$ of a space, which depends on $x \in U$. In particular, if $p\colon Y = X \times F \to X$ is the canonical projection, then $\mathrm{sec}_p = C(-, F)$.

Remark 2.15. In fact, if $\mathcal{F}$ is any sheaf on a topological space X, then there exists a continuous map $f\colon Y \to X$ and an isomorphism $\varphi\colon \mathcal{F} \to \sec_f$ of sheaves on X. (This is the "espace étalé" construction.) However, this is really not a useful point of view for us, but nevertheless, it influences our terminology in that we refer to elements $s \in \mathcal{F}(U)$ as sections of $\mathcal{F}$ over U.

We next define the stalks of a presheaf. This involves the filtered colimits of sets, which we have considered in Problem set 1. The slogan is:

- To understand a presheaf, you need to understand $\mathcal{F}(U)$ for all $U \subset X$ (plus restriction maps).
- To understand a sheaf is a bit easier: you only need to understand $\mathcal{F}(U)$ for all "sufficiently small $U \subset X$" (plus restriction maps), namely, for all $U_i \subset X$ that lie in some element of a fixed open covering $(U_i)_{i \in I}$ of X.
- However, you can get good, but not complete, knowledge by looking at "germs of sections defined in a neighborhood of every point $x \in X$."

We proceed to make sense of the last slogan. Let X be a topological space. The category of neighborhoods of $x \in X$ is the full subcategory of X_{Zar} spanned by the $U \subset X$ such that $x \in U$. Its opposite category is filtered.

Definition 2.16. Let X be a topological space. The stalk of a presheaf $\mathcal{F} \in \mathcal{P}(X)$ at the point $x \in X$ is the filtered colimit

$$\mathcal{F}_x = \operatorname{colim}_{x \in U \subset X} \mathcal{F}(U)$$

indexed by the opposite of the category of neighborhoods of $x \in X$.

In Problem set 1, we gave a description of a filtered colimit of sets. Spelling this out in the case at hand, we see that the stalk $\mathcal{F}_x$ of $\mathcal{F} \in \mathcal{P}(X)$ at $x \in X$ is given by the quotient of the set of pairs (U, s) of an open neighborhood $x \in U \subset X$ and a section $s \in \mathcal{F}(U)$ by the equivalence relation that identifies (U, s) and (V, t) if there exists $x \in W \subset U \cap V \subset X$ such that $s|_W = t|_W \in \mathcal{F}(W)$. We say that an element of $\mathcal{F}_x$ is a germ of sections of $\mathcal{F}$ at $x \in X$.

Notation 2.17. Let X be a topological space, let $\mathcal{F} \in \mathcal{P}(X)$, and let $x \in X$. For every open neighborhood $x \in V \subset X$, we have a canonical map

$$\mathcal{F}(V) \longrightarrow \mathcal{F}_x = \operatorname{colim}_{x \in U \subset X} \mathcal{F}(U),$$

which is part of what it means to be a colimit. Given $s \in \mathcal{F}(V)$, then we write $s_x \in \mathcal{F}_x$ for its image under this map and call it the germ of s at $x \in X$.

Example 2.18. The derivative of a smooth function $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ at $x \in \mathbb{R}$ depends only on its germ $\varphi_x \in C^\infty(-, \mathbb{R})_x$ at $x \in \mathbb{R}$.

Remark 2.19. We recall from Problem set 1 that, as a consequence of Grothendieck's theorem that filtered colimits and finite limits of sets commute, the forgetful functor

$$\mathsf{Ab} \xrightarrow{\text{fgt}} \mathsf{Set}$$

creates (small) filtered colimits. This means that if J is a (small) filtered category, then a diagram $\bar{p}\colon J^{\triangleright} \to \mathsf{Ab}$ of abelian groups is a colimit of $p = \bar{p}|_J\colon J \to \mathsf{Ab}$ if and only if the diagram $\text{fgt} \circ \bar{p}\colon J^{\triangleright} \to \mathsf{Set}$ is a colimit of $\text{fgt} \circ p\colon J \to \mathsf{Set}$. Therefore, if $\mathcal{F} \in \mathcal{P}(X, \mathsf{Ab})$ is a presheaf of abelian groups on X, then its stalk

$$\mathcal{F}_x = \operatorname{colim}_{x \in U \subset X} \mathcal{F}(U) \in \mathsf{Ab}$$

is given by the stalk of the underlying presheaf of sets with the unique structure of abelian group, which makes it the colimit in the category of abelian groups. The analogous statement is true for filtered colimits of presheaves of rings.

We now state some results, which show that stalks/germs give useful information about sheaves. We will prove them in the next lecture.

Proposition 2.20. *Let X be a topological space.*

(1) *Let $\mathcal{F}$ be a sheaf on X, and let $U \subset X$ be open. Two sections $s, t \in \mathcal{F}(U)$ are equal if and only if their germs $s_x, t_x \in \mathcal{F}_x$ are equal for all $x \in U$.*
(2) *A map $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves on X is an isomorphism if and only if the induced map of stalks $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for all $x \in X$.*

The statement (2) is referred to by saying that the topos of sheaves on X has enough points. We note that (2) does *not* say that if there exists an isomorphism between $\mathcal{F}_x$ and $\mathcal{G}_x$ for all $x \in X$, then there exists an isomorphism between $\mathcal{F}$ and $\mathcal{G}$. You need to have a map $\varphi\colon \mathcal{F} \to \mathcal{G}$. If you have the map, then you can test whether or not it is an isomorphism on stalks. But you cannot get the map of sheaves from having maps of stalks. This is a general phenomenon. You cannot make global constructions using stalks; you can only use stalks to test properties of global construction that you have already made.

Given $\mathcal{F}, \mathcal{G} \in \mathcal{P}(X)$, we write $\mathrm{Map}(\mathcal{F}, \mathcal{G})$ for the set of maps of presheaves.

Definition 2.21. Let X be a topological space. A map $\varphi\colon \mathcal{F} \to \mathcal{F}'$ from a presheaf on X to a sheaf on X is a sheafification if composition with φ induces a bijection

$$\mathrm{Map}(\mathcal{F}', \mathcal{G}) \longrightarrow \mathrm{Map}(\mathcal{F}, \mathcal{G}).$$

for every sheaf $\mathcal{G}$ on X.

It follows immediately that a sheafification of $\varphi\colon \mathcal{F} \to \mathcal{F}'$ of $\mathcal{F}$ is unique, up to unique isomorphism. We will show later that it exists. The following result is the main theorem of sheaf theory.

Theorem 2.22. *Let X be a topological space. A map $\varphi\colon \mathcal{F} \to \mathcal{F}'$ from a presheaf on X to a sheaf on X is a sheafification if and only if the induced map of stalks*

$$\mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{F}'_x$$

is a bijection for all $x \in X$.

3. Sheafification

We continue the discussion of sheaf theory. So let X be a topological space. We recall that the stalk of $\mathcal{F} \in \mathcal{P}(X)$ at $x \in X$ is the filtered colimit

$$\mathcal{F}_x = \operatorname{colim}_{x \in U \subset X} \mathcal{F}(U)$$

indexed by the opposite category of the category of open neighborhoods $x \in U \subset X$. It is given by the quotient of the set of pairs (U, s) with $x \in U \subset X$ open and $s \in \mathcal{F}(U)$ by the equivalence relation that identifies (U, s) and (V, t) if there exists $x \in W \subset U \cap V$ open such that $s|_W = t|_W \in \mathcal{F}(W)$. The canonical map

$$\mathcal{F}(U) \longrightarrow \mathcal{F}_x$$

takes the local section $s \in \mathcal{F}(U)$ to the germ $s_x \in \mathcal{F}_x$ given by the equivalence class of the pair (U, s). Moreover, if $\mathcal{F} \in \mathcal{P}(X, \mathsf{Ab})$ (resp. $\mathcal{F} \in \mathcal{P}(X, \mathrm{CAlg}(\mathsf{Ab}))$) if a presheaf of abelian groups (resp. a presheaf of rings), then there is a unique abelian group structure (resp. a unique ring structure) on $\mathcal{F}_x$ such that $\mathcal{F}(U) \to \mathcal{F}_x$ is a group homomorphism (resp. ring homomorphism) for all $x \in U \subset X$ open. More concretely, if $s_x, t_x \in \mathcal{F}_x$, then we choose representatives (U, s) and (V, t) of these equivalence classes as well as an open neighborhood $x \in W \subset U \cap V$ and define

$$s_x + t_x \in \mathcal{F}_x \qquad (\text{resp. } s_x + t_x, s_x \cdot t_x \in \mathcal{F}_x)$$

to be the equivalence class of the pair $(W, s|_W + t|_W)$ (resp. the equivalence classes of the pairs $(W, s|_W + t|_W)$ and $(W, s|_W \cdot t|_W)$), where we use the given addition (resp. the given addition and multiplication) on the set $\mathcal{F}(W)$.

Lemma 3.1. *Let X be a topological space, and let $x \in X$. The functor*

$$\mathcal{P}(X) \longrightarrow \mathsf{Set}$$

that to $\mathcal{F}$ assigns $\mathcal{F}_x$ preserves finite limits and all (small) colimits.

Proof. The statement concerning finite limits is a particular case of Grothendieck's theorem that finite limits and filtered colimits of sets commute. The statement about colimits is a particular case of the general fact that if $\mathcal{C}$ is any category that admits small colimits, and if $p \colon K \times L \to \mathcal{C}$ is a diagram indexed by a product of two (small) categories, then the canonical maps

$$\begin{tikzcd}[column sep=tiny] & \operatorname{colim}_{(k,l) \in K \times L} p(k,l) \arrow[dl] \arrow[dr] & \\ \operatorname{colim}_{k \in K} \operatorname{colim}_{l \in L} p(k,l) & & \operatorname{colim}_{l \in L} \operatorname{colim}_{k \in K} p(k,l) \end{tikzcd}$$

are isomorphisms. We refer to this statement as "colimits commute." □

A functor $F \colon \mathcal{A} \to \mathcal{B}$ between additive categories is additive if it preserves finite sums, or equivalently, if it preserves finite products. A functor $F \colon \mathcal{A} \to \mathcal{B}$ between abelian categories is left exact (resp. right exact) if it preserves finite limits (resp. finite colimits), or equivalently, if it is additive and preserves kernels (resp. it is additive and preserves cokernels). It is exact if it is both left exact and right exact.

Corollary 3.2. *Let X be a topological space, and let $x \in X$. The functor*

$$\mathcal{P}(X, \mathsf{Ab}) \longrightarrow \mathsf{Ab}$$

that to $\mathcal{F}$ assigns $\mathcal{F}_x$ is exact and preserves all (small) sums.

Proof. We give two proofs. First, since the forgetful functor fgt: $\mathsf{Ab} \to \mathsf{Set}$ creates all (small) limits and all (small) filtered colimits, it follows from Lemma 3.1 that the functor $\mathcal{P}(X, \mathsf{Ab}) \to \mathsf{Ab}$ preserves finite limits. It also preserves all (small) colimits, since "colimits commute." This completes the proof.

Second, we give a more pedestrian proof, using the characterization of exactness in terms of kernels and cokernels, which is perhaps more familiar. The functor preserves all (small) colimits, by the general fact that "colimits commute." So it is additive and preserves cokernels as well as all (small) sums. It remains to prove that it preserves kernels. So given a map $\varphi\colon \mathcal{F} \to \mathcal{G}$ of presheaves of abelian groups on X, we wish to prove that the canonical map

$$\ker(\mathcal{F} \xrightarrow{\varphi} \mathcal{G})_x \longrightarrow \ker(\mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x)$$

is an isomorphism. A map of abelian groups is an isomorphism if and only if the underlying map of sets is an isomorphism if and only if the underlying map of sets is both injective and surjective.

To prove injectively, let $s \in \ker(\varphi)(U)$ and suppose that $s_x \in \ker(\varphi_x) \subset \mathcal{F}_x$ is zero. In this case, there exists $x \in V \subset U$ such that $s|_V \in \mathcal{F}(V)$ is zero. Hence, the equivalence class $s_x \in \mathcal{F}_x$ of $(V, s|_V)$ is also zero, which is what we wanted to prove.

To prove surjectivity, let $s \in \mathcal{F}(U)$ and suppose that $\varphi_x(s_x) = \varphi_U(s)_x \in \mathcal{G}_x$ is zero. In this case, there exists $x \in V \subset U$ such that $\varphi_U(s)|_V \in \mathcal{G}(V)$ is zero. This shows that $s|_V \in \mathcal{F}(V)$ belongs to $\ker(\varphi)(V)$, which is what we wanted to prove. □

Lemma 3.3. *Let X be a topological space, and let $\mathcal{F}$ be a sheaf of sets on X. Let $U \subset X$ be an open subset, and let $s, t \in \mathcal{F}(U)$. If $s_x = t_x \in \mathcal{F}_x$ for all $x \in U$ then $s = t \in \mathcal{F}(U)$.*

Proof. Given $x \in U$, since $s_x = t_x \in \mathcal{F}_x$, there exists an open subset $x \in U_x \subset U$ such that $s|_{U_x} = t|_{U_x} \in \mathcal{F}(U_x)$. The family $(U_x)_{x \in U}$ is an open cover of U, and by our assumption, the local sections $s, t \in \mathcal{F}(U)$ have the same image by the map

$$\mathcal{F}(U) \xrightarrow{(\mathrm{res}^U_{U_x})} \textstyle\prod_{x \in U} \mathcal{F}(U_x).$$

Since $\mathcal{F}$ is a sheaf, this map is injective, so we conclude that $s = t$ as desired. □

Proposition 3.4. *Let X be a topological space, and let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves of sets on X. If the induced map $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is a bijection for all $x \in X$, then the map $\varphi\colon \mathcal{F} \to \mathcal{G}$ is an isomorphism.*

Proof. We wish to show that for all $U \subset X$ open, the map $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is a bijection. So we fix $U \subset X$ open and prove that the map in question is both injective and surjective.

To prove injectivity, let $s, t \in \mathcal{F}(U)$ and suppose that $\varphi_U(s) = \varphi_U(t) \in \mathcal{G}(U)$. This implies that $\varphi_x(s_x) = \varphi_U(s)_x = \varphi_U(t)_x = \varphi_x(t_x) \in \mathcal{G}_x$ for all $x \in U$, so by our assumption that $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is a bijection for all $x \in X$, we conclude that $s_x = t_x \in \mathcal{F}_x$ for all $x \in U$. Hence, by Lemma 3.3, we conclude that $s = t \in \mathcal{F}(U)$.

To prove surjectively, we let $t \in \mathcal{G}(U)$ and must find $s \in \mathcal{F}(U)$ with $\varphi_U(s) = t$. Now, for every $x \in U$, our assumption that $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is a bijection implies that there exists $x \in U_i \subset X$ and $s_i \in \mathcal{F}(U_i)$ such that $\varphi_{U_i}(s_i)_x = t_x \in \mathcal{G}_x$. Moreover, by shrinking $x \in U_i \subset X$, if necessary, we can assume that $\varphi_{U_i}(s_i) = t|_{U_i}$. We claim that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Indeed, we have

$$\varphi_{U_i \cap U_j}(s_i|_{U_i \cap U_j}) = t|_{U_i \cap U_j} = \varphi_{U_i \cap U_j}(s_j|_{U_i \cap U_j}),$$

and we have already proved that $\varphi_{U_i \cap U_j}$ is injective. Since $\mathcal{F}$ is a sheaf, the claim implies that there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i \in \mathcal{F}_{U_i}$ for all i. Finally,

$$\varphi_U(s)|_{U_i} = \varphi_{U_i}(s|_{U_i}) = \varphi_{U_i}(s_i) = t|_{U_i}$$

for all i, and since $\mathcal{G}$ is a sheaf, we conclude from Lemma 3.3 that $\varphi_U(s) = t$. $\square$

Remark 3.5. In the proof of Proposition 3.4, our proof that $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective used the fact that this map is injective, which we had proved first. In fact, if $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a map of sheaves of sets on X such that $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$, then $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is generally *not* surjective for all $U \subset X$ open!

The proofs above all use the following general principle: In a filtered colimit of sets, any finitary construction/property reduces to a construction/property at some stage of the filtered colimit. This, in turn, follows from the description of a filtered colimit of sets that we gave in Problem set 1.

We proceed to prove the main theorem of sheaf theory, which we stated at the end of Lecture 2. We first make a definition.

Definition 3.6. Let X be a topological space, and let $\mathcal{C}$ be a category that admits all (small) limits. The category of $\mathcal{C}$-valued sheaves on X is the full subcategory

$$\mathrm{Sh}(X, \mathcal{C}) \subset \mathcal{P}(X, \mathcal{C})$$

spanned by the $\mathcal{C}$-valued sheaves on X.

We write $\iota \colon \mathrm{Sh}(X, \mathcal{C}) \to \mathcal{P}(X, \mathcal{C})$ for the canonical inclusion functor. We recall that a map $\varphi \colon \mathcal{F} \to \iota(\mathcal{F}')$ with $\mathcal{F} \in \mathcal{P}(X, \mathcal{C})$ and $\mathcal{F}' \in \mathrm{Sh}(X, \mathcal{C})$ is defined to be a sheafification if for every $\mathcal{G} \in \mathrm{Sh}(X, \mathcal{C})$, the composite map

$$\mathrm{Map}(\mathcal{F}', \mathcal{G}) \xrightarrow{\iota} \mathrm{Map}(\iota(\mathcal{F}'), \iota(\mathcal{G})) \longrightarrow \mathrm{Map}(\mathcal{F}, \iota(\mathcal{G})),$$

where the second map is given by composition with $\varphi \colon \mathcal{F} \to \iota(\mathcal{F}')$, is a bijection.

Proposition 3.7. *Let X be a topological space. There exists a functor*

$$\mathcal{P}(X) \xrightarrow{\mathrm{ass}} \mathrm{Sh}(X)$$

and a natural transformation $\eta \colon \mathrm{id} \to \iota \circ \mathrm{ass}$ such that the following hold:

(1) *For every $x \in X$, the induced map of stalks*

$$\mathcal{F}_x \xrightarrow{\eta_{\mathcal{F},x}} (\iota \circ \mathrm{ass})(\mathcal{F})_x$$

is a bijection.

(2) *The map $\eta_{\mathcal{F}} \colon \mathcal{F} \to (\iota \circ \mathrm{ass})(\mathcal{F})$ is a sheafification.*

Proof. As a first approximation, we consider the Godement construction. It is given by the functor $\mathrm{God}\colon \mathcal{P}(X) \to \mathrm{Sh}(X)$ with

$$\mathrm{God}(\mathcal{F})(U) = \textstyle\prod_{x \in U} \mathcal{F}_x$$

and with restriction maps given by the canonical projections. It is clear that the presheaf $\mathrm{God}(\mathcal{F})$ is indeed a sheaf. Moreover, the maps

$$\mathcal{F}(U) \xrightarrow{\tilde{\eta}_{\mathcal{F},U}} (\iota \circ \mathrm{God})(U)$$

that to a local section $s \in \mathcal{F}(U)$ assigns the tuple $(s_x)_{x \in U} \in (\iota \circ \mathrm{God})(\mathcal{F})(U)$ form a natural transformation $\tilde{\eta}\colon \mathrm{id} \to \iota \circ \mathrm{God}$.

The Godement construction is too large. For example, if $\mathcal{F}$ is the presheaf of constant functions with values a set A, then $\mathrm{God}(\mathcal{F})$ is the sheaf of all functions with values in A, and we want $\mathrm{ass}(\mathcal{F})$ to be the subsheaf of locally constant functions with values in A. So for general $\mathcal{F} \in \mathcal{P}(X)$, we define

$$\mathrm{ass}(\mathcal{F}) \subset \mathrm{God}(\mathcal{F})$$

to be the subsheaf, where $\mathrm{ass}(\mathcal{F})(U) \subset \mathrm{God}(\mathcal{F})(U)$ is the subset that consists of tuples $(s(x))_{x \in U}$ with the property that there exists an open cover $(U_i)_{i \in I}$ of U and local sections $s_i \in \mathcal{F}(U_i)$ such that $s(x) = s_{i,x}$ for all $i \in I$ and $x \in U_i$. Since this is a local condition, it follows from Lemma 13 in Lecture 2 that $\mathrm{ass}(\mathcal{F})$ is indeed a sheaf. Moreover, the natural transformation $\tilde{\eta}\colon \mathrm{id} \to \iota \circ \mathrm{God}$ takes values in $\iota \circ \mathrm{ass} \subset \iota \circ \mathrm{God}$, so it induces a natural transformation $\eta\colon \mathrm{id} \to \iota \circ \mathrm{ass}$.

We next prove (1). Let us first remark that, in general, if $\varphi\colon q \to p$ is a natural transformation between diagrams $p, q\colon J \to \mathsf{Set}$ indexed by a filtered category J, and if $\varphi_j\colon q(j) \to p(j)$ is injective for all $j \in \mathrm{ob}(J)$, then also the induced map $\mathrm{colim}_J\, \varphi\colon \mathrm{colim}_J\, q \to \mathrm{colim}_j\, p$ is injective. Indeed, a map $f\colon Y \to X$ between sets is injective if and only if the diagram

$$\begin{tikzcd}
Y \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & Y \arrow[d, "f"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

is a limit diagram, so the statement follows from Grothendieck's theorem that, in the category of sets, finite limits and filtered colimits commute. Now, in order to prove (1), we consider the commutative diagram

$$\begin{tikzcd}
\mathcal{F}_x \arrow[rr, "\eta_{\mathcal{F},x}"] \arrow[dr, "\tilde{\eta}_{\mathcal{F},x}"'] & & (\iota \circ \mathrm{ass})(\mathcal{F})_x \arrow[dl] \\
& (\iota \circ \mathrm{God})(\mathcal{F})_x &
\end{tikzcd}$$

in which the two slanted maps are injective by the remark above. Therefore, the map $\eta_{\mathcal{F},x}$ is injective. But is also surjective, by the definition of $\mathrm{ass}(\mathcal{F})$, so (1) holds.

Finally, we prove (2). So let $\varphi\colon \mathcal{F} \to \iota(\mathcal{G})$ be a map with $\mathcal{G} \in \mathrm{Sh}(X)$. We must show that there exists a unique map $\bar{\varphi}\colon \operatorname{ass}(\mathcal{F}) \to \mathcal{G}$ that makes the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\eta_{\mathcal{F}}"] \arrow[dr, "\varphi"'] & (\iota \circ \operatorname{ass})(\mathcal{F}) \arrow[d, "\bar{\varphi}"] \\ & \iota(\mathcal{G}) \end{tikzcd}$$

commute. Uniqueness holds, since, locally, every locally section of $(\iota \circ \operatorname{ass})(\mathcal{F})$ comes from a local section of $\mathcal{F}$, and to prove existence, we consider the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\eta_{\mathcal{F}}"] \arrow[d, "\varphi"] & (\iota \circ \operatorname{ass})(\mathcal{F}) \arrow[d, "(\iota\circ\operatorname{ass})(\varphi)"] \\ \iota(\mathcal{G}) \arrow[r, "\eta_{\iota(\mathcal{G})}"] & (\iota \circ \operatorname{ass} \circ \iota)(\mathcal{G}). \end{tikzcd}$$

which commutes by the naturality of $\eta\colon \mathrm{id} \to \iota \circ \operatorname{ass}$. It follows from (1) that the bottom horizontal map induces a bijection on stalks, and therefore, it is an isomorphism by Proposition 3.4. So the composition of its inverse and the right-hand vertical map gives the desired map $\bar{\varphi}$. $\square$

We can now prove the "main theorem of sheaf theory," which we stated as Theorem 2.22.

Theorem 3.8. *Let X be a topological space, and let $\varphi\colon \mathcal{F} \to \iota(\mathcal{F}')$ be map from a presheaf on X to a sheaf on X. The following are equivalent:*

(1) *For every $x \in X$, the induced map of stalks*

$$\mathcal{F}_x \xrightarrow{\varphi_x} \iota(\mathcal{F}')_x$$

is a bijection.

(2) *For every sheaf $\mathcal{G}$ on X, the composite map*

$$\mathrm{Map}(\mathcal{F}', \mathcal{G}) \xrightarrow{\iota} \mathrm{Map}(\iota(\mathcal{F}'), \iota(\mathcal{G})) \longrightarrow \mathrm{Map}(\mathcal{F}, \iota(\mathcal{G})),$$

where the second map is given by composition with φ, is a bijection.

Proof. We first show that (2) implies (1). Since both the maps $\varphi\colon \mathcal{F} \to \iota(\mathcal{F}')$ and $\eta_{\mathcal{F}}\colon \mathcal{F} \to (\iota \circ \operatorname{ass})(\mathcal{F})$ satisfy (2), we have unique factorizations

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\varphi"] \arrow[dr, "\eta_{\mathcal{F}}"'] & & \iota(\mathcal{F}') \arrow[dl, "\bar{\eta}_{\mathcal{F}}"] \\ & (\iota \circ \operatorname{ass})(\mathcal{F}) & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{F} \arrow[rr, "\eta_{\mathcal{F}}"] \arrow[dr, "\varphi"'] & & (\iota \circ \operatorname{ass})(\mathcal{F}) \arrow[dl, "\bar{\varphi}"] \\ & \iota(\mathcal{F}') & \end{tikzcd}$$

and $\bar{\eta}$ and $\bar{\varphi}$ are each other's inverses. We consider the induced diagrams

$$\begin{tikzcd} \mathcal{F}_x \arrow[rr, "\varphi_x"] \arrow[dr, "\eta_{\mathcal{F},x}"'] & & \iota(\mathcal{F}')_x \arrow[dl, "\bar{\eta}_{\mathcal{F},x}"] \\ & (\iota \circ \operatorname{ass})(\mathcal{F})_x & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{F}_x \arrow[rr, "\eta_{\mathcal{F},x}"] \arrow[dr, "\varphi_x"'] & & (\iota \circ \operatorname{ass})(\mathcal{F})_x \arrow[dl, "\bar{\varphi}_x"] \\ & \iota(\mathcal{F}')_x & \end{tikzcd}$$

of stalks at $x \in X$. The maps $\bar{\varphi}_x$ and $\bar{\eta}_{\mathcal{F},x}$ are each other's inverses, because $\bar{\varphi}$ and $\bar{\eta}_{\mathcal{F}}$ are so, because taking stalks is a functor, and because any functor takes isomorphisms to isomorphisms. In particular, both maps are bijections. But the map $\eta_{\mathcal{F},x}$ is also a bijection, because $\eta_{\mathcal{F}}\colon \mathcal{F} \to (\iota \circ \mathrm{ass})(\mathcal{F})$ satisfies (1). It follows that also φ_x is a bijection, so $\varphi\colon \mathcal{F} \to \iota(\mathcal{F}')$ satisfies (1).

Conversely, suppose that $\varphi\colon \mathcal{F} \to \iota(\mathcal{F}')$ satisfies (1). Since $\eta\colon \mathrm{id} \to \iota \circ \mathrm{ass}$ is a natural transformation, the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\eta_{\mathcal{F}}"] \arrow[d, "\varphi"] & (\iota \circ \mathrm{ass})(\mathcal{F}) \arrow[d, "(\iota\circ\mathrm{ass})(\varphi)"] \\ \iota(\mathcal{F}') \arrow[r, "\eta_{\iota(\mathcal{F}')}"] & (\iota \circ \mathrm{ass})(\iota(\mathcal{F}')) \end{tikzcd}$$

commutes. Moreover, in the induced diagram

$$\begin{tikzcd} \mathcal{F}_x \arrow[r, "\eta_{\mathcal{F},x}"] \arrow[d, "\varphi_x"] & (\iota \circ \mathrm{ass})(\mathcal{F})_x \arrow[d, "(\iota\circ\mathrm{ass})(\varphi)_x"] \\ \iota(\mathcal{F}')_x \arrow[r, "\eta_{\iota(\mathcal{F}'),x}"] & (\iota \circ \mathrm{ass})(\iota(\mathcal{F}'))_x \end{tikzcd}$$

of stalks at $x \in X$, we know that all maps, except $(\iota \circ \mathrm{ass})(\varphi)_x$, are bijections, so therefore, this map is a bijection, too. Hence, we conclude from Proposition 3.4 that, in the top diagram, both $(\iota \circ \mathrm{ass})(\varphi)$ and $\eta_{\iota(\mathcal{F}')}$ are isomorphisms, and since $\eta_{\mathcal{F}}\colon \mathcal{F} \to (\iota \circ \mathrm{ass})(\mathcal{F})$ satisfies (2), the same is true for $\varphi\colon \mathcal{F} \to \iota(\mathcal{F}')$. □

Remark 3.9. There is a completely different way to construct the sheafification functor $\mathrm{ass}\colon \mathcal{P}(X) \to \mathrm{Sh}(X)$ than the one given in the proof of Proposition 3.7. However, what matters is the theorem as stated, not the precise construction. For example, it follows directly from (2) that sheafification is a functor, so Theorem 3.8 also implies Proposition 3.7.

Addendum 3.10. *Let X be a topological space.*

(1) *The category* $\mathrm{Sh}(X)$ *admits all (small) limits and colimits. Limits therein are calculated pointwise, whereas colimits are given by*

$$\mathrm{colim}_K\, p \simeq \mathrm{ass}(\mathrm{colim}_K(\iota \circ p)).$$

(2) *The sheafification functor* $\mathrm{ass}\colon \mathcal{P}(X) \to \mathrm{Sh}(X)$ *preserves finite limits and all (small) colimits.*

Proof. First, the full subcategory $\mathrm{Sh}(X) \subset \mathcal{P}(X)$ is preserved under limits, since the sheaf condition itself is expressed in terms of limits and since “limits commute.” It follows that $\mathrm{Sh}(X)$ admits all (small) limits and that they are calculated pointwise, since this is true in $\mathcal{P}(X)$.

Next, the statement that $\eta\colon \mathrm{id} \to \iota \circ \mathrm{ass}$ is a sheafification is precisely the statement that the functor $\mathrm{ass}\colon \mathcal{P}(X) \to \mathrm{Sh}(X)$ is a left adjoint of the functor $\iota\colon \mathrm{Sh}(X) \to \mathcal{P}(X)$. Also, in any category $\mathcal{C}$ that admits K-indexed colimits, the colimit functor $\mathrm{colim}_K\colon \mathrm{Fun}(K, \mathcal{C}) \to \mathcal{C}$ is left adjoint to the diagonal functor $\Delta\colon \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$. The statement that (small) colimits in $\mathrm{Sh}(X)$ exist and are given by the stated formula now follows from the canonical bijections

$$\begin{aligned} \mathrm{Map}(\mathrm{ass}(\mathrm{colim}_K(\iota \circ p)), \mathcal{G}) &\simeq \mathrm{Map}(\mathrm{colim}_K(\iota \circ p), \iota(\mathcal{G})) \simeq \mathrm{Map}(\iota \circ p, \Delta(\iota(\mathcal{G}))) \\ &\simeq \mathrm{Map}(\iota \circ p, \iota(\Delta(\mathcal{G}))) \simeq \mathrm{Map}(p, \Delta(\mathcal{G})), \end{aligned}$$

where the final bijection, because the functor $\iota\colon \mathcal{P}(X) \to \mathrm{Sh}(X)$ is fully faithful.

Finally, it is a general fact that any left adjoint functor preserves all colimits that exist in its domain category, and any right adjoint functor preserves all limits that exist in its domain category. So $\mathrm{ass}\colon \mathcal{P}(X) \to \mathrm{Sh}(X)$ preserves all (small) colimits, and $\iota\colon \mathrm{Sh}(X) \to \mathcal{P}(X)$ preserves all (small) limits. Thus, it remains only to prove that $\mathrm{ass}\colon \mathcal{P}(X) \to \mathrm{Sh}(X)$ preserves finite limits. So let $\mathcal{F}\colon J \to \mathcal{P}(X)$ be a diagram indexed by a finite category J. We wish to show that the canonical map

$$\mathrm{ass}(\lim_{j\in J} \mathcal{F}_j) \longrightarrow \lim_{j\in J} \mathrm{ass}(\mathcal{F}_j)$$

is an isomorphism. Since $\iota\colon \mathrm{Sh}(X) \to \mathcal{P}(X)$ is fully faithful and preserves all limits, this is equivalent to showing that the induced map

$$(\iota \circ \mathrm{ass})(\lim_{j\in J} \mathcal{F}_j) \longrightarrow \iota(\lim_{j\in J} \mathrm{ass}(\mathcal{F}_j)) \simeq \lim_{j\in J}(\iota \circ \mathrm{ass})(\mathcal{F}_j)$$

is an isomorphism. By Proposition 3.4, this, in turn, is equivalent to showing that for all $x \in X$, the induced map of stalks

$$(\iota \circ \mathrm{ass})(\lim_{j\in J} \mathcal{F}_j)_x \longrightarrow (\lim_{j\in J}(\iota \circ \mathrm{ass})(\mathcal{F}_j))_x$$

is a bijection. To this end, we embed this map in the commutative diagram

$$\begin{tikzcd}
(\lim_{j\in J} \mathcal{F}_j)_x \arrow[rr] \arrow[d, "(\eta_{\lim_{j\in J}\mathcal{F}_j})_x"] & & \lim_{j\in J}(\mathcal{F}_{j,x}) \arrow[d, "\lim_{j\in J}(\eta_{\mathcal{F}_j,x})"] \\
(\iota \circ \mathrm{ass})(\lim_{j\in J} \mathcal{F}_j)_x \arrow[r] & (\lim_{j\in J}(\iota \circ \mathrm{ass})(\mathcal{F}_j))_x \arrow[r] & \lim_{j\in J}((\iota \circ \mathrm{ass})(\mathcal{F}_j)_x),
\end{tikzcd}$$

where the vertical maps are bijections by part (1) of Theorem 3.8, and where the top horizontal and lower right-hand horizontal maps are bijections, by Grothendieck's theorem that finite limits and filtered colimits of sets commute. So also the lower left-hand horizontal map is a bijection, which is what we want to prove. $\square$

Since sheafification preserves finite limits, and hence, finite products, it follows that it induces a sheafification functor from presheaves of abelian groups (resp. presheaves of rings) to sheaves of abelian group (resp. sheaves of rings), which is left adjoint to the canonical inclusion functor.

Corollary 3.11. *Let X be a topological space.*

(1) *The category $\mathrm{Sh}(X, \mathsf{Ab})$ is abelian and admits all (small) sums and products. Moreover, products and kernels are calculated pointwise, whereas sums and cokernels are calculated by sheafifying the corresponding sums and cokernels of the underlying presheaves.*
(2) *The sheafification functor is exact and preserves all (small) sums.*
(3) *A sequence of sheaves $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is exact (at $\mathcal{F}$) if and only if the induced sequence of stalks $\mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x$ is exact (at $\mathcal{F}_x$) for all $x \in X$.*

Proof. Let $\mathcal{A}$ be an abelian category that admits all (small) sums and products, and let $\mathcal{B} \subset \mathcal{A}$ be a full subcategory. If the canonical inclusion $\iota\colon \mathcal{B} \to \mathcal{A}$ admits a left adjoint functor $L\colon \mathcal{A} \to \mathcal{B}$ that preserves kernels, then $\mathcal{B}$ is abelian; $\mathcal{B} \subset \mathcal{A}$ is closed under (small) products and kernels; and (small) sums and cokernels are obtained by applying $L\colon \mathcal{A} \to \mathcal{B}$ to the corresponding sums and cokernels in $\mathcal{A}$. So (1) and (2) follow from Theorem 3.8 and Addendum 3.10.

It remains to prove (3). Suppose that $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is exact. In order to proved that $\mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x$ is exact, it suffices to show that the functor $\mathcal{F} \mapsto \mathcal{F}_x$ preserves kernels and cokernels. (Recall: image = kernel of cokernel.) Now, more precisely, this functor is the composition

$$\mathrm{Sh}(X, \mathsf{Ab}) \xrightarrow{\iota} \mathcal{P}(X, \mathsf{Ab}) \xrightarrow{(-)_x} \mathsf{Ab}$$

of the canonical inclusion functor and the functor that to a presheaf assigns its stalk at $x \in X$. The former preserves all (small) limits, and the latter preserves finite limits, so in particular, the composite functor preserves kernels. To see that it also preserves cokernels, let $\varphi \colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves. By (1), we have

$$\mathrm{coker}(\varphi) \simeq \mathrm{ass}(\mathrm{coker}(\iota(\varphi))),$$

from which we conclude that

$$\iota(\mathrm{coker}(\varphi))_x \simeq (\iota \circ \mathrm{ass})(\mathrm{coker}(\iota(\varphi)))_x \simeq \mathrm{coker}(\iota(\varphi))_x \simeq \mathrm{coker}(\iota(\varphi)_x)$$

as we wanted to prove. Here the middle isomorphism follows from Theorem 3.8 (1), and the right-hand isomorphism follows from the fact that "colimits commute."

Conversely, suppose that $\mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x$ is exact for all $x \in X$. In order to prove that $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is exact, it suffices to prove that kernels and cokernels are detected on stalks. For kernels, we wish to prove that if we have maps of sheaves

$$\mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$$

such that $\varphi_x \colon \mathcal{F}'_x \to \mathcal{F}_x$ is a kernel of $\psi_x \colon \mathcal{F}_x \to \mathcal{F}''_x$ for all $x \in X$, then $\varphi \colon \mathcal{F}' \to \mathcal{F}$ is a kernel of $\psi \colon \mathcal{F} \to \mathcal{F}''$. Now, if we let $\gamma \colon \mathcal{K} \to \mathcal{F}$ be a kernel of $\psi \colon \mathcal{F} \to \mathcal{F}''$ and consider the unique factorization

$$\begin{tikzcd} \mathcal{F}' \arrow[rr, "\varphi"] \arrow[dr, "\bar{\varphi}"'] & & \mathcal{F} \\ & \mathcal{K} \arrow[ur, "\gamma"'] & \end{tikzcd}$$

of the map φ, then the statement "φ is a kernel of ψ" is equivalent to the statement "$\bar{\varphi}$ is an isomorphism." Since Proposition 3.4 shows that the latter statement can be checked on stalks, we conclude that kernels are detected on stalks. The proof that cokernels are detected on stalks is analogous. □

We need one final preliminary on sheaves, before we are ready to define the structure sheaf on the Zariski space. Let X be a topological space, and suppose that $\mathcal{B}$ is a basis for the topology on X. So $\mathcal{B}$ is a subset of the set of open subsets of X with the property that for every $x \in X$ and every $x \in U \subset X$ open, there exists $V \in \mathcal{B}$ such that $x \in V \subset U$. We let $\mathcal{B}_{\mathrm{Zar}} \subset X_{\mathrm{Zar}}$ be the full subcategory spanned by the $V \in \mathcal{B}$ and define

$$\mathcal{P}(\mathcal{B}) = \mathrm{Fun}(\mathcal{B}_{\mathrm{Zar}}^{\mathrm{op}}, \mathsf{Set}).$$

We say that $\mathcal{F} \in \mathcal{P}(\mathcal{B})$ satisfies the sheaf condition if for every $V \in \mathcal{B}$ and every cover $(V_i)_{i \in I}$ of V with $V_i \in \mathcal{B}$, the map

$$\mathcal{F}(V) \xrightarrow{(\mathrm{res}^V_{V_i})} \textstyle\prod_{i \in I} \mathcal{F}(V_i)$$

is injective and its image consists of the tuples $(s_i)_{i\in I}$ with the property that

$$s_i|_V = s_j|_V$$

for every pair $(i,j) \in I \times I$ and every $V \in \mathcal{B}$ such that $V \subset V_i \cap V_j$. We define

$$\mathrm{Sh}(\mathcal{B}) \subset \mathcal{P}(\mathcal{B})$$

to be the full subcategory spanned by the sheaves.

Remark 3.12. If $\mathcal{B}$ is closed under finite intersections, then $s \in \mathcal{P}(\mathcal{B})$ satisfies the sheaf condition if and only if for every $V \in \mathcal{B}$ and every cover $(V_i)_{i\in I}$ with $V_i \in \mathcal{B}$, the map $(\mathrm{res}^V_{V_i})\colon \mathcal{B} \to \prod_{i\in I} \mathcal{F}(V_i)$ is injective and its image consists of the tuples $(s_i)_{i\in I}$ with the property that $s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$ for all $(i,j) \in I \times I$.

Theorem 3.13. *Let X be a topological space, and let $\mathcal{B}$ be a basis for the topology on X. The restriction functor*

$$\mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(\mathcal{B})$$

is an equivalence of categories.

Proof. We obtain a quasi-inverse by mimicking the construction of the sheafification functor in the proof of Proposition 3.7: Given $\mathcal{F} \in \mathrm{Sh}(\mathcal{B})$ and $x \in X$, we define

$$\mathcal{F}_x = \mathrm{colim}_{x\in V, V\in\mathcal{B}} \mathcal{F}(V).$$

Now, if $U \subset X$ is open, then we define

$$\mathcal{F}'(U) \subset \textstyle\prod_{x\in U} \mathcal{F}_x$$

to be the subset consisting of the tuples $(s(x))_{x\in U}$ for which there exists a covering $(V_i)_{i\in I}$ of U with $V_i \in \mathcal{B}$ and $s_i \in \mathcal{F}(V_i)$ such that for all $x \in V_i$, the canonical map $\mathcal{F}(V_i) \to \mathcal{F}_x$ takes s_i to $s(x)$, and we define the restriction maps

$$\mathcal{F}'(U') \xrightarrow{\mathrm{res}^{U'}_U} \mathcal{F}'(U)$$

to be the maps induced by the canonical projections. We observe that $\mathcal{F}' \in \mathrm{Sh}(X)$. Moreover, if $V \in \mathcal{B}$, then the canonical map $\alpha_V \colon \mathcal{F}(V) \to \mathcal{F}'(V)$ is a bijection. So the family consisting of these maps define an isomorphism

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{F}'|_{\mathcal{B}}$$

in $\mathrm{Sh}(\mathcal{B})$. Similarly, if $\mathcal{G} \in \mathrm{Sh}(X)$ and if $U \subset X$ is open, then also the canonical map $\beta_U \colon \mathcal{G}(U) \to (\mathcal{G}|_{\mathcal{B}})'(U)$ is a bijection, so this family of maps define an isomorphism

$$\mathcal{G} \xrightarrow{\beta} (\mathcal{G}|_{\mathcal{B}})'$$

in $\mathrm{Sh}(X)$. This completes the proof. □

Remark 3.14. Let X be a topological space, and let $\mathcal{B}$ be a basis for the topology. We may similarly define $\mathrm{Sh}(\mathcal{B}, \mathsf{Ab})$ and show that the restriction functor

$$\mathrm{Sh}(X, \mathsf{Ab}) \longrightarrow \mathrm{Sh}(\mathcal{B}, \mathsf{Ab})$$

is an equivalence. A better point of view is that the category $\mathrm{Sh}(X, \mathsf{Ab})$ of sheaves of abelian groups is canonically equivalent to the category $\mathsf{Ab}(\mathrm{Sh}(X))$ of abelian group object in the category of sheaves of sets. So given the category $\mathrm{Sh}(X)$ of sheaves of sets, we obtain the category of sheaves of abelian groups by taking abelian group objects in this category, and the equivalence $\mathrm{Sh}(X) \to \mathrm{Sh}(\mathcal{B})$ induces an equivalence of the corresponding categories of abelian group objects.

Let R be a ring. In Lecture 1, we defined the Zariski space $|X|$ associated with R. In this lecture, we will define the structure sheaf $\mathcal{O}_X$ on this space. The pair

$$X = (|X|, \mathcal{O}_X)$$

is called the prime spectrum associated with R. This is Grothendieck's geometric object associated with R. It comes equipped with a natural ring homomorphism

$$R \xrightarrow{\epsilon_R} \mathcal{O}_X(|X|)$$

from the ring R to the ring of global sections of the structure sheaf, which is in fact an isomorphism. It allows us to view elements of R as global functions on X.

We first recall localization. Let R be a ring. A subset $S \subset R$ is defined to be multiplicative if $1 \in S$ and if $s, t \in S$ implies that $st \in S$.

Definition 4.1. Let R be a ring, and let $S \subset R$ be a multiplicative subset. A ring homomorphism $\gamma\colon R \to R'$ is a localization of R with respect to $S \subset R$ if

(1) for every $s \in S$, the element $\gamma(s) \in R'$ is invertible, and
(2) if $\varphi\colon R \to A$ is a ring homomorphism with the property (1), then there exists a unique ring homomorphism $\bar{\varphi}\colon R' \to A$ that makes the diagram

$$\begin{tikzcd} R \arrow[rr, "\gamma"] \arrow[dr, "\varphi"'] & & R' \arrow[dl, "\bar{\varphi}"] \\ & A & \end{tikzcd}$$

commute.

We refer to (2) by saying that $\gamma\colon R \to R'$ is initial among ring homomorphism with the property (1). If both $\gamma\colon R \to R'$ and $\gamma'\colon R \to R''$ are localizations of R with respect to $S \subset R$, then the unique ring homomorphism $\bar{\gamma}'\colon R' \to R''$ and $\bar{\gamma}\colon R'' \to R'$ are each other's inverses. In this way, a localization of R with respect to $S \subset R$ is a unique, up to unique isomorphism. We will write

$$R \xrightarrow{\gamma} S^{-1}R$$

for any choice of a localization of R with respect to $S \subset R$. Since any two are uniquely isomorphic, it does not matter, which one we choose. A localization exists by general category theoretical principles. But let us show that it can be constructed by means of (left) fractions.

So let $S\backslash S$ be the category with objects the elements of S, with morphisms from s_1 to s_2 the elements $t \in S$ such that $ts_1 = s_2$, and with composition of morphisms given by multiplication in S. We will use that $S\backslash S$ is a filtered category:

(i) For all $s_1, s_2 \in S$, there exists $t_1, t_2 \in S$ such that $t_1s_1 = t_2s_2$.
(ii) For all $s, s_1, s_2 \in S$ such that $s_1s = s_2s$, there exists $t \in S$ such that $ts_1 = ts_2$.

There is a functor from $S\backslash S$ to the category of right R-modules that takes each object s to R and that takes the morphism $t\colon s_1 \to s_2$ to the map $l_t\colon R \to R$ given by left multiplication by t. We define

$$B = \operatorname{colim}_{s \in S\backslash S} R$$

to be the colimit of this functor. The general description of a filtered colimit of sets, which we gave on Problem set 1, identifies B with the set of fractions $s^{-1}a$ with $s \in S$ and $a \in R$, where $s_1^{-1}a_1 = s_2^{-1}a_2$ if and only if there exists $t_1, t_2 \in S$ with $t_1a_1 = t_2a_2$ and $t_1s_1 = t_2s_2$. Now, for all $s \in S$, the map $r_s \colon B \to B$ given by right multiplication by s is an isomorphism. This means that:

(iii) Given $a \in R$ and $s \in S$ with $as = 0$, there exists $t \in S$ with $ta = 0$.
(iv) Given $a \in R$ and $s \in S$, there exists $b \in R$ and $t \in S$ such that $ta = bs$.

The assumptions (i)–(iv) are trivially verified.

Proposition 4.2. *Let R be a ring. If $\gamma \colon R \to S^{-1}R$ is the localization with respect to a multiplicative subset $S \subset R$, then the map*

$$B = \operatorname{colim}_{s \in S \setminus S} R \xrightarrow{u} S^{-1}R$$

that to $s^{-1}a$ assigns $\gamma(s)^{-1}\gamma(a)$ is an isomorphism of right R-modules. In particular, the ring homomorphism $\gamma \colon R \to S^{-1}R$ is flat.

Proof. The abelian group $\operatorname{End}(B)$ of endomorphisms of the underlying additive abelian group of B has a canonical noncommutative ring structure with multiplication given by composition of maps, and moreover, there is a ring homomorphism $r \colon R^{\mathrm{op}} \to \operatorname{End}(B)$ that to $a \in R$ assigns the map $r_a \colon B \to B$ given by right multiplication by a. Since (iii)–(iv) hold, the map r extends uniquely to a ring homomorphism $\bar{r} \colon S^{-1}R^{\mathrm{op}} \to \operatorname{End}(B)$. Thus, the structure of right R-module on B extends uniquely to a structure of right $S^{-1}R$-module. Moreover, the map u in question is $S^{-1}R$-linear, since every element of $S^{-1}R$ is a finite products of elements of the form $\gamma(s)^{-1}$ and $\gamma(a)$ with $s \in S$ and $a \in R$. Finally, since B is generated as a right $S^{-1}R$-module by the fraction $(1)^{-1}1$, and since u maps this generator to the identity element in $S^{-1}R$, we conclude that u is an isomorphism. $\square$

As motivation for the definition of the structure sheaf, we prove a lemma.

Lemma 4.3. *Let R be a ring, and let $|X|$ be its Zariski space. An element $f \in R$ is a unit if and only if $f(x) \neq 0$ in $k(x)$ for all $x \in |X|$.*

Proof. If $f \in R$ is a unit, then so is its image by every ring homomorphism. In particular, for every $x \in |X|$, the image $f(x) \in k(x)$ of $f \in R$ by the canonical ring homomorphism $\psi \colon R \to k(x)$ is a unit. But an element of a field is a unit if and only if it is nonzero. Conversely, the assumption that $f(x) \neq 0$ for all $x \in |X|$ implies that $|X_f| = |X|$. So $1 \in \sqrt{(f)}$, which implies that $1 \in (f)$, so there exists $g \in R$ such that $1 = fg$, which is what we wanted to show. $\square$

Now, given $f \in R$, the localization $\phi_f \colon R \to R_f$ with respect to the multiplicative subset $S = \{1, f, f^2, \dots\}$ is initial among ring homomorphisms $\varphi \colon R \to A$ that map $f \in R$ to a unit. Therefore, Lemma 4.3 suggests that we define

$$\mathcal{O}_X(|X_f|) = R_f.$$

To make this suggestion even more convinsing, we prove another lemma.

Lemma 4.4. *Let R be a ring, let $f \in R$, and let $\phi_f \colon R \to R_f$ be the localization with respect to $S = \{1, f, f^2, \dots\} \subset R$. The induced map of Zariski spaces*

$$|Y| = |\operatorname{Spec}(R_f)| \xrightarrow{j} |X| = |\operatorname{Spec}(R)|$$

is a homeomorphism onto the subspace $|X_f| \subset |X|$. Moreover, a subset $V \subset |Y|$ is a distinguished open subset if and only if $j(V) \subset |X|$ is a distinguished open subset that is contained in $|X_f| \subset |X|$.

Proof. We already know that j is continuous and if $U \subset |X|$ is a distinguished open subset, then so is $j^{-1}(U) \subset |Y|$. So it suffices to prove:

(1) The map $j\colon |Y| \to |X|$ is injective with image $|X_f| \subset |X|$.
(2) If $V \subset |Y|$ is a distinguished open subset, then so is $j(V) \subset |X|$.

To prove (1), we let $x \in |X|$ and consider the canonical map $\psi\colon R \to k(x)$. By the defining property of the localization $\phi_f\colon R \to R_f$, this map takes $f \in R$ to a unit if and only if it factors as a composition

$$\begin{tikzcd} R \arrow[rr, "\psi"] \arrow[dr, "\phi_f"'] & & k(x) \\ & R_f \arrow[ur, "\bar{\psi}"'] & \end{tikzcd}$$

so (1) ensues. To prove (2), let $V = |Y_h|$ with $h \in R_f$. We can write $h = g/f^N$ with $g \in R$ and $N \geq 1$, and since $f \in R_f$ is a unit, we have

$$|Y_h| = |Y_g| = j^{-1}(|X_{fg}|) \subset j^{-1}(|X_f|) = |Y|.$$

Finally, since j is a bijection by (1), this shows that

$$j(|Y_h|) = |X_{fg}| \subset |X_f|,$$

so $j(|Y_h|) \subset |X|$ is open, as we wanted to prove. $\square$

We use Lemma 4.4 to prove the following remarkable property of the distinguished open subsets of the Zariski space of a ring.

Proposition 4.5. *Let R be a ring, and let $|X|$ be its Zariski space. If $U \subset |X|$ is a distinguished open subset, then U is quasicompact.*

Proof. By Lemma 4.4, we may assume that $U = |X|$. So we let $(U_i)_{i \in I}$ be an open cover of $|X|$ and must show that it admits a finite subcover. Since the distinguished open subsets form a basis of the Zariski topology on $|X|$, we may further assume that each U_i is a distinguished open subset, say, $U_i = |X_{f_i}|$. Now, the statement that the family $(U_i)_{i \in I}$ of distinguished open subsets covers $|X|$ is equivalent to the statement that the family $(f_i)_{i \in I}$ generates the unit ideal $(1) = R$. In this case, there exists finitely many $i_1, \dots, i_n \in I$ and $g_1, \dots, g_n \in R$ such that

$$1 = f_{i_1} g_1 + \dots + f_{i_n} g_n \in R.$$

Therefore, the subfamily $(U_{i_k})_{1 \leq k \leq n}$ covers $|X|$, as we wanted to prove. $\square$

Remark 4.6. Let R be a ring, and let $|X|$ be its Zariski space. In general, an open subset $U \subset |X|$ is quasicompact if and only if there exists a finite set $S \subset R$ such that $|X| \smallsetminus U = V(S)$. In particular, if the ring R is noetherian (every ideal is finitely generated), then every open subset is quasicompact.

We will prove the following theorem, which produces the structure sheaf $\mathcal{O}_X$ on the Zariski space $|X|$.

Theorem 4.7. *Let R be a ring, and let $|X| = |\operatorname{Spec}(R)|$. Up to unique isomorphism, there is a unique pair $(\mathcal{O}_X, \epsilon_R\colon R \to \mathcal{O}_X(|X|))$ of a sheaf of rings on $|X|$ and ring homomorphism such that for every $f \in R$, the composite map*

$$R \xrightarrow{\epsilon_R} \mathcal{O}_X(|X|) \longrightarrow \mathcal{O}_X(|X_f|))$$

is a localization with respect to $S = \{1, f, f^2, \dots\} \subset R$.

Proof. Let X_{Zar} be the category of open subsets of $|X|$ and with a single map from U to V, if $U \subset V$, and with no maps from U to V, otherwise, and let $u\colon D_{\mathrm{Zar}} \to X_{\mathrm{Zar}}$ be the inclusion of the full subcategory spanned by the distinguished open subsets. The distinguished open subsets constitute a basis for the Zariski topology of $|X|$, and the intersection of two distinguished open subsets is again a distinguished open subset. Therefore, by Theorem 13 from Lecture 3, the forgetful functor

$$\mathrm{Sh}(X_{\mathrm{Zar}}) \xrightarrow{u^*} \mathrm{Sh}(D_{\mathrm{Zar}})$$

is an equivalence of categories. Here, the right-hand side is the full subcategory

$$\mathrm{Sh}(D_{\mathrm{Zar}}) \subset \mathcal{P}(D_{\mathrm{Zar}})$$

spanned by the presheaves $\mathcal{F}\colon D_{\mathrm{Zar}}^{\mathrm{op}} \to \mathsf{Set}$ that satisfy the sheaf condition:
For every $U \in D_{\mathrm{Zar}}$ and every covering $(U_i)_{i \in I}$ of U with $U_i \in D_{\mathrm{Zar}}$, the map

$$\mathcal{F}(U) \xrightarrow{(\mathrm{res}^U_{U_i})} \prod_{i \in I} \mathcal{F}(U_i)$$

is injective and its image consists of the tuples $(s_i)_{i \in I}$ with $s_i \in \mathcal{F}(U_i)$ with the property that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \in \mathcal{F}(U_i \cap U_j)$ for all $(i, j) \in I \times I$.
Therefore, it will suffice to show that, up to unique isomorphism, there is a unique pair $(\mathcal{O}_D, \epsilon_R\colon R \to \mathcal{O}_D(|X|))$ of a sheaf of rings on D_{Zar} and a ring homomorphism with the property that for every $f \in R$, the composite map

$$R \xrightarrow{\epsilon_R} \mathcal{O}_D(|X|) \longrightarrow \mathcal{O}_D(|X_f|)$$

is a localization with respect to $S = \{1, f, f^2, \dots\} \subset R$. The uniqueness, up to unique isomorphism, of a solution to this problem is clear, so it remains to prove existence. We first prove (a) that a pair $(\mathcal{O}_D, \epsilon_R\colon R \to \mathcal{O}_D(|X|))$, where $\mathcal{O}_D$ is a presheaf of rings on D_{Zar} with the desired properties, exists, and then prove (b) that the presheaf $\mathcal{O}_D$ satisfies the sheaf condition.

We first note that if $|X_f| \subset |X_g|$, then the localization $\phi_f\colon R \to R_f$ maps g to a unit. Indeed, this follows form Lemmas 4.3 and 4.4, because $g(x) \neq 0$ for all $x \in |X_f|$. Therefore, by the universal property of localization, if $|X_f| \subset |X_g|$, then there is a unique ring homomorphism $\phi_{f,g}\colon R_g \to R_f$ that makes the diagram

$$\begin{tikzcd} & R \arrow[dl, "\phi_g"'] \arrow[dr, "\phi_f"] & \\ R_g \arrow[rr, "\phi_{f,g}"] & & R_f \end{tikzcd}$$

commute. We claim that this proves (a). Indeed, if $U \subset |X|$ is a distinguished open subset, then we choose $f \in R$ with $U = |X_f|$ and a localization $\phi_f\colon R \to R_f$ and define $\mathcal{O}_D(U) = R_f$. Moreover, if $U \subset V \subset |X|$ is a pair of distinguished open subsets with $U = |X_f|$ and $V = |X_g|$, then we define

$$\mathcal{O}_D(V) \xrightarrow{\operatorname{res}^V_U} \mathcal{O}_D(U)$$

to be the map $\phi_{f,g} \colon R_g \to R_f$. This defines a functor

$$D^{\mathrm{op}}_{\mathrm{Zar}} \xrightarrow{\mathcal{O}_D} \mathrm{CAlg}(\mathsf{Ab}).$$

Indeed, if $|X_f| \subset |X_g| \subset |X_h|$, then $\phi_{f,h} = \phi_{f,g} \circ \phi_{g,h}$ by the uniqueness property of these maps. Finally, we define the map

$$R \xrightarrow{\epsilon_R} \mathcal{O}_D(|X|)$$

to be the localization $\phi_e \colon R \to R_e$ that corresponds to our choice of $e \in R$ such that $|X_e| = |X|$. We could of course choose $e = 1$ and $\phi_e = \mathrm{id}_R$, but there is no need to do so. But we note that, whatever the choice of $e \in R$, the map ϵ_R will be an isomorphism. This proves (a).

It remains to prove that the presheaf $\mathcal{O}_D$ satisfies the sheaf condition. It says that if $f \in R$ and if $(f_i)_{i \in I}$ is a family of elements $f_i \in R$ such that

$$|X_f| = \textstyle\bigcup_{i \in I} |X_{f_i}|,$$

then the map

$$R_f \xrightarrow{(\phi_{f,f_i})} \textstyle\prod_{i \in I} R_{f_i}$$

is injective and its image consists of those tuples $(g_i)_{i \in I}$ with $g_i \in R_{f_i}$ such that

$$\phi_{f_i,f_i f_j}(g_i) = \phi_{f_j,f_i f_j}(g_j) \in R_{f_i f_j}$$

for all $(i,j) \in I \times I$. To prove this, we make a series of reductions.

First reduction: We can assume that I is finite.

We must show that if $\mathcal{O}_D$ satisfies the sheaf for every finite cover, then it satisfies the sheaf condition in general. So let $U \in D_{\mathrm{Zar}}$ and let $(U_i)_{i \in I}$ be a cover of U with $U_i \in D_{\mathrm{Zar}}$. By Proposition 4.5, the topological space U is quasicompact, so we can find a finite subset $K \subset I$ such that $(U_i)_{i \in K}$ covers U. The same is therefore true for every finite subset $K \subset J \subset I$, and the set I is the union of the finite subsets $K \subset J \subset I$. Now, for each such J, we consider the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_D(U) \arrow[r, "(\operatorname{res}^U_{U_i})"] \arrow[d, equal] & \prod_{i \in I} \mathcal{O}_D(U_i) \arrow[d] \\
\mathcal{O}_D(U) \arrow[r, "(\operatorname{res}^U_{U_i})"] & \prod_{i \in J} \mathcal{O}_D(U_i)
\end{tikzcd}$$

where the right-hand vertical map is the canonical projection. Since $\mathcal{O}_D$ satisfies the sheaf condition for finite covers, the bottom horizontal map is injective, and hence, so is the top horizontal map. It remains to prove that if a tuple $(s_i)_{i \in I}$ satisfies $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $(i,j) \in I \times I$ then it is in the image of the top horizontal map. The image of $(s_i)_{i \in I}$ by the right-hand projection is $(s_i)_{i \in J}$, and it is in the image of the lower horizontal map, because $\mathcal{O}_D$ satisfies the sheaf condition for finite covers. So there exists a global section $s_J \in \mathcal{O}_D(U)$ such that $\operatorname{res}^U_{U_i}(s_J) = s_i$

for all $i \in J$. Moreover, if $K \subset J \subset J' \subset I$ are two such finite subsets and if $s_J, s_{J'} \in \mathcal{O}_D(U)$ are the corresponding global sections, then

$$\operatorname{res}^U_{U_i}(s_{J'}) = s_i = \operatorname{res}^U_{U_i}(s_J)$$

for all $i \in J$, so we conclude that $s_J = s_{J'}$. This shows that the global section $s = s_J \in \mathcal{O}_D(U)$ is independent of $K \subset J \subset I$ and satisfies $\operatorname{res}^U_{U_i}(s) = s_i$ for all $i \in I$. So $\mathcal{O}_D$ indeed satisfies the sheaf condition for the cover $(U_i)_{i \in I}$.

Second reduction: We can assume that $f = 1$ and $R_f = R$.

We recall from Lemma 4.4 that if $f \in R$, then the localization $\phi_f \colon R \to R_f$ induces an open embedding $j \colon |Y| \to |X|$ between the corresponding Zariski spaces. Moreover, if $D_{\mathrm{Zar}} \subset X_{\mathrm{Zar}}$ and $E_{\mathrm{Zar}} \subset Y_{\mathrm{Zar}}$ denote the full subcategories spanned by the distinguished open subsets, then the functor $v \colon E_{\mathrm{Zar}} \to D_{\mathrm{Zar}}$ that to $V \subset |Y|$ assigns $j(V) \subset |X|$ is fully faithful and its image consists for the distinguished open subsets $U \subset |X|$ such that $U \subset |X_f|$. Now, if we write $h \in R_f$ as $h = g/f^N$ with $g \in R$ and $N \geq 1$, then we have the commutative diagram

$$\begin{tikzcd} & R \arrow[dl, "\phi_f"'] \arrow[dr, "\phi_{fg}"] & \\ R_f \arrow[rr, "\phi_{f,fg}"] & & R_{fg} \end{tikzcd}$$

and the map $\phi_{f,fg}$ is a localization of R_f with respect to $T = \{1, h, h^2, \dots\}$. Hence, up to unique isomorphism, both $\mathcal{O}_D(|X_{fg}|)$ and $\mathcal{O}_E(|Y_h|)$ are given by R_{fg}, and since $v(|Y_h|) = |X_{fg}|$, we obtain an isomorphism

$$v^* \mathcal{O}_D \longrightarrow \mathcal{O}_E$$

of presheaves on E_{Zar}. Moreover, since the functor $v \colon E_{\mathrm{Zar}} \to D_{\mathrm{Zar}}$ is fully faithful, the functor v^* preserves sheaves. (Compare with Problem Set 3.) Therefore, if $\mathcal{O}_D$ satisfies the sheaf condition, then so does $\mathcal{O}_E$, which is what we wanted to prove.

After these reductions, it suffices to prove that $\mathcal{O}_D$ satisfies the sheaf condition for finite families $(f_i)_{i \in I}$ such that $|X| = \bigcup_{i \in I} |X_{f_i}|$. Given such a family, let us define $\mathrm{Glue}_{(f_i)_{i \in I}}(R)$ to be the equalizer of the two maps

$$\textstyle\prod_{i \in I} R_{f_i} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{(i,j) \in I \times I} R_{f_i f_j}$$

defined by $\mathrm{pr}_{(i,j)} \circ \alpha = \phi_{f_i, f_i f_j} \circ \mathrm{pr}_i$ and $\mathrm{pr}_{(i,j)} \circ \beta = \phi_{f_j, f_i f_j} \circ \mathrm{pr}_j$. The sheaf condition for the finite family $(f_i)_{i \in I}$ now amounts to the statement that the map

$$R \xrightarrow{(\phi_{f_i})} \mathrm{Glue}_{(f_i)_{i \in I}}(R)$$

is an isomorphism. We now prove three lemmas.

Lemma 4.8. *Let $\varphi \colon R \to R'$ be a flat ring homomorphism, let $(f_i)_{i \in I}$ be a finite family with $f_i \in R$, and let $(f'_i)_{i \in I}$ be the finite family with $f'_i = \varphi(f_i) \in R'$. In this situation, the map φ induces an isomorphism of R'-modules*

$$\mathrm{Glue}_{(f_i)_{i \in I}}(R) \otimes_R R' \longrightarrow \mathrm{Glue}_{(f'_i)_{i \in I}}(R').$$

Proof. By flatness, the functor $-\otimes_R R'$ preserves kernels. It also preserves finite products, since these are finite sums, and since $-\otimes_R R'$ preserves all sums. So it suffices to prove that for all $f \in R$, the map φ induces an isomorphism

$$R_f \otimes_R R' \longrightarrow R'_{\varphi(f)}.$$

But this follows immediately from the universal property of localization. □

Lemma 4.9. *Let R be a ring, and let $|X|$ be its Zariski space. Let $(f_i)_{i\in I}$ be a family with $f_i \in R$ such that $|X| = \bigcup_{i\in I} |X_{f_i}|$. A map of R-modules $\varphi\colon N \to M$ is an isomorphism if and only if the induced maps of R_{f_i}-modules*

$$N \otimes_R R_{f_i} \xrightarrow{\varphi\otimes\mathrm{id}} M \otimes_R R_{f_i}$$

are isomorphisms for all $i \in I$.

Proof. The "only if" statement holds, because $-\otimes_R R_{f_i}$ is a functor, and because any functor preserves isomorphisms, so we must prove the "if" statement. The functor $-\otimes_R R_{f_i}$ is exact, because $\phi_{f_i}\colon R \to R_{f_i}$ is flat by Proposition 4.2. Hence, by considering the kernel and cokernel of $\varphi\colon N \to M$, it suffices to show that if P is an R-module such that $P \otimes_R R_{f_i}$ is zero for all $i \in I$, then P is zero. Now, the assumption that $|X| = \bigcup_{i\in I} |X_{f_i}|$ is equivalent to the statement that $1 \in (f_i \mid i \in I)$. So it suffices to show that the subset

$$\mathrm{ann}_R(P) = \{f \in R \mid P \otimes_R R_f \text{ is zero}\} \subset R$$

is an ideal. We note that $P \otimes_R R_f$ is zero if and only if for all $x \in P$, there exists $n \geq 1$ such that $x \cdot f^n = 0$. It is clear that if $f \in \mathrm{ann}_R(P)$ and $g \in R$, then $f \cdot g \in \mathrm{ann}_R(P)$, and if $f, f' \in \mathrm{ann}_R(P)$ and $x \in P$, then there exists $n, n' \geq 1$ such that $x \cdot f^n = 0 = x \cdot f'^{n'}$. But in this situation, the binomial formula shows that

$$x \cdot (f + f')^{n+n'} = 0$$

so also $f + f' \in \mathrm{ann}_R(P)$. This completes the proof. □

Lemma 4.10. *Let X be a topological space, let $U \subset X$ be an open subset, and let $(U_i)_{i\in I}$ be a covering of U by open subsets. If $U_k = U$ for some $k \in I$, then every presheaf $\mathcal{F} \in \mathcal{P}(X)$ satisfies the sheaf condition for $(U_i)_{i\in I}$.*

Proof. Let $\mathcal{F} \in \mathcal{P}(X)$ be a presheaf. If $U_k = U$, then the map

$$\mathcal{F}(U) \xrightarrow{(\mathrm{res}^U_{U_i})} \textstyle\prod_{i\in I} \mathcal{F}(U_j)$$

is injective, since $\mathrm{res}^U_{U_k} = \mathrm{id}_{\mathcal{F}(U)}$ is so, and moreover, a tuple $(s_i)_{i\in I}$ with $s_i \in \mathcal{F}(U_j)$ satisfies $s_i|_{U_i\cap U_j} = s_i|_{U_i\cap U_j}$ for all $(i,j) \in I \times I$ if and only if

$$s_i = s_i|_{U_i} = s_i|_{U_i\cap U_k} = s_k|_{U_i\cap U_k} = s_k|_{U_i} = \mathrm{res}^U_{U_i}(s_k)$$

for all $j \in I$. So $\mathcal{F}$ satisfies the sheaf condition for $(U_j)_{j\in I}$. □

With these three lemmas in hand, we now let $(f_i)_{i\in I}$ be a finite family of elements of R such that $|X| = \bigcup_{i\in I} |X_{f_i}|$ and prove that the map

$$R \xrightarrow{(\phi_{f_i})} \mathrm{Glue}_{(f_i)_{i\in I}}(R)$$

is an isomorphism. By Lemma 4.9, if suffices to show that

$$R_{f_j} \xrightarrow{(f_i)\otimes \mathrm{id}} \mathrm{Glue}_{(f_i)_{i\in I}}(R) \otimes_R R_{f_j}$$

is an isomorphism for all $j \in I$. Moreover, by Lemma 4.8, the canonical map

$$\mathrm{Glue}_{(f_i)_{i\in I}}(R) \otimes_R R_{f_j} \longrightarrow \mathrm{Glue}_{(\phi_{f_j}(f_i))}(R_{f_j})$$

is an isomorphism, since $\phi_{f_j} : R \to R_{f_j}$ is flat. But the composition of these two maps is precisely the corresponding map

$$R_{f_j} \xrightarrow{(\phi_{f_j}(f_i))} \mathrm{Glue}_{(\phi_{f_j}(f_i))_{i\in I}}(R_{f_j})$$

for the ring R_{f_j} and the family $(\phi_{f_j}(f_i))_{i\in I}$. But $\phi_{\phi_{f_j}(f_j)} \colon R_{f_j} \to (R_{f_j})_{\phi_{f_j}(f_j)}$ is an isomorphism, so this map is an isomorphism by Lemma 4.10. $\square$

Remark 4.11. Let R be a ring, and let $|X|$ be its Zariski space. Theorem 4.7 specifies the value of the structure sheaf $\mathcal{O}_X$ on the distinguished open subsets $U \subset |X|$. Its value on a general open subset $V \subset |X|$ is specified by the sheaf condition. Indeed, if we write $V = \bigcup_{i\in I} U_i$ as the union of a family $(U_i)_{i\in I}$ of distinguished open subsets, then we have the equalizer diagram

$$\mathcal{O}_X(V) \xrightarrow{(\mathrm{res}^V_{U_i})} \textstyle\prod_{i\in I} \mathcal{O}_X(U_i) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{O}_X(U_i \cap U_j),$$

where $\mathrm{pr}_{(i,j)} \circ \alpha = \mathrm{res}^{U_i}_{U_i\cap U_j} \circ \mathrm{pr}_i$ and $\mathrm{pr}_{(i,j)} \circ \beta = \mathrm{res}^{U_j}_{U_i\cap U_j} \circ \mathrm{pr}_j$.

Remark 4.12. Let R be a ring, and let $|X|$ be its Zariski space. The proof of Theorem 4.7 did not use that $\mathcal{O}_X$ is a sheaf of rings, but only that it is a sheaf of R-modules. So given an R-module M, we get a sheaf $\widetilde{M}$ of R-modules on $|X|$ such that

$$\widetilde{M}(|X_f|) = M \otimes_R R_f$$

for every $f \in R$.

Remark 4.13. Let R be a ring, and let $|X|$ be its Zariski space. Why did we only specify the value of $\mathcal{O}_X$ directly on distinguished open subsets? Let us say that an R-algebra $\varphi\colon R \to A$ is distinguished if it is a localization with respect to the multiplicative subset $S = \{1, f, f^2, \dots\} \subset R$ for some $f \in R$. Now, if $U \subset |X|$ is a distinguished open subset, then the composite ring homomorphism

$$R \xrightarrow{\epsilon_R} \mathcal{O}_X(|X|) \longrightarrow \mathcal{O}_X(U),$$

is a distinguished R-algebra, and if $\varphi\colon R \to A$ is a distinguished R-algebra, then the image $U \subset |X|$ of the induced map of Zariski spaces

$$|Y| = |\operatorname{Spec}(A)| \xrightarrow{j} |X| = |\operatorname{Spec}(R)|$$

is a distinguished subset. In fact, the functor

$$\mathrm{CAlg}(\mathsf{Ab})^{\mathrm{dist}}_{R/} \longrightarrow D^{\mathrm{op}}_{\mathrm{Zar}}$$

given by the latter assignment is an equivalence of categories. In particular, every distinguished open subset $U \subset |X|$ is uniquely determined by the (distinguished) R-algebra $R \to \mathcal{O}_X(U)$. This is not true for open subsets $U \subset |X|$ in general. In fact, by Hartog's principle, in most situations, where $U \subset |X|$ is an open subset of codimension > 1, the restriction map

$$\mathcal{O}_X(|X|) \longrightarrow \mathcal{O}_X(U)$$

is an isomorphism! Simplest example: Let $R = k[x,y]$, and let

$$U = |X| \smallsetminus V(x,y) = |X_x| \cup |X_y| \subset |X|.$$

The sheaf condition tells us that the diagram of rings

$$\begin{tikzcd} \mathcal{O}_X(U) \arrow[r] \arrow[d] & \mathcal{O}_X(|X_x|) \arrow[d] \\ \mathcal{O}_X(|X_y|) \arrow[r] & \mathcal{O}_X(|X_{xy}|) \end{tikzcd}$$

is cartesian, which becomes

$$\begin{tikzcd} \mathcal{O}_X(U) \arrow[r] \arrow[d] & k[x^{\pm 1}, y] \arrow[d] \\ k[x, y^{\pm 1}] \arrow[r] & k[x^{\pm 1}, y^{\pm 1}], \end{tikzcd}$$

which shows that $\mathcal{O}_X(|X|) \to \mathcal{O}_X(U)$ is an isomorphism. However, a closer look, using cohomology, reveals that, in this case, the structure sheaf can in fact distinguish U from $|X|$. Without explaining cohomology, here is how it appears: The cartesian diagram above can be rewritten as an exact sequence

$$0 \longrightarrow \mathcal{O}_X(U) \longrightarrow k[x^{\pm 1}, y] \oplus k[x, y^{\pm 1}] \longrightarrow k[x^{\pm 1}, y^{\pm 1}].$$

The right-hand map, however, is *not* surjective, so the sequence is not short exact! Its cokernel is $H^1(U, \mathcal{O}_X)$, which is therefore nonzero. By contrast, for an open cover of a distinguished open subsets by distinguished open subsets, the corresponding exact sequence is short exact. So there is no cohomology. We will revisit this later.

We will now give Grothendieck's definition of a scheme. First, we define a ringed space to be a pair $X = (|X|, \mathcal{O}_X)$ of a topological space $|X|$ and a sheaf of rings $\mathcal{O}_X$ thereon, and we define a map of ringed spaces $f\colon Y \to X$ to be a pair

$$(|Y|, \mathcal{O}_Y) \xrightarrow{(p,\phi)} (|X|, \mathcal{O}_X)$$

of a continuous map $p\colon |Y| \to |X|$ and a map $\phi\colon \mathcal{O}_X \to p_*(\mathcal{O}_Y)$ of sheaves of rings on $|X|$. Here, by definition, $p_*(\mathcal{O}_Y)$ (called the pushforward of $\mathcal{O}_Y$ along f is the sheaf on $|X|$ with

$$(p_*\mathcal{O}_Y)(U) = \mathcal{O}_Y(f^{-1}U).$$

Intuitively speaking, the data ϕ tells us how we're supposed to pull back (locally defined) functions on X to (locally defined) functions on Y,

Example 4.14. The pair $V^{\mathrm{sm}} = (|V|, \mathcal{O}_V^{\mathrm{sm}})$ of an open subset $|V| \subset \mathbb{R}^d$ and the sheaf $\mathcal{O}_V^{\mathrm{sm}}$ of standard smooth functions on $|V|$ is a ringed space. By definition, a ringed space $X = (|X|, \mathcal{O}_X)$ is a smooth manifold if it is locally isomorphic to V^{sm} for some open subset $|V| \subset \mathbb{R}^d$. (It is common to also require $|X|$ to be Hausdorff and second countable, but let us not do so.) In 1956, Milnor made the remarkable discovery that there are 28 different isomorphism classes of smooth manifolds X such that $|X|$ is homeomorphic to the 7-sphere S^7!

Definition 4.15 (Grothendieck). The prime spectrum of a ring R is the ringed space $X = (|X|, \mathcal{O}_X)$ given by its Zariski space $|X|$ and the structure sheaf $\mathcal{O}_X$ thereon. A ringed space is a scheme if it is locally isomorphic to the prime spectrum of a ring.

So a ringed space $X = (|X|, \mathcal{O}_X)$ is a scheme if for every $x \in |X|$, there exists an open neighborhood $x \in U \subset |X|$ and an isomorphism of ringed spaces

$$(U, \mathcal{O}_X|_U) \longrightarrow (|V|, \mathcal{O}_V)$$

to the prime spectrum of some ring. Such a ring, if it exists, necessarily must vary with $x \in U \subset |X|$, since it must be isomorphic to $\mathcal{O}_X(U)$. We say that a scheme is affine if it is (globally) isomorphic to the prime spectrum of a ring. So, equivalently, a ringed space is a scheme if it is locally isomorphic to an affine scheme.

Example 4.16. Let $X = (|X|, \mathcal{O}_X)$ be the prime spectrum of a ring R. If $U \subset |X|$ is any open subset, then the ringed space $(U, \mathcal{O}_X|_U)$ is a scheme. Indeed, we can write $U = \bigcup_{i \in I} U_i$ for some family $(U_i)_{i \in I}$ of distinguished open subset $U_i \subset |X|$ and the ringed space $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to the prime spectrum of $\mathcal{O}_X(U_i)$. We say that $(U, \mathcal{O}_X|_U)$ is an open subscheme of $(|X|, \mathcal{O}_X)$. In the case of

$$U = |\operatorname{Spec}(k[x,y])| \smallsetminus V(x,y) \subset |X|$$

as before, the scheme $(U, \mathcal{O}_X|_U)$ is not affine. It is only quasiaffine in the sense that it is a quasicompact open subscheme of an affine scheme.

Caution. The notion of a map of schemes is more subtle than the definition of a scheme suggests. It is *not* true that the category of schemes is the full subcategory of the category of ringed spaces spanned by the schemes. We will discuss this in the next lecture.

5. The category of schemes

In the last lecture, we defined a scheme to be ringed space with the property that, locally, it is isomorphic to the prime spectrum of a ring. In this lecture, we will define the notion of maps (or morphisms) between schemes.

Let us first recall that we defined a ringed space to be a pair

$$X = (|X|, \mathcal{O}_X)$$

of a topological space $|X|$ and a sheaf of rings $\mathcal{O}_X$ on $|X|$, and we defined a map of ringed spaces $f\colon Y \to X$ to be a pair

$$(|Y|, \mathcal{O}_Y) \xrightarrow{(p,\phi)} (|X|, \mathcal{O}_X)$$

of a continuous map $p\colon |Y| \to |X|$ and a map $\phi\colon \mathcal{O}_X \to p_*(\mathcal{O}_Y)$ of sheaves of rings on $|X|$. The map ϕ determines and is determined by a family of ring homomorphisms

$$\mathcal{O}_X(U) \xrightarrow{\phi_U} p_*(\mathcal{O}_Y)(U) = \mathcal{O}_Y(p^{-1}(U))$$

indexed by the set of open subsets $U \subset |X|$ with the property that the diagram

$$\begin{tikzcd} \mathcal{O}_X(V) \arrow[r, "\phi_V"] \arrow[d, "\mathrm{res}^V_U"] & \mathcal{O}_Y(p^{-1}(V)) \arrow[d, "\mathrm{res}^{p^{-1}(V)}_{p^{-1}(U)}"] \\ \mathcal{O}_X(U) \arrow[r, "\phi_U"] & \mathcal{O}_Y(p^{-1}(U)) \end{tikzcd}$$

commutes for all open subsets $U \subset V \subset |X|$. We remark that since $\mathcal{O}_X$ and $\mathcal{O}_Y$ are sheaves, as opposed to presheaves, it suffices to specify the maps ϕ_U for the open subsets $U \subset |X|$ in a basis of the topology on $|X|$.

Let us also spell out that the composition of two composable maps of ringed spaces

$$(|Z|, \mathcal{O}_Z) \xrightarrow{(q,\psi)} (|Y|, \mathcal{O}_Y) \xrightarrow{(p,\phi)} (|X|, \mathcal{O}_X)$$

is the map of ringed spaces

$$(|Z|, \mathcal{O}_Z) \xrightarrow{(r,\xi)} (|X|, \mathcal{O}_X),$$

where $r = p \circ q$, and where ξ is the composite map

$$\mathcal{O}_X \xrightarrow{\phi} p_*(\mathcal{O}_Y) \xrightarrow{p_*(\psi)} p_*q_*(\mathcal{O}_Z) \simeq r_*(\mathcal{O}_Z)$$

of sheaves of rings on $|X|$. The map ξ determines and is determined by the family $(\xi_U)_{U\subset|X|}$ consisting of the composite ring homomorphisms

$$\mathcal{O}_X(U) \xrightarrow{\phi_U} \mathcal{O}_Y(p^{-1}(U)) \xrightarrow{\psi_{p^{-1}(U)}} \mathcal{O}_Z(q^{-1}p^{-1}(U)) = \mathcal{O}_Z(r^{-1}(U))$$

with $U \subset |X|$ open. With these definitions, ringed spaces and maps of ringed spaces form a category.

By definition, a ringed space $X = (|X|, \mathcal{O}_X)$ is a scheme, if for every $x \in |X|$, there exists an open subset $x \in U \subset |X|$ and an isomorphism of ringed spaces

$$(U, \mathcal{O}_X|_U) \longrightarrow (|V|, \mathcal{O}_V)$$

to the prime spectrum of some ring. To be a scheme is a *property* of a ringed space. This might suggest that we define the category of schemes to be the full subcategory of the category of ringed spaces, but this is *not* the correct definition.

Example 5.1. Let $Y = \operatorname{Spec}(\mathbb{Q})$ and $X = \operatorname{Spec}(\mathbb{Z})$, and let

$$(|Y|, \mathcal{O}_Y) \xrightarrow{(p,\phi)} (|X|, \mathcal{O}_X)$$

be the map ringed spaces, where $p\colon Y \to X$ sends the unique point $\eta_Y \in |Y|$ to the point $x = p(\eta_Y) \in X$ corresponding to some (arbitrarily chosen) prime number $p \in \mathbb{Z}$, and where $\phi_{|X_f|}\colon \mathcal{O}_X(|X_f|) \to \mathcal{O}_Y(p^{-1}(|X_f|))$ is the unique ring homorphism

$$\mathbb{Z}[\tfrac{1}{f}] \longrightarrow \mathbb{Q}$$

if $p \in |X_f|$ (that is, if p does not divide f) and zero map $\mathbb{Z}[\frac{1}{f}] \to \{0\}$ otherwise.

The map in Example 5.1 is a map ringed spaces, but it should not be a map of schemes. Two ways to understand the problem:
(1) The continuous map $p\colon |Y| \to |X|$ does not have the property that the inverse image of a distinguished open subset of $|X|$ is the corresponding distinguished open subset of $|Y|$. Indeed, we have $p^{-1}(|X_p|) = \emptyset$, which is not $|Y_{\phi(p)}| = |Y|$.
(2) The map $p\colon |Y| \to |X|$ sends the unique point $\eta_Y \in |Y|$ to the "wrong" point $x \in |X|$. The algebra of the structure sheaf suggests that $p\colon |Y| \to |X|$ should map $\eta_Y \in |Y|$ to the generic point $\eta_X \in |X|$, but that is not what is does. In particular, there is no induced map on residue fields, as there is no ring homomorphism $\mathbb{F}_p \to \mathbb{Q}$.

To understand the first problem, we extend the notion of distinguished open subsets to ringed spaces in general.

Definition 5.2. Let $X = (|X|, \mathcal{O}_X)$ be a ringed space. The distinguished open subset associated with $f \in \mathcal{O}_X(V)$, where $V \subset |X|$ is open, is the subset

$$|X_f| = \{x \in V \mid f_x \in \mathcal{O}_{X,x} \text{ is a unit}\} \subset V \subset |X|,$$

where $\mathcal{O}_{X,x}$ is the stalk of $\mathcal{O}_X$ at $x \in |X|$, and f_x is the germ of f at $x \in |X|$.

We will show that $|X_f| \subset U$ is characterized as the "largest" open subset on which the local section $f \in \mathcal{O}_X(V)$ is invertible.

Lemma 5.3. *Let $X = (|X|, \mathcal{O}_X)$ be a ringed space, let $V \subset |X|$ be an open subset, and let $f \in \mathcal{O}_X(V)$ be a local section.*

(1) *The subset $|X_f| \subset V$ is open.*
(2) *If $U \subset V$ is open, then $U \subset |X_f|$ if and only if $f|_U \in \mathcal{O}_X(U)$ is a unit.*

Proof. (1) By definition, if $x \in |X_f|$, then $f_x \in \mathcal{O}_{X,x}$ is a unit, so there exists $g_x \in \mathcal{O}_{X,x}$ such that $f_x \cdot g_x = 1$ in $\mathcal{O}_{X,x}$. Since this stalk is the colimit

$$\mathcal{O}_{X,x} = \operatorname{colim}_{x \in U \subset V} \mathcal{O}_X(U)$$

indexed by the opposite category of the filtered category of open neighborhoods $x \in U \subset V$, we conclude that there exists $x \in U \subset V$ open and $g \in \mathcal{O}_X(U)$ such that $f|_U \cdot g = 1$ in $\mathcal{O}_X(U)$. But this implies that $f_y \cdot g_y = 1$ in $\mathcal{O}_{X,y}$ for all $y \in U$, so $x \in U \subset |X_f|$, which proves (1).
(2) If $f \in \mathcal{O}_X(U)$ is a unit, then so is $f_x \in \mathcal{O}_{X,x}$ for all $x \in U$, so $U \subset |X_f|$. To prove the converse implication, it suffices to prove that $f \in \mathcal{O}_X(|X_f|)$ is a unit.

Now, as in the proof of (1), for every $x \in |X_f|$, we can find $x \in U_x \subset |X_f|$ open and $g_{U_x} \in \mathcal{O}_X(U_x)$ such that $f|_{U_x} \cdot g_{U_x} = 1$ in $\mathcal{O}_X(U_x)$. Moreover, for every pair of elements $(x, y) \in |X_f| \times |X_f|$, we automatically have

$$(g_{U_x})|_{U_x \cap U_y} = (g_{U_y})|_{U_x \cap U_y},$$

since both are inverses of $f|_{U_x \cap U_y}$ and inverses are unique! Therefore, since $\mathcal{O}_X$ is a sheaf, the family of local sections $(g_{U_x})_{x \in |X_f|}$ glues to give a global section $g \in \mathcal{O}_X(|X_f|)$. Finally, for every $x \in |X_f|$, we have

$$(f \cdot g)|_{U_x} = f|_{U_x} \cdot g|_{U_x} = f|_{U_x} \cdot g_{U_x} = 1 = 1|_{U_x},$$

and since $\mathcal{O}_X$ is a sheaf, this implies that $f \cdot g = 1$. □

We next let $X = \operatorname{Spec}(R)$ and prove that the stalk $\mathcal{O}_{X,x}$ of the structure sheaf at $x \in |X|$ corresponding to $\mathfrak{p} \subset R$ indeed is the local ring $R_{\mathfrak{p}}$.

Lemma 5.4. *Let $X = \operatorname{Spec}(R)$ be the prime spectrum of a ring, let $x \in |X|$, and let $\mathfrak{p} \subset R$ be the corresponding prime ideal. In this situation, the composite map*

$$R \xrightarrow{\epsilon_R} \mathcal{O}_X(|X|) \longrightarrow \mathcal{O}_{X,x}$$

is a localization with respect to the multiplicative subset $S = R \smallsetminus \mathfrak{p}$. In particular, the stalk $\mathcal{O}_{X,x}$ is a local ring.

Proof. Since the set of distinguished open subsets is a basis for the topology on the Zariski space $|X|$, the canonical map

$$\operatorname{colim}_{x \in |X_f| \subset |X|} R_f \longrightarrow \operatorname{colim}_{x \in U \subset |X|} \mathcal{O}_X(|X_f|) = \mathcal{O}_{X,x}$$

is an isomorphism. But $x \in |X_f|$ if and only if $f \in S$, so we conclude that the composite map in the statement exactly has the universal property of a localization with respect to $S \subset R$. □

Corollary 5.5. *If $X = (|X|, \mathcal{O}_X)$ is a scheme, then the stalk $\mathcal{O}_{X,x}$ is a local ring for every $x \in |X|$.*

Proof. Indeed, given $x \in |X|$, there exists an open neighborhood $x \in U \subset |X|$ and an isomorphism of ringed spaces

$$(U, \mathcal{O}_X|_U) \xrightarrow{(p,\phi)} (|V|, \mathcal{O}_V) = \operatorname{Spec}(R).$$

Hence, if $v = p(x) \in |V|$, then $\phi_v \colon \mathcal{O}_{V,v} \to \mathcal{O}_{X,x}$ is an isomorphism, and Lemma 5.4 shows that $\mathcal{O}_{V,v}$ is a local ring. □

Suppose that R and R' are local rings and that $\mathfrak{m} \subset R$ and $\mathfrak{m}' \subset R'$ are their respective maximal ideals. We recall that a ring homomorphism $\phi \colon R \to R'$ is defined to be local, if the following equivalent conditions hold:

(a) $\mathfrak{m} \subset \phi^{-1}(\mathfrak{m}')$, or equivalently, $\phi(\mathfrak{m}) \subset \mathfrak{m}'$.
(b) $\mathfrak{m} = \phi^{-1}(\mathfrak{m}')$.
(c) If $f \in R$ and $\phi(f) \in R'$ is a unit, then $f \in R$ is a unit.

We note that the condition (c) is meaningful also if R and R' are not local rings. So we will say, more generally, that a ring homomorphism $\phi \colon R \to R'$ between general rings is local if the condition (c) is satisfied.

Example 5.6. The unique ring homomorphism $\mathbb{Z}_{(p)} \to \mathbb{F}_p$ is local, whereas the unique ring homomorphism $\mathbb{Z}_{(p)} \to \mathbb{Q}$ is not.

Geometrically, the spectrum of a local ring has a unique closed point, and all the other points are generalizations of it. A local map of local rings is one whose induced map on spectra (in the opposite direction) sends the closed point to the closed point.

Proposition 5.7. *Let $f = (p, \phi)\colon (|Y|, \mathcal{O}_Y) \to (|X|, \mathcal{O}_X)$ be a map of ringed spaces. The following are equivalent:*

(1) *For all $U \subset |X|$ open and $f \in \mathcal{O}_X(U)$,*
$$p^{-1}(|X_f|) = |Y_{\phi_U(f)}| \subset p^{-1}(U).$$
(2) *For all $y \in Y$ with image $x = p(y) \in X$, the ring homomorphism*
$$\mathcal{O}_{X,x} \xrightarrow{\phi_y} \mathcal{O}_{Y,y}$$
is local.

Proof. Suppose that (1) holds. We let $y \in Y$ with image $x = p(y) \in X$. We must show that if $f_x \in \mathcal{O}_{X,x}$ and if $\phi_y(f_x) \in \mathcal{O}_{Y,y}$ is a unit, then so is $f_x \in \mathcal{O}_{X,x}$. To this end, we lift $f_x \in \mathcal{O}_{X,x}$ to $f \in \mathcal{O}_X(U)$ for some $x \in U \subset X$ open. Now, if
$$\phi_U(f)_y = \phi_y(f_x) \in \mathcal{O}_{Y,y}$$
is a unit, then $y \in |Y_{\phi_U(f)}|$, and since we assume that (1) holds, we conclude that $x = p(y) \in |X_f|$. So $f_x \in \mathcal{O}_{X,x}$ is a unit by the definition of $|X_f| \subset U$.

Conversely, suppose that (2) holds. If $U \subset |X|$ is open and $f \in \mathcal{O}_X(U)$, then, by definition, $y \in |Y_{\phi_U(f)}|$ if and only if $\phi_U(f)_y \in \mathcal{O}_{Y,y}$ is a unit. By the assumption that (2) holds, this happens if and only if $f_x \in \mathcal{O}_{X,x}$ is a unit if and only if $x = p(y) \in |X_f|$ if and only if $y \in p^{-1}(|X_f|)$. So we conclude that (1) holds. $\square$

Definition 5.8. A map of ringed spaces is local if it satisfies the equivalent conditions of Proposition 5.7.

Remark 5.9. (1) Given two composable maps of ringed spaces
$$(|Z|, \mathcal{O}_Z) \xrightarrow{(q,\psi)} (|Y|, \mathcal{O}_Y) \xrightarrow{(p,\phi)} (|X|, \mathcal{O}_X),$$
if both are local, then so is their composite map
$$(|Z|, \mathcal{O}_Z) \xrightarrow{(r,\xi)} (|X|, \mathcal{O}_X).$$
Indeed, if the given maps satisfy (1) of Proposition 5.7, then so does the composite map. So ringed spaces and local maps of ringed spaces form a category. We remark that this category is a *non-full* subcategory of the category of ringed spaces and all maps of ringed spaces. By imposing the additional requirement that maps be local, we have discarded the troublesome maps.

(2) An isomorphism of ringed spaces is local: If $\phi\colon R \to R'$ is a ring homomorphism and if $f \in R$ is a unit, then $\phi(f) \in R'$ is a unit. If ϕ is invertible, then the converse is true, since also $\phi^{-1}\colon R' \to R$ is a ring homomorphism. So an isomorphism of ringed spaces automatically satisfies (2) in Proposition 5.7.

We define a map of schemes to be a local map of ringed spaces. Equivalently:

Definition 5.10. The category of schemes is the full subcategory of the category of ringed spaces and *local* maps spanned by the schemes.

Why is this the correct definition? Let $f = (p, \phi)\colon Y \to X$ be a map of schemes, and let $y \in |Y|$ with image $x = p(y) \in |X|$. The requirement that the induced map $\phi_x\colon \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ be local is equivalent to the requirement that there exists a (necessarily unique) map $\phi(x)$ making the following diagram commute.

$$\begin{tikzcd}
\mathcal{O}_{X,x} \arrow[r, "\phi_x"] \arrow[d] & \mathcal{O}_{Y,y} \arrow[d] \\
k(x) \arrow[r, "\phi(x)"] & k(y)
\end{tikzcd}$$

Therefore, if $x \in U \subset |X|$ is an open neighborhood and $f \in \mathcal{O}_X(U)$, then the value of $g = \phi(f)$ at $y \in |Y|$ only depends on the value of f at $x \in |X|$, namely,

$$g(y) = \phi(x)(f(x)) \in k(y).$$

We prove two lemmas, which reduce the problem of understanding maps between general schemes to understanding maps between affine schemes. We write

$$\mathrm{Map}(Y, X) = \mathrm{Map}((|Y|, \mathcal{O}_Y), (|X|, \mathcal{O}_X))$$

for the set of maps between two schemes X and Y. We also recall from Example 16 of Lecture 4 that if $|V| \subset |Y|$ is an open subset, then the ringed space $(|V|, \mathcal{O}_Y|_{|V|})$ is a scheme. We denote this scheme by $V = (|V|, \mathcal{O}_V)$ and say that it is an open subscheme of $Y = (|Y|, \mathcal{O}_Y)$. We have a map of schemes

$$V = (|V|, \mathcal{O}_V) \xrightarrow{j=(h,\eta)} Y = (|Y|, \mathcal{O}_Y),$$

where $h\colon |V| \to |Y|$ the canonical inclusion, and where

$$\mathcal{O}_Y \xrightarrow{\eta} h_*h^*(\mathcal{O}_Y) \simeq h_*(\mathcal{O}_Y|_V) \simeq h_*(\mathcal{O}_V)$$

is the unit of the adjunction (h^*, h_*) from Problem set 3. We say that the map of schemes $j\colon V \to Y$ is the open immersion of V in Y. We refer to the following result by saying that "maps of schemes are local on the source."

Lemma 5.11. *Let $X = (|X|, \mathcal{O}_X)$ and $Y = (|Y|, \mathcal{O}_Y)$ be schemes, and let $\mathcal{F}$ be the presheaf on $|Y|$ that to $V \subset |Y|$ open assigns the set of maps of schemes*

$$\mathcal{F}(V) = \mathrm{Map}((V, \mathcal{O}_Y|_V), (|X|, \mathcal{O}_X))$$

and that to $V \subset W \subset |Y|$ assign the map $\mathrm{res}^W_V \colon \mathcal{F}(W) \to \mathcal{F}(V)$ given by composition with the open immersion $j\colon (V, \mathcal{O}_Y|_V) \to (W, \mathcal{O}_Y|_W)$. The presheaf $\mathcal{F}$ is a sheaf.

Proof. Let $V \subset |Y|$ be an open subset, and let $(V_i)_{i\in I}$ be a covering of V by open subsets. We wish to prove that $\mathcal{F}$ satisfies the sheaf condition: The diagram

$$\mathcal{F}(V) \xrightarrow{(\mathrm{res}^V_{V_i})} \prod_{i\in I} \mathcal{F}(V_i) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(V_i \cap V_j),$$

where $\mathrm{pr}_{(i,j)} \circ \alpha = \mathrm{res}^{V_i}_{V_i\cap V_j} \circ \mathrm{pr}_i$ and $\mathrm{pr}_{(i,j)} \circ \beta = \mathrm{res}^{V_j}_{V_i\cap V_j} \circ \mathrm{pr}_j$, is a limit diagram. So we suppose that for every $i \in I$, we are given a map of schemes

$$(V_i, \mathcal{O}_Y|_{V_i}) \xrightarrow{(p_i,\phi_i)} (|X|, \mathcal{O}_X)$$

and that for all $(i,j) \in I \times I$, the diagram of maps of schemes

$$\begin{tikzcd}
(V_i \cap V_j, \mathcal{O}_Y|_{V_i \cap V_j}) \arrow[r] \arrow[d] & (V_i, \mathcal{O}_Y|_{V_i}) \arrow[d, "(p_i,\phi_i)"] \\
(V_j, \mathcal{O}_Y|_{V_j}) \arrow[r, "(p_j,\phi_j)"] & (|X|, \mathcal{O}_X)
\end{tikzcd}$$

commutes. The top horizontal map and the left-hand vertical map are the open immersions. We must show that there exists a unique map of schemes

$$(V, \mathcal{O}_Y|_V) \xrightarrow{(p,\phi)} (|X|, \mathcal{O}_X)$$

such that for all $i \in I$, the diagram of maps of schemes

$$\begin{tikzcd}
& (V, \mathcal{O}_Y|_V) \arrow[dr, "(p,\phi)"] & \\
(V_i, \mathcal{O}_Y|_{V_i}) \arrow[ur] \arrow[rr, "(p_i,\phi_i)"] & & (|X|, \mathcal{O}_X)
\end{tikzcd}$$

commutes. The left-hand slanted map is the open immersion. We have already seen in Example 14 in Lecture 2 that there exists a unique continuous map $p \colon V \to |X|$ with this property. Moreover, for every $i \in I$ and $U \subset |X|$ open, we are given a ring homomorphism $\phi_{i,U} \colon \mathcal{O}_X(U) \to \mathcal{O}_Y(p^{-1}(U) \cap V_i)$ such that the composite maps

$$\mathcal{O}_X(U) \xrightarrow{(\phi_{i,U})} \prod_{i \in I} \mathcal{O}_Y(p^{-1}(U) \cap V_i) \overset{\alpha_U}{\underset{\beta_U}{\rightrightarrows}} \prod_{(i,j) \in I \times I} \mathcal{O}_Y(p^{-1}(U) \cap V_i \cap V_j)$$

are equal. Since $\mathcal{O}_Y$ is a sheaf, we conclude:

(a) For every $U \subset |X|$ open, there exists a unique map

$$\mathcal{O}_X(U) \xrightarrow{\phi_U} \mathcal{O}_Y(p^{-1}(U))$$

such that for all $i \in I$, the diagram

$$\begin{tikzcd}
& \mathcal{O}_Y(p^{-1}(U)) \arrow[dr, "\mathrm{res}^{p^{-1}(U)}_{p^{-1}(U) \cap V_i}"] & \\
\mathcal{O}_X(U) \arrow[ur, "\phi_U"] \arrow[rr, "\phi_{i,U}"] & & \mathcal{O}_Y(p^{-1}(U) \cap V_i)
\end{tikzcd}$$

commutes.

(b) For every $U \subset U' \subset |X|$, the diagram

$$\begin{tikzcd}
\mathcal{O}_X(U') \arrow[r, "\phi_{U'}"] \arrow[d, "\mathrm{res}^{U'}_U"] & \mathcal{O}_Y(p^{-1}(U')) \arrow[d, "\mathrm{res}^{p^{-1}(U')}_{p^{-1}(U)}"] \\
\mathcal{O}_X(U) \arrow[r, "\phi_U"] & \mathcal{O}_Y(p^{-1}(U))
\end{tikzcd}$$

commutes.

(c) For every $i \in I$, the map ϕ_U is a ring homomorphism.

It remains only to prove that the map (p,ϕ) is local. But this is clear from the characterization (2) in Proposition 5.7. □

Lemma 5.12. *Suppose that $(p,\phi) \colon (|Y|, \mathcal{O}_Y) \to (|X|, \mathcal{O}_X)$ is a map of schemes, and that $U \subset |X|$ is an open subset. The following are equivalent:*

(1) *The image of* $p\colon |Y| \to |X|$ *is contained in* $U \subset |X|$.
(2) *There exists a unique map of schemes* (q,ψ) *that makes the diagram*

$$\begin{tikzcd} & (U, \mathcal{O}_X|_U) \arrow[dr, "(h,\eta)"] & \\ (|Y|, \mathcal{O}_Y) \arrow[ur, "(q,\psi)"] \arrow[rr, "(p,\phi)"] & & (|X|, \mathcal{O}_X) \end{tikzcd}$$

commute.

Proof. It is clear that (2) implies (1). Conversely, the statement (1) is equivalent to the statement that there exists a map $q\colon |Y| \to U$ such that $p = h \circ q$. The map q is unique with this property, because h is injective, and it is continuous, by the universal property of the subspace topology. Moreover, if $V \subset |X|$ is open, then

$$p^{-1}(V) = (h \circ q)^{-1}(V) = q^{-1}(h^{-1}(V)) = q^{-1}(U \cap V) = p^{-1}(U \cap V),$$

and the diagram following diagram commutes.

$$\begin{tikzcd} \mathcal{O}_X(V) \arrow[r, "\phi_V"] \arrow[d, "\mathrm{res}^V_{U\cap V}"] & \mathcal{O}_Y(p^{-1}(V)) \arrow[d, equal] \\ \mathcal{O}_X(U \cap V) \arrow[r, "\phi_{U\cap V}"] & \mathcal{O}_Y(p^{-1}(U \cap V)) \end{tikzcd}$$

So we conclude that there is a unique map of sheaves ψ that makes the diagram

$$\begin{tikzcd} & h_*(\mathcal{O}_X|_U) \arrow[dr, "h_*(\psi)"] & \\ \mathcal{O}_X \arrow[ur, "\eta"] \arrow[r, "\phi"] & p_*(\mathcal{O}_Y) \simeq h_*q_*(\mathcal{O}_Y) & \end{tikzcd}$$

commute, namely, the map ψ with $\psi_V = \phi_V$ for $V \subset U$ open. Finally, it is clear from the characterization (2) in Proposition 5.7 that the map (q,ψ) is local. □

It follows from Lemmas 5.11 and 5.12 that we can understand maps between all schemes if we understand maps between affine schemes. Indeed:

Lemma 5.13. *Let* $f\colon Y \to X$ *be a map of schemes. For every* $y \in |Y|$, *there exists open immersions* $j_U\colon U \to X$ *and* $j_V\colon V \to Y$ *with* U *and* V *affine such that* y *is contained in the image of* $j_V\colon V \to Y$ *and such that* $f \circ j_V$ *factors uniquely as*

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "j_V"] & U \arrow[d, "j_U"] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

Proof. Let $f = (p,\phi)\colon Y \to X$. We first choose a cover $(U_i)_{i\in I}$ of $|X|$ be affine open subsets. This is possible, because X is a scheme. Next, for every $i \in I$, we choose a cover $(V_{i,j})_{j\in J_i}$ of $p^{-1}(U_i)$ be affine open subsets. This is possible, since $(p^{-1}(U_i), \mathcal{O}_Y|_{p^{-1}(U_i)})$ is a scheme. So we can choose $V = V_{i,j}$ such that $y \in V$ and let $U = U_i$. That f factors uniquely as stated follows form Lemma 5.12. □

It remains to understand maps between affine schemes. We will a more general result, describing arbitrary maps with target an affine scheme. Let $\mathsf{Schemes}$ be the category of schemes and maps of schemes.

Theorem 5.14. *There is an adjunction*

$$\mathsf{Schemes} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{CAlg}(\mathsf{Ab})^{\mathrm{op}}$$

where $F(Y) = \mathcal{O}_Y(|Y|)$ *and* $G(R) = \operatorname{Spec}(R)$*, and where the counit is the ring homomorphism* $\epsilon_R \colon R \to \mathcal{O}_{\operatorname{Spec}(R)}(|\operatorname{Spec}(R)|)$ *from Theorem 7 of Lecture 4.*

Proof. The theorem amounts to the statement that for every scheme Y and every ring R with prime spectrum $X = \operatorname{Spec}(R)$, the composition

$$\operatorname{Map}(Y, X) \longrightarrow \operatorname{Map}(\mathcal{O}_X(|X|), \mathcal{O}_Y(|Y|)) \longrightarrow \operatorname{Map}(R, \mathcal{O}_Y(|Y|))$$

of the map that is part of the functor F and the map induced by ϵ_R is a bijection. To produce an inverse map, we let $\varphi \colon R \to \mathcal{O}_Y(|Y|)$ be a ring homomorphism and proceed to produce a map of schemes $(p, \phi) \colon (|Y|, \mathcal{O}_Y) \to (|X|, \mathcal{O}_X)$. We define $p \colon |Y| \to |X|$ to be the map that to $y \in |Y|$ assigns the prime ideal $\mathfrak{p} \subset R$ given by the kernel of the composite ring homomorphism

$$R \xrightarrow{\varphi} \mathcal{O}_Y(|Y|) \xrightarrow{\psi} k(y).$$

We note that this definition is forced upon us by the requirement that (p, ϕ) be a local map that maps to φ by the composite map in question. We claim that

$$p^{-1}(|X_f|) = |Y_{\varphi(f)}|$$

for $f \in R$. Indeed, by definition, we have $p(y) \in |X_f|$ if and only $(\psi \circ \varphi)(f) \neq 0$ in $k(y)$ if and only if $\psi(\varphi(f)) \neq 0$ in $k(y)$ if and only if $y \in |Y_{\varphi(f)}|$. In particular, the map $p \colon |Y| \to |X|$ is continuous. By Lemma 5.3, $\varphi(f) \in \mathcal{O}_Y(|Y_{\varphi(f)}|)$ is invertible, so there is a unique map $\phi_{|X_f|}$ that makes the following diagram commute.

$$\begin{tikzcd}
R \arrow[r, "\epsilon_R"] \arrow[d, "\varphi"] & \mathcal{O}_X(|X|) \arrow[r, "\mathrm{res}^{|X|}_{|X_f|}"] & \mathcal{O}_X(|X_f|) \arrow[d, "\phi_{|X_f|}"] \\
\mathcal{O}_Y(|Y|) \arrow[rr, "\mathrm{res}^{|Y|}_{|Y_{\varphi(f)}|}"] & & \mathcal{O}_Y(|Y_{\varphi(f)}|)
\end{tikzcd}$$

Given $f, g \in R$ such that $|X_g| \subset |X_f|$, we must show that $\phi_{|X_f|}$ and $\phi_{|X_g|}$ are compatible with the respective restriction maps. But this follows automatically from the uniqueness of these maps. So (p, ϕ) is a map of ringed spaces. Finally, this map is local, by construction. Similarly, the map that to φ assigns (p, ϕ) is an inverse to the map in question, since the definition of (p, ϕ) was forced upon us by the requirement that it be so. $\square$

The most important special of this theorem says that maps of affine schemes $\operatorname{Spec}(R) \to \operatorname{Spec}(R')$ are in natural bijection with homomorphisms of rings $R' \to R$.

Remark 5.15. The proof of Theorem 5.14 does not use that Y is a scheme, but only that it is a locally ringed space, that is, a ringed space with the property that for all $y \in |Y|$, the stalks $\mathcal{O}_{Y,y}$ are local rings. So in fact we have an adjunction

$$\mathsf{LocallyRingedSpaces} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathrm{CAlg}(\mathsf{Ab})^{\mathrm{op}},$$

Moreover, since the functor G' takes values in the full subcategory spanned by the schemes, this adjunction induces the adjunction in Theorem 5.14.

Remark 5.16. The unit of the adjunction in Theorem 5.14 is a natural transformation

$$Y \xrightarrow{\eta_Y} \operatorname{Spec}(\mathcal{O}_Y(|Y|)).$$

Moreover, it follows from the theorem that this map is the universal map of schemes from X to an affine scheme in the sense that the composition

$$\operatorname{Map}(R, \mathcal{O}_Y(|Y|)) \longrightarrow \operatorname{Map}(\operatorname{Spec}(\mathcal{O}_Y(|Y|)), \operatorname{Spec}(R)) \longrightarrow \operatorname{Map}(Y, \operatorname{Spec}(R))$$

of the map that is part of the functor G and the map induced by η_Y is a bijection, namely, the inverse of the composite map in the statement of the theorem.

Example 5.17. In any adjunction, the left adjoint functor preserves all colimits that exist in its domain, whereas the right adjoint functor preserves all limits that exist in its domain. In the case of the adjunction in Theorem 5.14, it follows that Spec takes all colimits that exist in the category of rings to limits in the category of schemes. In particular, it takes the initial object $\mathbb{Z}$ in the category of rings to a final object $\operatorname{Spec}(\mathbb{Z})$ in the category of schemes. So for every scheme Y, there is a unique map of schemes $(p, \phi)\colon Y \to \operatorname{Spec}(\mathbb{Z})$, and the proof of Theorem 5.14 shows that the continuous map $p\colon |Y| \to |\operatorname{Spec}(\mathbb{Z})|$ is given by

$$p(y) = \operatorname{char}(k(y))\mathbb{Z} \subset \mathbb{Z},$$

where $\operatorname{char}(k)$ is the characteristic of the field k.

Example 5.18. We recall that the polynomial ring $\mathbb{Z}[T]$ is the free ring on a single generator T in the sense that the map

$$\operatorname{Map}(\mathbb{Z}[T], R) \xrightarrow{\alpha} R$$

defined by $\alpha(\varphi) = \varphi(T)$ is a bijection. The affine scheme

$$\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}[T])$$

is called the affine line. Composing the two bijections

$$\operatorname{Map}(Y, \mathbb{A}^1_{\mathbb{Z}}) \longrightarrow \operatorname{Map}(\mathbb{Z}[T], \mathcal{O}_Y(|Y|)) \longrightarrow \mathcal{O}_Y(|Y|),$$

we see that the affine line rescues the idea that every element $g \in \mathcal{O}_Y(|Y|)$ can be considered as a function on Y, namely, the map of schemes $f\colon Y \to \mathbb{A}^1_{\mathbb{Z}}$ that corresponds to g under this composite bijection.

Moreover, we conclude from Theorem 5.14 that the natural ring structure on the set $\mathcal{O}_Y(|Y|)$ gives rise to a structure of ring object on the scheme $\mathbb{A}^1_{\mathbb{Z}}$. We denote this ring scheme by $\mathcal{O}$. It follows that we have a natural ring isomorphism

$$\operatorname{Map}(Y, \mathcal{O}) \longrightarrow \mathcal{O}_Y(|Y|).$$

Example 5.19. Similarly, the polynomial ring $\mathbb{Z}[T_1, \dots, T_n]$ is the free ring generated by the set $\{T_1, \dots, T_n\}$, and the affine scheme

$$\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}[T_1, \dots, T_n])$$

is called the affine space of dimension n. A map of schemes $f\colon Y \to \mathbb{A}^n_{\mathbb{Z}}$ determines and is determined by a family $(f_1, \dots, f_n)$ of n elements in $\mathcal{O}_Y(|Y|)$.

Example 5.20. The open subscheme $\mathbb{A}^1_{\mathbb{Z}} \smallsetminus \{0\}_{\mathbb{Z}} \subset \mathbb{A}^1_{\mathbb{Z}}$ defined by the open complement of closed subset $V(T) \subset |\mathbb{A}^1_{\mathbb{Z}}|$ is itself an affine scheme, namely,

$$\mathbb{A}^1_{\mathbb{Z}} \smallsetminus \{0\}_{\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}[T^{\pm 1}]).$$

Indeed, the open subset $|\mathbb{A}^1_{\mathbb{Z}} \smallsetminus \{0\}_{\mathbb{Z}}| \subset |\mathbb{A}^1_{\mathbb{Z}}|$ is the distinguished open subset $|(\mathbb{A}^1_{\mathbb{Z}})_T|$. Arguing as in Example 5.18, we obtain a natural bijection

$$\operatorname{Map}(Y, \mathbb{A}^1_{\mathbb{Z}} \smallsetminus \{0\}_{\mathbb{Z}}) \longrightarrow \mathcal{O}_Y(|Y|)^{\times}$$

from the set of maps of schemes $f \colon Y \to \mathbb{A}^1_{\mathbb{Z}} \smallsetminus \{0\}_{\mathbb{Z}}$ onto the set $\mathcal{O}_Y(|Y|)^{\times}$ of units in the ring $\mathcal{O}_Y(|Y|)$. Again, the natural group structure gives rise to a group object structure on the scheme $\mathbb{A}^1_{\mathbb{Z}} \smallsetminus \{0\}_{\mathbb{Z}}$. We write $\mathbb{G}_m$ for this group scheme and call it the multiplicative group (scheme). So have a natural group isomorphism

$$\operatorname{Map}(Y, \mathbb{G}_m) \longrightarrow \mathcal{O}_Y(|Y|)^{\times}.$$

Example 5.21. The open subscheme $\mathbb{A}^n_{\mathbb{Z}} \smallsetminus \{0\}_{\mathbb{Z}} \subset \mathbb{A}^n_{\mathbb{Z}}$ defined by the open complement of the closed subset $V(T_1, \dots, T_n) \subset |\mathbb{A}^n_{\mathbb{Z}}|$ is not affine for $n \geq 2$, but instead

$$|\mathbb{A}^n_{\mathbb{Z}} \smallsetminus \{0\}_{\mathbb{Z}}| = |(\mathbb{A}^n_{\mathbb{Z}})_{T_1}| \cup \dots \cup |(\mathbb{A}^n_{\mathbb{Z}})_{T_n}| \subset |\mathbb{A}^n_{\mathbb{Z}}|.$$

It follows from Lemma 5.12 that for every scheme Y, the open immersion

$$\mathbb{A}^n_{\mathbb{Z}} \smallsetminus \{0\}_{\mathbb{Z}} \xrightarrow{\; j \;} \mathbb{A}^n_{\mathbb{Z}}$$

induces an injective map

$$\operatorname{Map}(Y, \mathbb{A}^n_{\mathbb{Z}} \smallsetminus \{0\}_{\mathbb{Z}}) \longrightarrow \operatorname{Map}(Y, \mathbb{A}^n_{\mathbb{Z}}) \simeq \mathcal{O}_Y(|Y|)^n$$

and that its image consists of the tuples $(f_1, \dots, f_n)$ with the property that for every $y \in |Y|$, there exists $1 \leq i \leq n$ such that $f_i(y) \in k(y)$ is nonzero.

Example 5.22. The projective line $\mathbb{P}^1_{\mathbb{Z}}$ is defined to be the pushout

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{Z}} \smallsetminus \{0\}_{\mathbb{Z}} \arrow[r, "j_1"] \arrow[d, "j_2"] & \mathbb{A}^1_{\mathbb{Z}} \arrow[d, "j_2'"] \\
\mathbb{A}^1_{\mathbb{Z}} \arrow[r, "j_1'"] & \mathbb{P}^1_{\mathbb{Z}}
\end{tikzcd}$$

in the category of schemes, where the open immersions j_1 and j_2 correspond to the ring homomorphisms $\mathbb{Z}[T] \to \mathbb{Z}[T^{\pm 1}]$ that to T assign T and T^{-1}, respectively.[1] It follows from Lemmas 5.11 and 5.12 that a map of schemes

$$Y \xrightarrow{\; h \;} \mathbb{P}^1_{\mathbb{Z}}$$

determines and is determined by a quadruple (U, V, f, g), where $U, V \subset |Y|$ are open subsets and $f \in \mathcal{O}_Y(U)$ and $g \in \mathcal{O}_Y(V)$ are local sections such that

$$|Y_f| = U \cap V = |Y_g|$$

[1] One has to prove that this pushout exists! It is a general fact, which is not difficult to prove, that the pushout of two open immersions of schemes exists. Informally, we can always "glue" two schemes along a common open subscheme. However, if we want to "glue" more than two schemes along open immersions, then the "transition functions" must satisfy the "cocycle condition."

and $f|_{U\cap V} \cdot g|_{U\cap V} = 1$ in $\mathcal{O}_Y(U \cap V)$. However, two such quadruples can give rise to the same map, so this is not the best possible description of maps to $\mathbb{P}^1$. We will return to this later.

In this lecture, we will assign to every scheme X an abelian category $\mathrm{QCoh}(X)$ of "quasicoherent $\mathcal{O}_X$-modules" based on the following two ideas:

(1) If $X \simeq \mathrm{Spec}(R)$, then $\mathrm{QCoh}(X) \simeq \mathrm{Mod}_R$.
(2) The assigment $X \mapsto \mathrm{QCoh}(X)$ is "local" on X.

Here Mod_R is the category of R-modules. We will define $\mathrm{QCoh}(X)$ to be a full subcategory of the larger category $\mathrm{Mod}_{\mathcal{O}_X}$ of $\mathcal{O}_X$-modules.

Definition 6.1. Let $X = (|X|, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-module is a pair $(\mathcal{M}, \mu)$ of a sheaf of abelian groups $\mathcal{M}$ on $|X|$ and a map $\mu\colon \mathcal{O}_X \times \mathcal{M} \to \mathcal{M}$ of sheaves of sets on $|X|$ with the property that for all $U \subset |X|$ open, the map

$$\mathcal{O}_X(U) \times \mathcal{M}(U) \simeq (\mathcal{O}_X \times \mathcal{M})(U) \xrightarrow{\mu_U} \mathcal{M}(U)$$

makes $(\mathcal{M}(U), \mu_U)$ an $\mathcal{O}_X(U)$-module. A map of $\mathcal{O}_X$-modules $h\colon (\mathcal{M}, \mu) \to (\mathcal{M}', \mu')$ is a map $h\colon \mathcal{M} \to \mathcal{M}'$ of sheaves of abelian groups on $|X|$ such that for every $U \subset |X|$ open, the map $h_U\colon \mathcal{M}(U) \to \mathcal{M}'(U)$ is $\mathcal{O}_X(U)$-linear.

We will abuse notation and write $\mathcal{M}$ instead of $(\mathcal{M}, \mu)$ for an $\mathcal{O}_X$-module, just as we abuse notation and write $\mathcal{O}_X$ for the ring object $(\mathcal{O}_X, +, \cdot)$.

Remark 6.2. In Grothendieck's philosophy, the category $\mathrm{Sh}(|X|)$ of sheaves of sets on $|X|$ behaves just like the category Set of sets with the one exception that, in general, the axiom of choice fails in $\mathrm{Sh}(|X|)$. So sheaves of abelian groups, sheaves of rings, and $\mathcal{O}_X$-modules are simply the abelian group objects, ring objects, and $\mathcal{O}_X$-module objects in $\mathrm{Sh}(|X|)$.

Just as the category $\mathrm{Sh}(|X|, \mathsf{Ab})$ of sheaves of abelian groups and the category Mod_R of R-modules both are abelian, so is the category $\mathrm{Mod}_{\mathcal{O}_X}$ of $\mathcal{O}_X$-modules. Limits, or equivalently, products and kernels, of $\mathcal{O}_X$-modules are calculated sectionwise, whereas colimits, or equivalently, sums and cokernels, of $\mathcal{O}_X$-modules are calculated by sheafifying the sectionwise colimits. Finally, the sheafification of an $\mathcal{O}_X$-module presheaf is calculated by the "same" formula as the sheafification of a presheaf of sets,[2] and a sequence of $\mathcal{O}_X$-modules

$$\mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}''$$

is exact if and only if the sequence of $\mathcal{O}_{X,x}$-modules

$$\mathcal{M}'_x \longrightarrow \mathcal{M}_x \longrightarrow \mathcal{M}''_x$$

is exact for all $x \in |X|$.

You may think, based on the above, that products (and limits) of $\mathcal{O}_X$-modules are better behaved than sums (and colimits). In fact, the opposite is true:

(1) Sums (and colimits) commute with taking stalks: Given $(\mathcal{M}_i)_{i\in I}$, the map

$$\textstyle\bigoplus_{i\in I}(\mathcal{M}_i)_x \longrightarrow (\bigoplus_{i\in I}\mathcal{M}_i)_x$$

[2] Indeed, sheafification preserves finite limits, in general, and finite products, in particular, and an $\mathcal{O}_X$-module structure on a (pre)sheaf of sets $\mathcal{M}$ is given by a pair of maps between finite products of (pre)sheaves, namely, $+\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and $\mu\colon \mathcal{O}_X \times \mathcal{M} \to \mathcal{M}$.

is an isomorphism. This is true presheaves, and sheafification preserves all (small) colimits, so it is also true for sheaves. By contrast, the corresponding statement for infinite products (and infinite limits) is false.

(2) Sums are exact: Given $(h_i\colon \mathcal{M}_i \to \mathcal{M}'_i)_{i\in I}$, the maps

$$\textstyle\bigoplus_{i\in I} \ker(h_i) \longrightarrow \ker(\bigoplus_{i\in I} h_i)$$

$$\textstyle\bigoplus_{i\in I} \operatorname{coker}(h_i) \longrightarrow \operatorname{coker}(\bigoplus_{i\in I} h_i)$$

are isomorphisms. Indeed, we can check this on stalks, where the statements follow from the analogous statements for families of maps of abelian groups. The cokernel statement fails for infinite products.

(3) Sums (and colimits) commute with restriction to open subsets: Given a family of $\mathcal{O}_X$-modules $(\mathcal{M}_i)_{i\in I}$, the canonical map

$$\textstyle\bigoplus_{i\in I}(\mathcal{M}_i)|_U \longrightarrow (\bigoplus_{i\in I} \mathcal{M}_i)|_U$$

is an isomorphism. The same is true for products (and limits), but for sums (and colimits), a more general statement is true: For every map of schemes $f\colon Y \to X$, the inverse image functor

$$\mathrm{Mod}_{\mathcal{O}_X} \xrightarrow{\ f^*\ } \mathrm{Mod}_{\mathcal{O}_Y},$$

which is left adjoint to the pushforward fucntor, preserves sums (and colimits). This can be checked on stalks, or it can be proved by showing that f^* admits a right adjoint f_*, and hence, preserves all colimits that exist in its domain. The corresponding statement for infinite products (and infinite limits) is false.

Finally, if X is a scheme, then a special feature of the topology on $|X|$ makes sums (and colimits) even more manageable:

Lemma 6.3. *Let $|X|$ be a topological space such that the set of quasicompact open subsets $U \subset |X|$ form a basis for the topology and is closed under finite intersections. Given a family $(\mathcal{M}_i)_{i\in I}$ of sheaves of abelian groups on $|X|$, the canonical map*

$$\textstyle\bigoplus_{i\in I} \mathcal{M}_i(U) \longrightarrow (\bigoplus_{i\in I} \mathcal{M}_i)(U)$$

is an isomorphism for every quasicompact open subset $U \subset |X|$.

Proof. As the category of sheaves on $|X|$ is equivalent to the category of sheaves on the basis of quasicompact open subsets, it suffices to show that the presheaf sum satisfies the sheaf condition on this basis. Moreover, since the basis elements are quasicompact, it suffices to check the sheaf condition for finite covers. But the sheaf condition for finite covers involves kernels and finite products, and finite products are the same as finite sums, since we consider (pre)sheaves of abelian groups, and both kernels and sums preserve sums. □

Remark 6.4. Lemma 6.3 shows that, to calculate the value of a sum of sheaves of abelian groups on a quasicompact open subset, we do not need to sheafify. The same is *not true* for cokernels! Indeed, taking kernels and cokernels do generally not commute, as the example

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "2\,\mathrm{id}"] \arrow[d, "4\,\mathrm{id}"] & \mathbb{Z} \arrow[d, "2\,\mathrm{id}"] \\ \mathbb{Z} \arrow[r, "\mathrm{id}"] & \mathbb{Z} \end{tikzcd}$$

shows. If we first take horizontal cokernels and then vertical kernels, then we get $\mathbb{Z}/2\mathbb{Z}$, but if we do this in the opposite order, then we get 0.

Corollary 6.5. *Let X be a scheme, and let $(\mathcal{M}_i)_{i\in I}$ be a family of $\mathcal{O}_X$-modules. For every affine open subscheme $U \subset X$, the canonical map*

$$\bigoplus_{i\in I} \mathcal{M}_i(U) \longrightarrow (\bigoplus_{i\in I} \mathcal{M}_i)(U)$$

is an isomorphism.

Proof. We have already seen that restriction to any open subset preserves sums. So we can assume that $X = U$ is affine, and in this case, Lemma 6.3 applies. □

We make one final general observation:

Lemma 6.6. *If X is a ringed space, and if $\mathcal{M}$ is an $\mathcal{O}_X$-module, then the map*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}) \longrightarrow \mathcal{M}(|X|)$$

that to $h\colon \mathcal{O}_X \to \mathcal{M}$ assigns $h_X(1) \in \mathcal{M}(|X|)$ is a bijection.

Proof. If R is a ring, and if M is an R-module, then the map

$$\mathrm{Hom}_R(R, M) \longrightarrow M$$

that to an R-linear map $h\colon R \to M$ assigns $h(1) \in M$ is a bijection. Thus, the map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}) \longrightarrow \prod_{U \,\subset\, |X| \text{ open}} \mathcal{M}(U)$$

that to $h\colon \mathcal{O}_X \to \mathcal{M}$ assigns $(h_U(1))_{U \,\subset\, |X| \text{ open}}$ is injective and its image consists of the tuples $(s_U)_{U \,\subset\, |X| \text{ open}}$ such that for all $U \subset V \subset |X|$ open, $s_V|_U = s_U$. But such a tuple determines and is determined by the component s_X. □

Remark 6.7. To explain Lemma 6.6 in more detail, there are adjunctions

$$\begin{tikzcd} \mathsf{Set} \arrow[r, shift left, "p^*"] & \mathrm{Sh}(|X|) \arrow[l, shift left, "p_*"] \arrow[r, shift left, "\phi^*"] & \mathrm{Mod}_{\mathcal{O}_X}, \arrow[l, shift left, "\phi_*"] \end{tikzcd}$$

where p_* is the functor that to a sheaf $\mathcal{F}$ assigns its set $p_*(\mathcal{F}) = \mathcal{F}(|X|)$ of global sections, and where ϕ_* is the forgetful functor that to an $\mathcal{O}_X$-module $\mathcal{M}$ assigns its underlying sheaf of sets $\phi_*(\mathcal{M})$. The left adjoint functor p^* takes a set S to the constant sheaf $p^*(S)$ given by the sheafification of the presheaf $U \mapsto S$, and the left adjoint functor ϕ^* takes a sheaf of sets $\mathcal{F}$ to the "free $\mathcal{O}_X$-module spanned by $\mathcal{F}$" given by the sheafification of the presheaf that to U assigns the free $\mathcal{O}_X(U)$-module spanned by $\mathcal{F}(U)$. If 1 is a set with a single element, then $\phi^*p^*(1) \simeq \mathcal{O}_X$, so the lemma is the special case $S = 1$ of the statement that the natural map

$$\mathrm{Hom}_{\mathcal{O}_X}(\phi^*p^*(S), \mathcal{M}) \longrightarrow \mathrm{Map}(S, p_*\phi_*(\mathcal{M})),$$

which is part of the data of an adjunction, is a bijection.

Let us now look at $\mathcal{O}_X$-modules on $X = \mathrm{Spec}(R)$. The adjunction

$$\mathsf{Set} \underset{p_*}{\overset{p^*}{\rightleftarrows}} \mathrm{Sh}(|X|)$$

from Remark 15.3 gives rise to an adjunction

$$\mathrm{Mod}_R \underset{p_*}{\overset{p^*}{\rightleftarrows}} \mathrm{Mod}_{p^*(R)}$$

between categories of modules. Moreover, we have the ring homomorphism

$$R \xrightarrow{\epsilon} \mathcal{O}_X(|X|) \simeq p_*(\mathcal{O}_X),$$

which was part of the definition of the structure sheaf, and its adjunct

$$p^*(R) \xrightarrow{\psi} \mathcal{O}_X$$

is a a map of sheaves of rings on $|X|$. So we also have an adjunction

$$\mathrm{Mod}_{p^*(R)} \underset{\psi_*}{\overset{\psi^*}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{O}_X},$$

where ψ_* is given by restriction of scalars along ψ, and where its left adjoint ψ^* is given by extension of scalars along ψ. In the composite adjunction

$$\mathrm{Mod}_R \underset{p_*\psi^*}{\overset{\psi^* p^*}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{O}_X},$$

the right adjoint $p_*\psi_*$ takes an $\mathcal{O}_X$-module $\mathcal{M}$ to the set $\mathcal{M}(|X|)$ of global sections with the R-module structure obtained by restriction of scalars along $R \to \mathcal{O}_X(|X|)$, and the left adjoint $\psi^* p^*$ takes an R-module M to the $\mathcal{O}_X$-module

$$\widetilde{M} \simeq \psi^* p^*(M)$$

that we defined in Lecture 4. The unit of the composite adjunction

$$M \xrightarrow{\eta} p_*\psi_*\psi^* p^*(M) \simeq \widetilde{M}(|X|)$$

was part of our definition of $\widetilde{M}$: the pair $(\widetilde{M}, \eta)$ is characterized uniquely, up to unique isomorphism, by the requirement that for every $f \in R$, the composition

$$M \xrightarrow{\eta} \widetilde{M}(|X|) \longrightarrow \widetilde{M}(|X_f|)$$

is a localization with respect to $S = \{1, f, f^2, \dots\} \subset R$. Since the localization with respect to S is given by a filtered colimit, it preserves finite limits (and colimits). So the functor

$$M \mapsto \widetilde{M} \simeq \phi^* p^*(M)$$

is exact! The counit

$$\widetilde{\mathcal{M}(|X|)} \simeq \psi^* p^* p_* \psi_*(\mathcal{M}) \xrightarrow{\epsilon} \mathcal{M}$$

is a map of $\mathcal{O}_X$-modules. So the R-module of global sections $\mathcal{M}(|X|)$ it is trying to tell the $\mathcal{O}_X$-module $\mathcal{M}$ via this map what its sections should be over every open subset $U \subset |X|$. But $\mathcal{M}$ does not have to listen.

Theorem 6.8. *Let $X \simeq \mathrm{Spec}(R)$ be an affine scheme, and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. The following are equivalent:*

(1) *The counit $\epsilon\colon \widetilde{\mathcal{M}(X)} \to \mathcal{M}$ is an isomorphism.*
(2) *There exists an exact sequence of $\mathcal{O}_X$-modules*

$$\bigoplus_{j\in J} \mathcal{O}_X \longrightarrow \bigoplus_{i\in I} \mathcal{O}_X \longrightarrow \mathcal{M} \longrightarrow 0.$$

(3) *For every $x \in |X|$, there exists $x \in U \subset |X|$ open and an exact sequence*

$$\bigoplus_{j\in J} \mathcal{O}_X|_U \longrightarrow \bigoplus_{i\in I} \mathcal{O}_X|_U \longrightarrow \mathcal{M}|_U \longrightarrow 0$$

of $\mathcal{O}_X|_U$-modules.

Moreover, the full subcategory $\mathrm{QCoh}(X) \subset \mathrm{Mod}_{\mathcal{O}_X}$ spanned by the $\mathcal{O}_X$-modules that satisfy these conditions is closed under kernels and cokernels, and hence, is abelian. It is also closed under arbitrary sums.

Proof. We assume (1), so that $\mathcal{M} \simeq \widetilde{M}$ with $M \simeq \mathcal{M}(|X|)$. We choose a presentation

$$\bigoplus_{j\in J} R \longrightarrow \bigoplus_{i\in I} R \longrightarrow M \longrightarrow 0$$

and apply the exact functor $\widetilde{(-)} \simeq \psi^* p^*$ to get

$$\bigoplus_{j\in J} \mathcal{O}_X \longrightarrow \bigoplus_{i\in I} \mathcal{O}_X \longrightarrow \mathcal{M} \longrightarrow 0.$$

This proves (2). It is clear that (2) implies (3), so it remains to show that (3) implies (1), which is the hard part. Let us first show that (2) implies (1). So we assume that there exists an exact sequence of $\mathcal{O}_X$-modules

$$\bigoplus_{j\in J} \mathcal{O}_X \xrightarrow{\ h\ } \bigoplus_{i\in I} \mathcal{O}_X \longrightarrow \mathcal{M} \longrightarrow 0$$

and show that this sequence arises by applying the exact functor $\widetilde{(-)} \simeq \psi^* p^*$ to an exact sequence of R-modules

$$\bigoplus_{j\in J} R \xrightarrow{\ g\ } \bigoplus_{i\in I} R \longrightarrow M \longrightarrow 0.$$

By exactness, for this it suffices to show that every map h as above arises from some map g as above by applying $\widetilde{(-)}$. Because maps out of a direct sum are the same as separate maps out of each factor, this reduces to the case where J has cardinality 1. By Lemma 6.6, in this case maps h amount to elements of $(\oplus_{i\in I}\mathcal{O}_X)(X)$ and maps g amoung to elements of $\oplus_{i\in I}R$. Thanks to Lemma 6.5, these agree and we get the claim.

Finally, to prove that (3) implies (2), it suffices to show that if $(|X_{f_i}|)_{i\imath I}$ is a finite family of distinguished open subsets of $|X|$ that cover $|X|$, and if the map

$$\widetilde{\mathcal{M}(|X_{f_i}|)} \xrightarrow{\ \epsilon\ } \mathcal{M}|_{|X_{f_i}|}$$

is an isomorphism for all $i \in I$, then so is the map

$$\widetilde{\mathcal{M}(|X|)} \xrightarrow{\ \epsilon\ } \mathcal{M}.$$

It suffices to show that for all $i \in I$, the restriction of the latter map

$$\widetilde{\mathcal{M}(|X|)}|_{|X_{f_i}|} \xrightarrow{\ \epsilon\ } \mathcal{M}|_{|X_{f_i}|}$$

to $|X_{f_i}| \subset |X|$ is an isomorphism. This map factors as a composition

$$\widetilde{\mathcal{M}(|X|)}|_{|X_{f_i}|} \longrightarrow \widetilde{\mathcal{M}(|X_{f_i}|)} \longrightarrow \mathcal{M}|_{|X_{f_i}|},$$

where the right-hand map is an isomorphism, by assumption. So it suffices to show that the left-hand map is an isomorphism, or equivalently, that if $|X_g| \subset |X_{f_i}|$, then the induced map of R_g-modules

$$\mathcal{M}(|X|)_g \longrightarrow \mathcal{M}(|X_{f_i}|)_g$$

is an isomorphism. Clearly, it suffices to consider $g = f_i$. Now, since $\mathcal{M}$ is a sheaf, we have an exact sequence of R-modules

$$0 \longrightarrow \mathcal{M}(|X|) \longrightarrow \prod_{j\in I} \mathcal{M}(|X_{f_j}|) \longrightarrow \prod_{(j,k)\in I\times I} \mathcal{M}(|X_{f_j f_k}|),$$

and the two products are finite products. Hence, they agree with the corresponding sums, and therefore, they commute with localization. It follows that that in the following commutative diagram, the two horizontal sequence is exact.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}(|X|)_{f_i} \arrow[r] \arrow[d] & \prod_{j\in I} \mathcal{M}(|X_{f_j}|)_{f_i} \arrow[r] \arrow[d] & \prod_{(j,k)\in I\times I} \mathcal{M}(|X_{f_j f_k}|)_{f_i} \arrow[d] \\
0 \arrow[r] & \mathcal{M}(|X_{f_i}|) \arrow[r] & \prod_{j\in I} \mathcal{M}(|X_{f_i f_j}|) \arrow[r] & \prod_{(j,k)\in I\times I} \mathcal{M}(|X_{f_i f_j f_k}|)
\end{tikzcd}$$

The bottom horizontal sequence is also exact, because $\mathcal{M}$ is a sheaf, and the middle and right-hand vertical maps are isomorphisms by assumption. Hence, the left-hand vertical map is an isomorphism, too, as we wanted to show.

We have now proved that (1)–(3) are equivalent. Since $\widetilde{(-)} \simeq \psi^* p^*$ is exact and since the global sections functor $p_*\psi_*$ preserves limits, we conclude from (1) that $\mathrm{QCoh}(X) \subset \mathrm{Mod}_{\mathcal{O}_X}$ is closed under kernels. Finally, it is clear from (2) that it is also closed under colimits. This completes the proof. $\square$

Remark 6.9. In the proof of Theorem 6.8, we saw that the functor

$$\mathrm{Mod}_R \longrightarrow \mathrm{QCoh}(X)$$

that to an R-module M assigns the quasicoherent $\mathcal{O}_X$-module $\widetilde{M}$ is an equivalence of categories. A quasi-inverse is given by the functor that takes a quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ to its R-module of global sections $\mathcal{M}(|X|)$.

Example 6.10. Let V be a discrete valuation ring with maximal ideal $\mathfrak{m} \subset V$. Its Zariski space is the Sierpinski space $|X| = \{\eta, s\}$, where η corresponds to the zero ideal $\{0\} \subset V$ and is a generic point, and where s corresponds to the maximal ideal $\mathfrak{m} \subset V$ and is a closed point. So the open subsets of $|X|$ are $|X| \supset \{\eta\} \supset \emptyset$, and the values of the structure sheaf on these open subsets are $V \to K \to 0$, where K is the quotient field of V and 0 is the zero ring. Now, an $\mathcal{O}_X$-module $\mathcal{M}$ determines and is determined, up to unique isomorphism, by the V-module $\mathcal{M}(|X|)$, the K-vector space $\mathcal{M}(\{\eta\})$, and the K-linear restriction map

$$K \otimes_V \mathcal{M}(|X|) \longrightarrow \mathcal{M}(\{\eta\}).$$

It is a quasicoherent $\mathcal{O}_X$-module if and only if this map is an isomorphism.

Theorem 6.11. *Let $X = (|X|, \mathcal{O}_X)$ be a scheme, and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. The following are equivalent:*

(1) *For every $x \in |X|$, there exists $x \in U \subset |X|$ affine open such that*

$$\mathcal{M}|_U \simeq \widetilde{N}$$

for some $\mathcal{O}_X(U)$-module N.

(2) *For every affine open subset $U \subset |X|$, there exists an isomorphism*

$$\mathcal{M}|_U \simeq \widetilde{N}$$

for some $\mathcal{O}_X(U)$-module N.

(3) *For every $x \in |X|$, there exists $x \in U \subset |X|$ open and an exact sequence*

$$\bigoplus_{j\in J} \mathcal{O}_X|_U \longrightarrow \bigoplus_{i\in I} \mathcal{O}_X|_U \longrightarrow \mathcal{M}|_U \longrightarrow 0$$

of $\mathcal{O}_X|_U$-modules.

Moreover, the full subcategory $\mathrm{QCoh}(X) \subset \mathrm{Mod}_{\mathcal{O}_X}$ spanned by the $\mathcal{O}_X$-modules that satisfy these conditions is closed under kernels and cokernels, and hence, is abelian. It is also closed under arbitrary sums.

Proof. That (1) implies (3) follows from the affine case, and that (2) implies (1) is trivial. To prove that (3) implies (2), we note that (3) for $\mathcal{M}$ implies (3) for $\mathcal{M}|_U$, since restriction along an open immersion $j\colon U \to X$ preserves colimits, or equivalently, preserves sums and cokernels. So that (3) implies (2) also follows from the affine case. In fact, restriction along an open immersion $j\colon U \to X$ preserves both limits and colimits, so the final statement concerning $\mathrm{QCoh}(X) \subset \mathrm{Mod}_{\mathcal{O}_X}$ also reduces to the corresponding statement for X affine. □

Remark 6.12. Let X be a scheme. The canonical inclusion

$$\mathrm{QCoh}(X) \longrightarrow \mathrm{Mod}_{\mathcal{O}_X}$$

preserves all colimits and finite limits.[3] In particular, a sequence of quasicoherent $\mathcal{O}_X$-modules is exact in $\mathrm{QCoh}(X)$ if and only if it is exact in $\mathrm{Mod}_{\mathcal{O}_X}$. So we can check exactness on stalks. The following are equivalent:

(1) The sequence of quasicoherent $\mathcal{O}_X$-modules

$$\mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}''$$

is exact.

(2) For every *affine* open $U \subset |X|$, the induced sequence of $\mathcal{O}_X(U)$-modules

$$\mathcal{M}'(U) \longrightarrow \mathcal{M}(U) \longrightarrow \mathcal{M}''(U)$$

is exact.

[3] But it but it does not preserve infinite products.

(3) For some covering $(U_i)_{i \in I}$ of $|X|$ by *affine* open subsets, the sequences

$$\mathcal{M}'(U_i) \longrightarrow \mathcal{M}(U_i) \longrightarrow \mathcal{M}''(U_i)$$

of $\mathcal{O}_X(U_i)$-modules are exact for all $i \in I$.

It is clear that (2) implies (3) and that (3) implies (1). To see that (1) implies (2), we use that if $j \colon U \to X$ is open immersion with U affine, then the functor

$$\mathrm{QCoh}(U) \longrightarrow \mathrm{Mod}_{\mathcal{O}_X(U)}$$

that to $\mathcal{N}$ assigns $\mathcal{N}(U)$ is an equivalence of categories, so the sequence

$$\mathcal{M}'|_U \longrightarrow \mathcal{M}|_U \longrightarrow \mathcal{M}''|_U$$

in $\mathrm{QCoh}(U)$ is exact if and only if the sequence in $\mathrm{Mod}_{\mathcal{O}_X(U)}$ obtained by taking global sections is exact.

Last time, we associated to a scheme $X = (|X|, \mathcal{O}_X)$ the full subcategory

$$\mathrm{QCoh}(X) \subset \mathrm{Mod}_{\mathcal{O}_X}$$

of the abelian category of $\mathcal{O}_X$-module objects in the category $\mathrm{Sh}(|X|)$ spanned by the $\mathcal{O}_X$-modules $\mathcal{M}$ that satisfy the following equivalent conditions:

(1) For every $x \in |X|$, there exists $x \in U \subset |X|$ open and an exact sequence

$$\textstyle\bigoplus_{j\in J} \mathcal{O}_X|_U \longrightarrow \bigoplus_{i\in I} \mathcal{O}_X|_U \longrightarrow \mathcal{M}|_U \longrightarrow 0$$

of $\mathcal{O}_X|_U$-modules.

(2) For every affine open subset $U \subset |X|$, there exists an isomorphism

$$\mathcal{M}|_U \simeq \widetilde{N}$$

of $\mathcal{O}_X|_U$-modules.

We also saw that the $\mathcal{O}_X(U)$-module N in (2) necessarily is isomorphic to $\mathcal{M}(U)$.

As an application of this technology, we will define affine maps of schemes and the relative prime spectrum of a quasicoherent $\mathcal{O}_X$-algebra. For this purpose, we need gluing of schemes, which is the following result.

Theorem 7.1. *The category of schemes admits quotients by equivalence relations*

$$R \xrightarrow{(s,t)} Y \times Y$$

such that $Y = \coprod_{i\in I} Y_i$ *and* $R = \coprod_{(i,j)\in I\times I} U_{i,j}$ *and such that* s *and* t *restrict to open immersions* $s|_{U_{i,j}} \colon U_{i,j} \to Y_i$ *and* $t|_{U_{i,j}} \colon U_{i,j} \to Y_j$*.*

Proof. Given an equivalence relation as is the statement, we let $p\colon |Y| \to |X|$ be a coequalizer of $r, s\colon |R| \to |Y|$ in the category of topological spaces and continuous maps. We claim that the map $p|_{Y_i}\colon |Y_i| \to p(|Y_i|)$ is a homeomorphism for all $i \in I$. First, it is a bijection, because the maps $s|_{U_{i,i}}\colon |U_{i,i}| \to |Y_i|$ and $t|_{U_{i,i}}\colon |U_{i,i}| \to |Y_i|$ necessarily are equal. Indeed, they are both open embeddings, and the diagonal map $\Delta\colon Y_i \to Y_i \times Y_i$ factors through $(s,t)|_{U_{i,i}}\colon U_{i,i} \to Y_i \times Y_i$, because (s,t) is an equivalence relation. Second, it is an open map. For if $|V| \subset |Y_i|$ is open, then so is

$$p^{-1}(p(|V|)) = \textstyle\coprod_{j\in I} (t \circ s^{-1})(|V| \cap |U_{i,j}|) \subset \coprod_{j\in I} Y_j = |Y|.$$

This proves the claim. Finally, the sheaf of rings $\mathcal{O}_X$ given by the equalizer

$$\mathcal{O}_X \xrightarrow{\phi} p_*\mathcal{O}_Y \underset{p_*\tau}{\overset{p_*\sigma}{\rightrightarrows}} q_*\mathcal{O}_R,$$

where $h = f \circ s = f \circ t$, makes $(|X|, \mathcal{O}_X)$ a scheme and makes the diagram

$$R \underset{t}{\overset{s}{\rightrightarrows}} Y \xrightarrow{f} X$$

a coequalizer in the category of schemes. $\square$

Example 7.2. Let us show that, in the category of schemes, every scheme X is the colimit of its affine open subschemes $V \subset X$. Let us write $V \subset X$ with the understanding that we only consider affine open subschemes. We have

$$\mathrm{colim}_{V\subset X} V \simeq Y/R$$

with $Y = \coprod_{V \subset X} Y_V$ and $R = \coprod_{V,W \subset X} U_{V,W}$, where $Y_V = V$ and $U_{V,W} = V \cap W$, and with $s|_{U_{V,W}} : U_{V,W} \to Y_V$ and $t|_{U_{V,W}} : U_{V,W} \to Y_W$ given by the respective open immersions $V \cap W \to V$ and $V \cap W \to W$. Hence, we conclude from Theorem 7.1 that the colimit exists. Moreover, the open immersions $V \to X$ for $V \subset X$ define a unique map of schemes

$$\operatorname{colim}_{V \subset X} V \xrightarrow{f} X$$

and we claim that it is an isomorphism. First, the underlying map of topological spaces is a homeomorphism, since continuity is a local property. Second, the map of structure sheaves is an isomorphism, or equivalently, the diagram

$$\mathcal{O}_X \xrightarrow{\phi} p_*\mathcal{O}_Y \overset{p_*\sigma}{\underset{p_*\tau}{\rightrightarrows}} q_*\mathcal{O}_R,$$

is a limit diagram. Taking sections over $U \subset X$ affine open, we get

$$\mathcal{O}_X(U) \longrightarrow \prod_{V \subset X} \mathcal{O}_X(U \cap V) \rightrightarrows \prod_{V,W \subset X} \mathcal{O}_X(U \cap V \cap W)$$

which is a limit diagram of rings, because $\mathcal{O}_X$ is a sheaf. Finally, since the inclusion of sheaves in presheaves reflects limits, we conclude that the diagram of sheaves in question is a limit diagram.

In general, if $\mathcal{C}$ is a category and X and object of $\mathcal{C}$, then the slice category $\mathcal{C}_{/X}$ is the category, whose objects are the maps $f \colon Y \to X$ in $\mathcal{C}$ with target X, and whose maps are commutative triangular diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[dr, "g"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

with h a map from g to f. If $\mathcal{C} = \mathsf{Sch}$ is the category of schemes, then we say that the slice category $\mathsf{Sch}_{/X}$ is the category of schemes over X. We now construct the relative prime spectrum functor.

Proposition 7.3. *Let X be a scheme. There exists a functor*

$$\mathrm{CAlg}(\mathrm{QCoh}(X))^{\mathrm{op}} \xrightarrow{\mathrm{Spec}} \mathsf{Sch}_{/X}$$

that to a quasicoherent $\mathcal{O}_X$-algebra $\phi \colon \mathcal{O}_X \to \mathcal{A}$ assigns the map of schemes

$$\mathrm{Spec}(\mathcal{A}) \simeq \operatorname{colim}_{V \subset X} \mathrm{Spec}(\mathcal{A}(V)) \xrightarrow{f} X$$

exhibited during the course of the proof.

Proof. We first argue as in Example 7.2 that the colimit in the statement exists. The colimit is again indexed by the partially ordered set of affine open subschemes $V \subset X$, and we must show that if $V \subset W \subset X$ are affine open, then the map

$$\mathrm{Spec}(\mathcal{A}(V)) \longrightarrow \mathrm{Spec}(\mathcal{A}(W))$$

induced by the restriction is an open immersion. Now, it follows from the assumption that $\mathcal{A}$ is quasicoherent that the diagram of rings

$$\begin{tikzcd}
\mathcal{O}_X(W) \arrow[r] \arrow[d, "\phi_W"] & \mathcal{O}_X(V) \arrow[d, "\phi_V"] \\
\mathcal{A}(W) \arrow[r] & \mathcal{A}(V)
\end{tikzcd}$$

is cocartesian. It follows that the induced diagram of schemes

$$\begin{tikzcd}
\operatorname{Spec}(\mathcal{A}(V)) \arrow[r, "j'"] \arrow[d] & \operatorname{Spec}(\mathcal{A}(W)) \arrow[d] \\
\operatorname{Spec}(\mathcal{O}_X(V)) \arrow[r, "j"] & \operatorname{Spec}(\mathcal{O}_X(W))
\end{tikzcd}$$

is cartesian. Indeed, we saw in Lecture 4 that the prime spectrum functor

$$\mathrm{CAlg}(\mathsf{Ab})^{\mathrm{op}} \xrightarrow{\mathrm{Spec}} \mathsf{Sch}$$

is a right adjoint, and therefore, it preserves all limits that exist in its domain. The lower horizontal map is canonically isomorphic to $j\colon V \to W$, and hence, is an open immersion. But open immersions are preserved under base-change, so also the upper horizontal map j' is an open immersion, as we wanted to prove. Hence, we conclude from Theorem 7.1 that the colimit exists. The map $f\colon \operatorname{Spec}(\mathcal{A}) \to X$ is the unique map from the colimit, whose Vth component is the composition

$$\operatorname{Spec}(\mathcal{A}(V)) \longrightarrow \operatorname{Spec}(\mathcal{O}_X(V)) \simeq V \longrightarrow X$$

of the map induced by $\phi_V\colon \mathcal{O}_X(V) \to \mathcal{A}(V)$, the canonical isomorphism, and the open immersion of V in X. Finally, given a map of quasicoherent $\mathcal{O}_X$-algebras $\psi\colon \mathcal{A} \to \mathcal{B}$, there is a unique map $\operatorname{Spec}(\psi)$ that makes the diagram

$$\begin{tikzcd}
\operatorname{Spec}(\mathcal{B}(V)) \arrow[r, "\operatorname{Spec}(\psi_V)"] \arrow[d] & \operatorname{Spec}(\mathcal{A}(V)) \arrow[d] \\
\operatorname{Spec}(\mathcal{B}) \arrow[r, "\operatorname{Spec}(\psi)"] & \operatorname{Spec}(\mathcal{A})
\end{tikzcd}$$

commute for all $V \subset X$ affine open. The uniqueness statement implies that this defines a functor, as stated. □

Definition 7.4. A map of schemes $f\colon Y \to X$ is affine if it satisfies the following equivalent conditions:

(1) The map $f\colon Y \to X$ is in the essential image of the relative prime spectrum functor.
(2) For every affine open subscheme $V \subset X$, the open subscheme $f^{-1}(V) \subset Y$ is affine.
(3) There exists a covering $(V_i)_{i \in I}$ of X by affine open subschemes such that for all $i \in I$, the open subscheme $f^{-1}(V_i) \subset Y$ is affine.

Let us define $(\mathsf{Sch}_{/X})^{\mathrm{aff}} \subset \mathsf{Sch}_{/X}$ to be the full subcategory spanned by the affine maps $f\colon Y \to X$. The relative prime spectrum functor

$$\mathrm{CAlg}(\mathrm{QCoh}(X))^{\mathrm{op}} \xrightarrow{\mathrm{Spec}} (\mathsf{Sch}_{/X})^{\mathrm{aff}}$$

is an equivalence of categories. A quasi-inverse assigns to $f\colon Y \to X$ affine the quasicoherent $\mathcal{O}_X$-algebra $\phi\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$.

Example 7.5. Let Y be a scheme. The unique map of schemes $f\colon Y \to \mathrm{Spec}(\mathbb{Z})$ is an affine map if and only if the scheme Y is affine.

We next consider the special case of closed immersions.

Proposition 7.6. *Let $f\colon Y \to X$ be a map of scheme. The following are equivalent:*

(1) *The map $f\colon Y \to X$ is affine, and the map of sheaves $\phi\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.*
(2) *The underlying map of topological space $p\colon |Y| \to |X|$ is a closed embedding, and the map of sheaves $\phi\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.*
(3) *For every affine open subscheme $V \subset X$, the open subscheme $f^{-1}(V) \subset Y$ is affine and isomorphic to* $\mathrm{Spec}(R/I)$ *for some ideal $I \subset R \simeq \mathcal{O}_X(V)$.*
(4) *There exists a cover $(V_i)_{i\in I}$ of X by affine open subschemes such that for all $i \in I$, the open subscheme $f^{-1}(V_i) \subset Y$ is affine and isomorphic to* $\mathrm{Spec}(R_i/I_i)$ *for some ideal $I_i \subset R_i \simeq \mathcal{O}_X(V_i)$.*

Proof. It is clear that (3) implies (4).

If (4) holds, then $f\colon Y \to X$ is affine, since $f^{-1}(V_i) \subset Y$ is affine for all $i \in I$. Moreover, since $f^{-1}(V_i) \to V_i$ is isomorphic to $\mathrm{Spec}(R_i/I_i) \to \mathrm{Spec}(R_i)$, the map $\phi_{V_i}\colon \mathcal{O}_X(V_i) \to (f_*\mathcal{O}_Y)(V_i)$ is isomorphic to the canonical projection $R_i \to R_i/I_i$, and hence, is surjective. It follows that the map of sheaves $\phi\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective. In fact, as a map of presheaves, $\phi\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is locally surjective.

If (1) holds, then the map $f\colon Y \to X$ is recovered as the relative prime spectrum of $\phi\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$. The latter is a surjective map of quasicoherent $\mathcal{O}_X$-modules, so for $V \subset X$ affine open, the map $\phi_V\colon \mathcal{O}_X(V) \to (f_*\mathcal{O}_Y)(V) \simeq \mathcal{O}_Y(f^{-1}(V))$ is surjective. So $p\colon |Y| \to |X|$ is locally on $|X|$ a closed embedding, and therefore, it is globally a closed embedding.

The final implication that (2) implies (3) is more tricky. We skip the proof. □

Definition 7.7. A map of schemes $f\colon Y \to X$ is a closed immersion if it satisfies the equivalent conditions of Proposition 7.6.

Corollary 7.8. *Let $f\colon Y \to X$ and $f'\colon Y' \to X$ be closed immersions. The following are equivalent:*

(1) *There exists a isomorphism $h\colon Y \to Y'$ of schemes over X. If so, then h is unique.*
(2) *The closed subsets $p(|Y|) \subset |X|$ and $p'(|Y'|) \subset |X|$ are equal, and the maps $\phi\colon \mathcal{O}_X \to p_*\mathcal{O}_X$ and $\phi'\colon \mathcal{O}_X \to p'_*\mathcal{O}_X$ have the same kernel.*

Proof. Clear from Proposition 7.6. □

Definition 7.9. A closed subscheme of a scheme X is an isomorphism class of closed immersions $f\colon Y \to X$.

It follows form Corollary 7.8 that there are mutually inverse maps

$$\{\text{closed subschemes of } X\} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \{\text{quasicoherent ideals } \mathcal{I} \subset \mathcal{O}_X\},$$

where α takes a closed immersion $f\colon Y \to X$ to the quasicoherent ideal given by the kernel of $\phi\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$, and where β take a quasicoherent ideal $\mathcal{I} \subset \mathcal{O}_Y$ to the closed immersion $f\colon \operatorname{Spec}(\mathcal{O}_X/\mathcal{I}) \to \operatorname{Spec}(\mathcal{O}_X) \simeq X$.

If X is a scheme, and $Z \subset |X|$ a closed subset, then there are in general many different closed subschemes $f\colon Y \to X$ with $p(|Y|) = Z \subset |X|$. For example, we have $|\operatorname{Spec}(R/I)| = |\operatorname{Spec}(R/\sqrt{I})|$ for any ideal $I \subset R$.

However, there is always a minimal choice. The nilradical of a ring R is the ideal

$$\sqrt{(0)} \subset R$$

which consists of the elements $f \in R$ such that $f^N = 0$ for some $N \geq 0$. The ring R is said to be reduced if its nilradical is equal to $\{0\}$.

Definition 7.10. A scheme X is reduced if for every open subscheme $U \subset X$, the ring $\mathcal{O}_X(U)$ is reduced, or equivalently, if for every affine open subscheme $V \subset X$, the ring $\mathcal{O}_X(V)$ is reduced.

Example 7.11. A ring R is reduced if and only if its prime spectrum X is reduced. If X is reduced, then so is $R \simeq \mathcal{O}_X(X)$. Conversely, if R is reduced, then so is every localization $S^{-1}R$ of R, so X is reduced.

So for every scheme X, there exists a canonical closed immersion

$$X_{\text{red}} \longrightarrow X$$

corresponding to the quasicoherent ideal $\mathcal{N} \subset \mathcal{O}_X$ such that $\mathcal{N}(U) \subset \mathcal{O}_X(U)$ is the nilradical for every affine open subscheme $U \subset X$. We have $|X_{\text{red}}| = |X|$ and

$$\mathcal{O}_{X_{\text{red}}}(U) \simeq \mathcal{O}_X(U)/\mathcal{N}(U).$$

If $|Z| \subset |X|$ is a closed subset, then there exists a unique reduced closed subscheme structure on $|Z|$. We write $Z_{\text{red}} \subset X$ for this reduced subscheme.

Example 7.12. Let $X = \operatorname{Spec}(\mathbb{Z})$, and let $|Z| \subset |X|$ be the closed subset consisting of the single closed point z corresponding to the maximal ideal $p\mathbb{Z} \subset \mathbb{Z}$. The map

$$Z^{(n)} = \operatorname{Spec}(\mathbb{Z}/p^{n+1}\mathbb{Z}) \xrightarrow{i^{(n)}} X = \operatorname{Spec}(\mathbb{Z})$$

induced by the canonical projection is a closed immersion for all $n \geq 0$. It is reduced if and only if $n = 0$. Grothendieck tells us to think of the closed immersions

$$Z_{\text{red}} = Z^{(0)} \longrightarrow Z^{(1)} \longrightarrow \cdots \longrightarrow Z^{(n)} \longrightarrow \cdots \longrightarrow X$$

as increasing "infinitesimal neighborhoods" of the closed point $Z^{(0)} = \operatorname{Spec}(\mathbb{F}_p)$. These schemes all have the same underlying topological space $|Z| \subset |X|$, but their structure sheaves vary with $n \geq 0$.

Finally, we will discuss vector bundles on schemes. It is possible to mimic the geometric definition of a vector bundle, but this requires more technology than we currently have available.[4] So we are going to give an algebraic definition, which, in the end, turns out to be equivalent to the geometric definition.

Definition 7.13. Let X be a scheme. A quasicoherent $\mathcal{O}_X$-module $\mathcal{E}$ is a vector bundle if for every $x \in |X|$, there exists an open neighborhood $x \in U \subset |X|$ and an isomorphism of quasicoherent $\mathcal{O}_X|_U$-modules $\varphi\colon \mathcal{E}|_U \to (\mathcal{O}_X|_U)^{\oplus d}$ for some $d \geq 0$.

We also say, more precisely, that a quasicoherent $\mathcal{O}_X$-module $\mathcal{E}$ that satisfies the condition in Definition 7.13 is a locally free $\mathcal{O}_X$-module of finite rank. Note that it's not necessary to require in advance that $\mathcal{E}$ is quasicoherent, as the defining condition for being a vector bundle clearly implies quasicoherence by definition.

A vector bundle $\mathcal{E}$ on a scheme X gives rise to the family $(\mathcal{E}(x))_{x\in|X|}$ of finite dimensional vector spaces, where

$$\mathcal{E}(x) \simeq \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} k(x).$$

It is a vector space over the residue field $k(x)$ at $x \in |X|$, and its dimension

$$\dim_{k(x)} \mathcal{E}(x) = d$$

is the rank of the locally free $\mathcal{O}_X$-module $\mathcal{E}$ in some open neighborhood of $x \in |X|$. Moreover, these vector spaces vary nicely with $x \in |X|$. For instance:

Lemma 7.14. *If $\mathcal{E}$ is a vector bundle on a scheme X, then the function*

$$|X| \xrightarrow{\quad d \quad} \mathbb{Z}_{\geq 0}$$

defined by $d(x) = \dim_{k(x)} \mathcal{E}(x)$ is continuous for the discrete topology on $\mathbb{Z}_{\geq 0}$.

Proof. To say that d is continuous for the discrete topology on $\mathbb{Z}_{\geq 0}$ is equivalent to saying that d is locally constant. But given $x \in |X|$, there exists by definition an open subset $x \in U \subset |X|$ and an isomorphism of quasicoherent $\mathcal{O}_X|_U$-modules $\varphi\colon \mathcal{E}_U \to (\mathcal{O}_X|_U)^{\oplus d}$, and therefore, we have $d(y) = d$ for all $y \in U$. ☐

Corollary 7.15. *If $\mathcal{E}$ is a vector bundle on a scheme X such that the underlying topological space $|X|$ is connected, then $d\colon |X| \to \mathbb{Z}_{\geq 0}$ is constant.*

Proof. This is clear from the definition of a connected topological space. ☐

Example 7.16. An irreducible topological space is connected.

A vector bundle $\mathcal{E}$ on a scheme X is, in particular, a quasicoherent $\mathcal{O}_X$-module, and therefore, its restriction $\mathcal{E}|_U$ to an affine open subset $U \subset |X|$ determines and is determined by the $\mathcal{O}_X(U)$-module $\mathcal{E}(U)$. This begs the question as to which R-modules correspond to vector bundles on $\operatorname{Spec}(R)$.

Theorem 7.17. *Let R be a ring with Zariski spectrum X, and let M be an R-module. The following are equivalent:*

(1) *The quasicoherent $\mathcal{O}_X$-module $\mathcal{E} = \widetilde{M}$ is a vector bundle on X.*

[4] The problem is that maps between vector bundles should be fiberwise linear. To encode this, we need to have a category of schemes over X with $\mathbb{G}_m$-action.

(2) *There exists a finite family $(f_1, \dots, f_n)$ of elements of R that generates the unit ideal such that for every $1 \leq i \leq n$, the R_{f_i}-module M_{f_i} is free of finite rank.*
(3) *The R-module M is finitely presented and flat.*
(4) *The R-module M is finitely generated and projective.*
(5) *The R-module M is a direct summand of a free R-module of finite rank.*

Proof. We first show that (1) implies (2). Since $\mathcal{E}$ is a vector bundle, given $x \in |X|$, we can find $x \in U_x \subset |X|$ open and an isomorphism $\varphi_x \colon \mathcal{E}|_{U_x} \to (\mathcal{O}_X|_{U_x})^{\oplus d_x}$ for some $d_x \geq 0$, and we can assume that $U_x = |X_{f_x}| \subset |X|$ is a distinguished open subset. The family $(U_x)_{x \in |X|}$ covers $|X|$, and since $|X|$ is compact, some finite subfamily $(U_{x_1}, \dots, U_{x_n})$ covers $|X|$, or equivalently, the finite family $(f_{x_1}, \dots, f_{x_n})$ generates the unit ideal. Finally, the isomorphisms φ_{x_i} induces isomorphisms

$$M_{f_{x_i}} \simeq \mathcal{E}(U_{f_{x_i}}) \longrightarrow (\mathcal{O}_X|_{U_{x_i}})^{\oplus d_{x_i}}(U_{x_i}) \simeq (R_{f_{x_i}})^{\oplus d_{x_i}}$$

of R_{f_i}-modules, so (2) follows.

Conversely, let us show that (2) implies (1). By assumption, there exists an isomorphism of R_{f_i}-modules $\varphi_i \colon M_{f_i} \to (R_{f_i})^{\oplus d_i}$ for all $1 \leq i \leq n$. But $\mathcal{E}$ is a quasicoherent $\mathcal{O}_X$-module and $U_i = |X_{f_i}| \subset |X|$ is affine open, so this isomorphism determines an isomorphism of quasicoherent $\mathcal{O}_X|_{U_i}$-modules $\mathcal{E}|_{U_i} \to (\mathcal{O}_X|_{U_i})^{\oplus d_i}$. Finally, since $(f_1, \dots, f_n)$ generates the unit ideal, the family $(U_1, \dots, U_n)$ covers $|X|$, so (1) follows.

We have now proved that the local definitions (1) and (2) are equivalent, and we proceed to show that the commutative algebra definitions (3)–(5) are equivalent.

We first show that (5) implies both (3) and (4). If M is a summand of $R^{\oplus n}$, then M is a quotient of $R^{\oplus n}$, and hence, finitely generated. But the kernel of projection $R^{\oplus n} \to M$ is the complementary summand, so it is also a quotient of $R^{\oplus n}$, and hence, finitely generated. This shows that M is finitely presented. Finally, as R is flat and projective, so is $R^{\oplus n}$, and hence, so is any summand of $R^{\oplus n}$. This proves that (3) and (4) hold.

Suppose next that (4) holds. By the assumption that M is finitely generated, we can choose a surjective map of R-modules $p \colon R^{\oplus n} \to M$, and by the assumption that M is projective, this map admits an R-linear section $s \colon M \to R^{\oplus n}$. This shows that (5) holds. So (4) and (5) are equivalent and imply (3). To prove that (3) implies (5), we will use the following results:

Lemma 7.18 (Lazard)**.** *Let R be a ring. An R-module M is flat if and only if it is the colimit of a filtered diagram of finitely generated free R-modules.*

Proof. We give a proof in the appendix. □

Lemma 7.19 (Grothendieck)**.** *Let R be a ring. An R-module M is finitely presented if and only if the functor* $\operatorname{Hom}_R(M, -) \colon \operatorname{Mod}_R \to \operatorname{Mod}_R$ *preserves filtered colimits.*

Proof. A finite presentation of M identifies $\operatorname{Hom}_R(M, N)$ with a set of finitely many elements of N satisfying finitely many equations. So the description of a filtered colimit of sets (and hence, of R-modules) that we gave on problem set 1 shows that such colimits are preserved by the functor $\operatorname{Hom}_R(M, -)$.

Conversely, we first observe that every R-module M can be written as a filtered colimit of finitely presented R-modules. Indeed, we may write

$$\begin{aligned} M &\simeq \operatorname{coker}(\ \textstyle\bigoplus_{j\in J} R \xrightarrow{h} \textstyle\bigoplus_{i\in I} R\ ) \\ &\simeq \operatorname{colim}_{(I_0,J_0)} \operatorname{coker}(\ \textstyle\bigoplus_{j\in J_0} R \xrightarrow{h'} \textstyle\bigoplus_{i\in I_0} R\ ), \end{aligned}$$

where the colimit is indexed by the filtered partially ordered set of pairs (I_0, J_0) of finite subsets $I_0 \subset I$ and $J_0 \subset J$ with the property that h restricts to a map

$$\begin{tikzcd} \bigoplus_{j\in J_0} R \arrow[r, "h'"] \arrow[d] & \bigoplus_{i\in I_0} R \arrow[d] \\ \bigoplus_{j\in J} R \arrow[r, "h"] & \bigoplus_{i\in I} R. \end{tikzcd}$$

So we write $M \simeq \operatorname{colim}_{k\in K} M_k$ as a filtered colimit of finitely presented R-modules and use the assumption that $\operatorname{Hom}_R(M, -)$ preserves filtered colimits to conclude that the canonical map

$$\operatorname{colim}_{k\in K} \operatorname{Hom}_R(M, M_k) \longrightarrow \operatorname{Hom}_R(M, \operatorname{colim}_{k\in K} M_k) \longrightarrow \operatorname{Hom}_R(M, M)$$

is an isomorphism. In particular, the identity map of M is in the image. So by the description of filtered colimits for problem set 1, we conclude that there exists a map $r\colon M \to M_k$ such that composition

$$M \xrightarrow{r} M_k \xrightarrow{i} M$$

is equal to the identity map of M. So M is a summand of the finitely presented R-module M_k, and hence, is itself finitely presented. $\square$

Now we use the two lemmas to see that (3) implies (5). Since M is flat, Lazard's lemma shows that we can write M as a filtered colimit

$$M \simeq \operatorname{colim}_{k\in K} R^{\oplus d_k}$$

of finitely generated free R-modules. And since M is finitely presented, we conclude from Grothendieck's lemma that $\operatorname{Hom}_R(M, -)$ preserves filtered colimits, so

$$\operatorname{colim}_{k\in K} \operatorname{Hom}_R(M, R^{\oplus d_k}) \longrightarrow \operatorname{Hom}_R(M, \operatorname{colim}_{k\in K} R^{\oplus d_k}) \longrightarrow \operatorname{Hom}_R(M, M)$$

is an isomorphism. In particular, the identity map of M is in the image, so we conclude as above that M is a summand of $R^{\oplus d_k}$ for some $k \in K$, as desired.

We have proved that (1)–(2) are equivalent, and that (3)–(5) are equivalent, so let us now prove that (4) implies (2). Since M is finitely generated and projective R-module, it follows that also M_x is a finitely generated and projective R_x-module for all $x \in |X|$. But Nakayama's lemma shows that a finitely generated projective module over a local ring is finitely generated free. Therefore, for every $x \in |X|$, we can choose an isomorphism of R_x-modules

$$M_x \xrightarrow{\varphi_x} (R_x)^{\oplus d_x}$$

for some $d_x \geq 0$. Moreover, using that both M and $R^{\oplus d_x}$ are finitely presented, we can lift φ_x to an isomorphism of R_f-modules

$$M_f \xrightarrow{\varphi_f} (R_f)^{\oplus d_x}$$

for some $f \in R$ with $x \in |X_f| \subset |X|$. Indeed, an isomorphism between finitely presented modules if *finitary*, so if it appears in a filtered colimit, then it must appear at some finite stage. Concretely, we can take $f \in R$ to be the product all denominators which appear in $\varphi_x \colon M_x \to (R_x)^{\oplus d_x}$. This proves (2).

Finally, we prove that (2) implies (3). This follows from the following lemma.

Lemma 7.20. *Let R be a ring, and let $(f_i)_{i \in I}$ be a family of elements, which generate the unit ideal.*

(1) *If M_{f_i} is a flat R_{f_i}-module for all $i \in I$, then M is a flat R-module.*
(2) *If M_{f_i} is a finitely presented R_{f_i}-module for all $i \in I$, then M is a finitely presented R-module.*

Proof. To prove (1), we let $N' \to N$ be an injective map of R-modules and must show that the induced map $M \otimes N' \to M \otimes N$ is injective, or equivalently, that its kernel is zero. But begin zero is a local property, so it suffices to show that $(M \otimes N')_{f_i} \to (M \otimes N)_{f_i}$ is injective for all $i \in I$. This, in turn, is equivalent to showing that $M_{f_i} \otimes N'_{f_i} \to M_{f_i} \otimes N_{f_i}$ is injective for all $i \in I$, which follows from the assumption that the M_{f_i} is a flat R_{f_i}-module for all $i \in I$.

To prove (2), we use that, by the equivalent of categories

$$\mathrm{Mod}_R \xrightarrow{\widetilde{(-)}} \mathrm{QCoh}(X)$$

and the sheaf property for $\widetilde{(-)}$, the diagram

$$\mathrm{Hom}_R(M, N) \longrightarrow \textstyle\prod_{i \in I} \mathrm{Hom}_R(M_{f_i}, N_{f_i}) \rightrightarrows \textstyle\prod_{(i,j) \in I \times I} \mathrm{Hom}_R(M_{f_i f_j}, N_{f_i f_j})$$

is a limit diagram for all $M, N \in \mathrm{Mod}_R$. Now, the assumption that M_{f_i} is a finitely presented R_{f_i}-module for all $i \in I$ implies that also $M_{f_i f_j}$ is a finitely presented $R_{f_i f_j}$-module for all $(i,j) \in I \times I$. Moreover, both I and $I \times I$ are finite. So by Lemma 7.19 and by Grothendieck's theorem that, in the category of sets, finite limits and filtered colimits commute, we conclude that the functor $\mathrm{Hom}_R(M, -)$ preserves filtered colimits. So by Lemma 7.19, the R-module M is a finitely presented. □

This completes the proof of the theorem. □

Finally, we give a proof of Lazard's lemma, which shows that three different characterization of flatness are equivalent. The second of these characterizations is known as the "equational criterion for flatness."

Lemma 7.21 (Lazard)**.** *Let R be a ring, and let M be an R-module. The following are equivalent.*

(1) *The R-module M is flat.*
(2) *Given maps of R-modules*

$$F' \xrightarrow{a} F \xrightarrow{x} M$$

with F and F' finitely generated free and $xa = 0$, there exists a factorization

$$\begin{tikzcd} F \arrow[rr, "b"] \arrow[dr, "x"'] & & F'' \arrow[dl, "y"] \\ & M & \end{tikzcd}$$

with F'' finitely generated free and $ba = 0$.

(3) *The R-module M is a filtered colimit of finitely generated free R-modules.*

Proof. We first assume (1) and prove (2). We have a diagram of R-modules

$$\begin{tikzcd} F' \arrow[r, "a"] & F \arrow[r, "p"] \arrow[d, "x"] & C \arrow[r] \arrow[dl, "\bar{x}"] & 0 \\ & M & & \end{tikzcd}$$

with the top row exact. Moreover, since F and F' are dualizable, and since M is flat, this diagram determines and is determined by the diagram

$$\begin{tikzcd} M \otimes F'^{\vee} & M \otimes F^{\vee} \arrow[l, "M\otimes a^{\vee}"'] & M \otimes C^{\vee} \arrow[l, "M\otimes p^{\vee}"'] & 0. \arrow[l] \\ & R \arrow[u, "\tilde{x}"] \arrow[ur, "\tilde{\bar{x}}"'] & & \end{tikzcd}$$

The map $\tilde{\bar{x}}$ determines and is determined by the element $\tilde{\bar{x}}(1) \in M \otimes C^{\vee}$. This element, in turn, can be written, non-canonically, as a finite sum pure tensors, and therefore, the map $\tilde{\bar{x}}$ admits a factorization

$$\begin{tikzcd} M \otimes F'^{\vee} & M \otimes F^{\vee} \arrow[l, "M\otimes a^{\vee}"'] & M \otimes C^{\vee} \arrow[l, "M\otimes p^{\vee}"'] & 0 \arrow[l] \\ & A \arrow[u, "\tilde{x}"] \arrow[r, "\tilde{y}"] & M \otimes F''^{\vee} \arrow[u, "M\otimes c"'] & \end{tikzcd}$$

with F'', and hence, its dual finitely generated free. So if $b\colon F \to F''$ the unique map such that $b^{\vee} = p^{\vee}c$, then $x = yb$ with $ba = 0$, which proves (2).

We next prove that (2) implies (3). Let $\mathrm{Mod}_R^{\mathrm{ff}} \subset \mathrm{Mod}_R$ be the full subcategory of the category R-modules spanned by the finitely generated free R-modules. Given any R-module M, the diagram

$$((\mathrm{Mod}_R^{\mathrm{ff}})_{/M})^{\triangleright} \xrightarrow{\bar{p}} \mathrm{Mod}_R$$

that to $(F, x\colon F \to M)$ assigns F and to the cone point assigns M is a colimit diagram. But the category $(\mathrm{Mod}_R^{\mathrm{ff}})_{/M}$ is additive, so (2) is exactly the statement that it is filtered. Hence, (2) implies (3), and it is clear that (3) implies (1). $\square$

Last time, we associated to a scheme $X = (|X|, \mathcal{O}_X)$ the full subcategory

$$\mathrm{Vect}(X) \subset \mathrm{QCoh}(X)$$

of vector bundles on X. A quasicoherent $\mathcal{O}_X$-module $\mathcal{E}$ is a vector bundle if, locally on X, there exists an isomorphism $\varphi\colon \mathcal{E} \to \mathcal{O}_X^{\oplus d}$. We say that a vector bundle $\mathcal{E}$ is trivial (of rank d) if such an isomorphism exists globally.

We proved that for $X \simeq \mathrm{Spec}(R)$, the quasicoherent $\mathcal{O}_X$-module $\widetilde{M}$ associated with an R-module M is a vector bundle if and only if M satisfies the following equivalent conditions:

(1) The R-module M is finitely presented and flat.
(2) The R-module M is finitely generated and projective.
(3) The R-module M is a direct summand of a free R-module of finite rank.

Let us give some examples of non-trivial vector bundles on affine schemes.

The first example is rather strange from the perspective of scheme theory, but still: Let X be a compact Hausdorff topological space, and let

$$R = C(X, \mathbb{R})$$

be the ring of continuous real valued functions on X. A real vector bundle on X is the data of a continuous map $p\colon V \to X$ and a structure of real vector spaces on the fiber $V_x = p^{-1}(x) \subset V$ for every $x \in X$ (so in particular the fiber is non-empty), and this data much satisfy the condition that there exists a covering $(U_i)_{i \in I}$ of X by open subsets and a family $(\varphi_i)_{i \in I}$ of homeomorphisms

$$p^{-1}(U_i) \xrightarrow{\varphi_i} U_i \times \mathbb{R}^{d_i}$$

with the property that (1) the diagram

$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "\varphi_i"] \arrow[d, "p"] & U_i \times \mathbb{R}^{d_i} \arrow[d, "\mathrm{pr}_1"] \\ U_i \arrow[r, equal] & U_i \end{tikzcd}$$

commutes, and (2) for every $x \in U_i$, the induced map of fibers

$$V_x = p^{-1}(x) \xrightarrow{\varphi_{i,x}} \mathrm{pr}_1^{-1}(x) = \mathbb{R}^{d_i}$$

is $\mathbb{R}$-linear. It is common to abuse language and say that $p\colon V \to X$ is a real vector bundle, although this ignores the structure of real vector space on the fibers of this map. A map of real vector bundles over X from $q\colon W \to X$ to $p\colon V \to X$ is a continuous map $h\colon W \to V$ with the property that (1) the diagram

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "q"] & V \arrow[d, "p"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commutes, and (2) for every $x \in X$, the induced map of fibers

$$W_x = q^{-1}(x) \xrightarrow{h_x} V_x = p^{-1}(x)$$

is $\mathbb{R}$-linear. This defines the category $\mathrm{Vect}(X)$ of real vector bundles on X.

If $p\colon V \to X$ is a real vector bundle, then the set of sections

$$\Gamma(X,V) = \{s\colon X \to V \mid s \text{ is continuous and } p \circ s = \mathrm{id}_X\}$$

is given an R-module structure with addition and scalar multiplication defined as follows. If $s,t \in \Gamma(X,V)$ and $\varphi \in R$, then

$$\begin{aligned}(s+t)(x) &= s(x)+t(x)\\ (\varphi \cdot s)(x) &= \varphi(x)\cdot s(x),\end{aligned}$$

where the sum and scalar multiplication on the right-hand side uses the real vector structure on V_x. If $h\colon W \to V$ is a map of real vector bundles, then the map

$$\Gamma(X,W) \xrightarrow{\Gamma(X,h)} \Gamma(X,V)$$

that to $t\colon X \to W$ assigns $s = h \circ t\colon X \to V$ is R-linear. This defines a functor

$$\mathrm{Vect}(X) \xrightarrow{\Gamma(X,-)} \mathrm{Mod}_R,$$

and by using a partition of unity, one shows that its essential image is contained in the full subcategory of vector bundles on $\mathrm{Spec}(R)$. In fact:

Theorem 8.1 (Serre–Swan)**.** *If X is a compact Hausdorff spaces, then the functor*

$$\mathrm{Vect}(X) \xrightarrow{\widetilde{\Gamma(X,-)}} \mathrm{Vect}(\mathrm{Spec}(C(X,\mathbb{R})))$$

is an equivalence of categories.

Proof. This is proved by showing that, since X is compact Hausdorff, every real vector bundle on X is a summand of a trivial vector bundle $\mathrm{pr}_1\colon X \times \mathbb{R}^d \to X$. □

Example 8.2. The Möbius band is a non-trivial real vector bundle of rank 1 on the circle S^1, so by Theorem 8.1, we get a non-trivial vector bundle of rank 1 on the scheme $\mathrm{Spec}(C(S^1,\mathbb{R}))$.

Remark 8.3. Let X be a compact Hausdorff space, and let $R = C(X,\mathbb{R})$ be the ring of continuous real valued functions on X. There is a map

$$X \longrightarrow |\,\mathrm{Spec}(R)|$$

that to $x \in X$ assigns the closed point in the Zariski space given by the maximal ideal $\mathfrak{m} \subset R$ consisting of the continuous functions $\varphi\colon X \to \mathbb{R}$ such that $\varphi(x) = 0$. This map is continuous and is an embedding onto the subspace of the Zariski space consisting of the closed points. There are many non-closed points in the Zariski space, however, so this map is very far from being a bijection, let alone a homeomorphism. The structure of the non-closed points is extremely weird and depends on our model of set theory, even for $X = S^1$.

The second example is much better: Dedekind domains.

Definition 8.4. A ring R is a Dedekind domain, if it is a noetherian integral domain and if the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ of the local ring at every closed point $x \in |X|$ of its Zariski space is a nonzero principal ideal.

By definition, if R is a Dedekind domain, then the local ring $\mathcal{O}_{X,x}$ at a closed point $x \in |X|$ of its Zariski space is a noetherian local ring, whose maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is principal. This implies that every nonzero ideal $I \subset \mathcal{O}_{X,x}$ is of the form $I = \mathfrak{m}_x^n$ for some $n \geq 0$; see [5, Theorem 11.2] for a proof. So if $f \in \mathcal{O}_{X,x}$ is a nonzero element, then there is a unique $n \geq 0$ such that $f \in \mathfrak{m}_x^n$ and $f \notin \mathfrak{m}_x^{n+1}$. We say that this non-negative integer n is the order of vanishing of f at x and write

$$\mathrm{ord}_x(f) = n.$$

So in particular, a nonzero element $f \in \mathcal{O}_{X,x}$ is a unit if and only if $\mathrm{ord}_x(f) = 0$.

Let X be any scheme, and let $x \in |X|$ be a closed point. The problem set this week will construct a natural map of schemes

$$\mathrm{Spec}(\mathcal{O}_{X,x}) \xrightarrow{f_{X,x}} X$$

and show that the image of the map of underlying topological spaces precisely consists of the elements $\eta \in |X|$ such that x is contained in the closure of $\{\eta\} \subset |X|$. (This is the set of points $\eta \in |X|$ that specialize to $x \in |X|$.) If $x \in U \subset |X|$ for some affine open subset, then the map $f_{X,x}$ is equal to the composition

$$\mathrm{Spec}(\mathcal{O}_{X,x}) \longrightarrow \mathrm{Spec}(\mathcal{O}_X(U)) \longrightarrow X$$

of the map induced by the localization $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ and the canonical inclusion. In particular, the image of $f_{X,x}$ is contained in an affine open subset of X.

Proposition 8.5. *If R is a Dedekind domain, then every prime ideal $\mathfrak{p} \subset R$ is either zero or maximal.*

Proof. Let $|X|$ be the Zariski space of R. The zero ideal $\{0\} \subset R$ is a prime ideal, since R is an integral domain. The corresponding point $\eta \in |X|$ is a generic point. The prime ideals of the local ring $\mathcal{O}_{X,x}$ at a closed point $x \in |X|$ are $\{0\}$ and $\mathfrak{m}_x$. Now, let $\mathfrak{p} \subset R$ be a prime ideal, and let $\mathfrak{p} \subset \mathfrak{m} \subset R$ be a maximal ideal that contains it. Hence, if $x, y \in |X|$ are the points corresponding to $\mathfrak{m}, \mathfrak{p} \subset R$, then x is a specialization of y. Therefore, the point y is in the image of the map

$$\mathrm{Spec}(\mathcal{O}_{X,x}) \xrightarrow{f_{X,x}} X$$

from this week's problem set. It follows that either $y = x$ or $y = \eta$, or equivalently, either $\mathfrak{p} = \mathfrak{m}$ or $\mathfrak{p} = \{0\}$, as we wanted to show. $\square$

So if R is a Dedekind domain, then its Krull dimension is equal to 1. In general, if R is a ring and M an R-module, then we say that M is torsion free if for $f \in R$ and $y \in M$, $fy = 0$ implies that $f = 0$ or $y = 0$.

Lemma 8.6. *If R is a Dedekind domain and M an R-module, then M is flat if and only if M is torsion free.*

Proof. We abuse notation and write $f\colon R \to R$ for the map given by multiplication by $f \in R$. If $f \in R$ is nonzero, then $f\colon R \to R$ is injective, because R is an integral domain. Hence, if M is flat, then also $f\colon M \to M$ is injective, which shows that M is torsion free.

It remains to prove that if M is torsion free, then M is flat. We may assume that M is finitely generated, since, we can write a general torsion free R-module as the filtered colimit of its finitely generated (and torsion free) submodules. To prove that M is flat R-module, it suffices to prove that its localization M_x is a flat $\mathcal{O}_{X,x}$-module for all $x \in |X| = |\operatorname{Spec}(R)|$. Since R is a Dedekind domain, the only point in $|X|$, which is not closed, is the generic point η, and $\mathcal{O}_{X,\eta}$ is a field, so flatness at η is automatic. So it suffices to show that M_x is a flat $\mathcal{O}_{X,x}$-module for every closed point $x \in |X|$. Since we assume that M is a finite generated R-module, it follows that M_x is a finitely generated $\mathcal{O}_{X,x}$-module. But $\mathcal{O}_{X,x}$ is a principal ideal domain, so to prove that M_x is flat, it suffices to show that

$$\operatorname{Tor}_i^{\mathcal{O}_{X,x}}(M_x, k(x)) = 0$$

for all $i > 0$. For this purpose, we may use the resolution

$$0 \longrightarrow \mathcal{O}_{X,x} \xrightarrow{f} \mathcal{O}_{X,x} \longrightarrow k(x) \longrightarrow 0$$

of the $\mathcal{O}_{X,x}$-module $k(x)$ by free $\mathcal{O}_{X,x}$-modules. So we see that the assumption that M is torsion free implies that the Tor-groups in question vanish. So M is flat. $\square$

We conclude that if R is a Dedekind domain, then every finitely generated torsion free R-module determines a vector bundle on $X = \operatorname{Spec}(R)$. Indeed, since R is noetherian, every finitely generated R-module is finitely presented.

Example 8.7. If R is a Dedekind domain, then every nonzero ideal $I \subset R$ determines a vector bundle of rank 1 on $X = \operatorname{Spec}(R)$. Indeed, it is finitely generated, since R is noetherian, and it is torsion free, since $I \subset R$ and R is torsion free. Moreover, if F is the quotient field of R, then the inclusion $I \to R$ induces an isomorphism

$$I \otimes_R F \longrightarrow R \otimes_R F.$$

For the map is injective, because F is a flat R-module, and it is surjective, because if $f \in I$ is any nonzero element, then $f \otimes 1$ is mapped to a nonzero element in the 1-dimensional F-vector space in the target. It follows that

$$d(\eta) = \dim_{k(\eta)} \widetilde{I}(\eta) = 1,$$

so $\widetilde{I}$ has rank 1 at $\eta \in |X|$. But then $\widetilde{I}$ has rank 1 at every $x \in |X|$, since $|X|$ is irreducible, and hence, connected.

We conclude that every non-principal ideal $I \subset R$ gives rise to a non-trivial vector bundle of rank 1 on $X = \operatorname{Spec}(R)$.

Example 8.8. Let $\mathbb{Q} \to F$ be a finite field extension. The integral closure $\mathcal{O}_F \subset F$ of $\mathbb{Z} \subset \mathbb{Q}$ is a Dedekind domain. If $F = \mathbb{Q}(\sqrt{-5})$, then $\mathcal{O}_F = \mathbb{Z}[\sqrt{-5}]$, and in this case, the ideal $I = (3, 1 + \sqrt{-5}) \subset \mathcal{O}_F$ is not a principal ideal.

Example 8.9. Let k be a field, and let $f \colon X \to \operatorname{Spec}(k)$ be a smooth affine curve. In this case, the coordinate ring $\mathcal{O}_X(|X|)$ is a Dedekind domain. Moreover, if k is algebraically closed, then, by Hilbert's Nullstellensatz, the closed points of X are in one-to-one correspondence with the k-valued points of $f \colon X \to \operatorname{Spec}(k)$, that is, the set $X(k)$ of sections $x \colon \operatorname{Spec}(k) \to X$ of $f \colon X \to \operatorname{Spec}(k)$. The maximal ideal $\mathfrak{m} \subset \mathcal{O}_X(|X|)$ corresponding to $x \in X(k)$ is very rarely principal.

The Dedekind domain examples are all line bundles:

Definition 8.10. Let X be a scheme. A vector bundle $\mathcal{L}$ on X is a line bundle if its rank is constant equal to 1.

Equivalently, a line bundle on a scheme X is a quasicoherent $\mathcal{O}_X$-module $\mathcal{L}$ that, locally on X, is isomorphic to $\mathcal{O}_X$.

If $\mathcal{L}$ is a line bundle on a scheme X, then how can we understand that it is not possible to find a global isomorphism $\varphi\colon \mathcal{L} \to \mathcal{O}$? The answer is that we can do so by "descent" which allows us build quasicoherent $\mathcal{O}_X$-modules from local data.

We fix a scheme X and a cover $(U_i)_{i\in I}$ of $|X|$ by open subsets. The following statement is true, but useless: A quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ determines and is determined by a family $(\mathcal{M}_i)_{i\in I}$ of quasicoherent $\mathcal{O}_X|_{U_i}$-modules such that

$$\mathcal{M}_i|_{U_i\cap U_j} = \mathcal{M}_j|_{U_i\cap U_j}$$

for all $(i,j) \in I \times I$. (To go backwards, note that the set of open subsets $U \subset |X|$ such that $U \subset U_i$ for some $i \in I$ is a basis for the topology on $|X|$.) Why is this statement useless? Because it is not invariant under isomorphism: If we replace the $\mathcal{M}_i$ by isomorphic quasicoherent $\mathcal{O}_X|_{U_i}$-modules $\mathcal{M}'_i$, then the family $(\mathcal{M}'_i)_{i\in I}$ will typically no longer have the property that

$$\mathcal{M}'_i|_{U_i\cap U_j} = \mathcal{M}'_J|_{U_i\cap U_j}$$

for all $(i,j) \in I \times I$. This we cannot tolerate. For one thing, isomorphisms show up all the time, and for another, the very definition of a vector bundle involves such local isomorphisms, so to study them, we need to have this flexibility.

The solution to this problem is to relax the above requirement to the requirement that there an exist isomorphism of quasicoherent $\mathcal{O}_X|_{U_i\cap U_j}$-modules

$$\mathcal{M}_j|_{U_i\cap U_j} \xrightarrow{\alpha_{i,j}} \mathcal{M}_i|_{U_i\cap U_j},$$

for all $(i,j) \in I \times I$. However, there is a subtlety: It is not enough to impose the condition that such isomorphisms exist. We need to specify the isomorphisms as part of the data. Without the specific choice of isomorphisms, we cannot assemble the quasicoherent $\mathcal{O}_X|_{U_i}$-modules $\mathcal{M}_i$ to a quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$. So the isomorphisms are *structure*, not a *property*. Moreover, there is an additional subtlety: This extra data needs to satisfy a condition over triple intersections. For all $(i,j,k) \in I \times I \times I$, the diagram

$$\begin{tikzcd}
\mathcal{M}_k|_{U_i\cap U_j\cap U_k} \arrow[rr, "\alpha_{i,k}|_{U_i\cap U_j\cap U_k}"] \arrow[dr, "\alpha_{j,k}|_{U_i\cap U_j\cap U_k}"'] & & \mathcal{M}_i|_{U_i\cap U_j\cap U_k} \\
& \mathcal{M}_j|_{U_i\cap U_j\cap U_k} \arrow[ur, "\alpha_{i,j}|_{U_i\cap U_j\cap U_k}"'] &
\end{tikzcd}$$

must commute. We refer to this condition as the "cocycle condition."

Definition 8.11. Let X be a scheme and $(U_i)_{i\in I}$ a cover of $|X|$ by open subsets. A descent datum for quasicoherent $\mathcal{O}_X$-modules with respect to the given cover is the data of $(\mathcal{M}_i)_{i\in I}$ and $(\alpha_{i,j})_{(i,j)\in I\times I}$, where $\mathcal{M}_i$ is a quasicoherent $\mathcal{O}_X|_{U_i}$-module and $\alpha_{i,j}$ is an isomorphism of quasicoherent $\mathcal{O}_X|_{U_i\cap U_j}$-modules

$$\mathcal{M}_j|_{U_i\cap U_j} \xrightarrow{\alpha_{i,j}} \mathcal{M}_i|_{U_i\cap U_j}$$

subject to the requirement that the family of isomorphism $(\alpha_{i,j})_{(i,j)\in I\times I}$ satisfies the cocycle condition.

Remark 8.12. The cocycle condition implies that $\alpha_{i,i}$ is the identity map and that $\alpha_{i,j}$ and $\alpha_{j,i}$ are each other's inverses. Hence, if we specify an order $<$ on I, then it suffices to give $\alpha_{i,j}$ for $(i,j) \in I \times I$ with $i < j$.

Descent data form a category $\mathrm{Desc}(X, (U_i)_{i \in I})$: A map of descent data

$$(\mathcal{M}_i, \alpha_{i,j}) \longrightarrow (\mathcal{N}_i, \beta_{i,j})$$

is a family $(h_i \colon \mathcal{M}_i \to \mathcal{N}_i)_{i \in I}$ of maps of quasicoherent $\mathcal{O}_X|_{U_i}$-modules such that the diagram

$$\begin{tikzcd} \mathcal{M}_j|_{U_i \cap U_j} \arrow[r, "h_j|_{U_i \cap U_j}"] \arrow[d, "\alpha_{i,j}"] & \mathcal{N}_j|_{U_i \cap U_j} \arrow[d, "\beta_{i,j}"] \\ \mathcal{M}_i|_{U_i \cap U_j} \arrow[r, "h_i|_{U_i \cap U_j}"] & \mathcal{N}_i|_{U_i \cap U_j} \end{tikzcd}$$

commutes for all $(i,j) \in I \times I$.

Theorem 8.13. *Let X be a scheme, and let $(U_i)_{i \in I}$ a cover of $|X|$ by open subsets. The functor $\mathrm{QCoh}(X) \to \mathrm{Desc}(X, (U_i)_{i \in I})$ that to a quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ assigns the descent datum $(\mathcal{M}|_{U_i}, \mathrm{id})$ is an equivalence of categories.*

Proof. To go backwards, given a descent datum $(\mathcal{M}_i, \alpha_{i,j})$, we wish to construct a quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$. As already noted, it suffices to specify $\mathcal{M}(U)$ for all open subsets $U \subset |X|$ with the property that $U \subset U_i$ for some $i \in I$. Given such an open subset $U \subset |X|$, we choose arbitrarily an $i \in I$ such that $U \subset U_i$ and define

$$\mathcal{M}(U) = \mathcal{M}_i(U).$$

Given two such open subsets $V \subset U \subset |X|$, we may have chosen different $i, j \in I$ with $V \subset U_i$ and $U \subset U_j$. So we define the restriction map

$$\mathcal{M}(U) \xrightarrow{\mathrm{res}^U_V} \mathcal{M}(V)$$

to be the composite map

$$\mathcal{M}(U) = \mathcal{M}_i(U) \xrightarrow{\mathrm{res}^U_V} \mathcal{M}_i(V) \xrightarrow{\alpha_{j,i}} \mathcal{M}_j(V) = \mathcal{M}(V).$$

Finally, given three such open subsets $W \subset V \subset U \subset |X|$, we must verify that

$$\mathrm{res}^U_W = \mathrm{res}^V_W \circ \mathrm{res}^U_V,$$

and this precisely holds, by the assumption that $(\alpha_{i,j})_{(i,j) \in I \times I}$ satisfies the cocycle condition. This defines the functor in the opposite direction. We must also define natural isomorphisms between the two compositions of the two functors and the respective identity functors. You can do that. $\square$

What does this tell us about line bundles? Given a scheme X and a cover $(U_i)_{i \in I}$ of $|X|$ by open subsets, Theorem 8.13 gives an equivalence of categories

$$\begin{aligned} &\{\text{ Line bundles on } X\text{, which can be trivialized on } (U_i)_{i \in I}\} \\ &\simeq \{\text{Descent data } (\mathcal{M}_i, \alpha_{i,j}) \text{ such that } \mathcal{M}_i \simeq \mathcal{O}_X|_{U_i}\}, \end{aligned}$$

and the canonical inclusion functor gives a further equivalence of categories

$$\begin{aligned} &\{\text{Descent data } (\mathcal{M}_i, \alpha_{i,j}) \text{ such that } \mathcal{M}_i = \mathcal{O}_X|_{U_i}\} \\ &\simeq \{\text{Descent data } (\mathcal{M}_i, \alpha_{i,j}) \text{ such that } \mathcal{M}_i \simeq \mathcal{O}_X|_{U_i}\}. \end{aligned}$$

What does such a descent datum amount to? For any scheme X, the map

$$\mathrm{End}_{\mathcal{O}_X}(\mathcal{O}_X) \longrightarrow \mathcal{O}_X(|X|)$$

that to $h\colon \mathcal{O}_X \to \mathcal{O}_X$ assigns $h_X(1)$ is an isomorphisms of rings. The multiplication in the endomorphism ring is given by the composition of maps. Any isomorphism of rings induces an isomorphism of the respective groups of units, which, in the case that we consider, is an isomorphism of groups

$$\mathrm{Aut}_{\mathcal{O}_X}(\mathcal{O}_X) \longrightarrow \mathcal{O}_X(|X|)^\times$$

from the group of $\mathcal{O}_X$-linear automorphism of $\mathcal{O}_X$ to the group of global units.

Using this translation, we see that the category of line bundles on X, which can be trivialized on $(U_i)_{i\in I}$ is equivalent to the category, where an object is a family

$$(\alpha_{i,j})_{(i,j)\in I\times I}$$

with $\alpha_{i,j} \in \mathcal{O}_X(U_i \cap U_j)^\times$ such that the identity

$$\alpha_{i,k}|_{U_i\cap U_j\cap U_k} = \alpha_{i,j}|_{U_i\cap U_j\cap U_k} \cdot \alpha_{j,k}|_{U_i\cap U_j\cap U_k}$$

holds in $\mathcal{O}_X(U_i \cap U_j \cap U_k)^\times$ for all $(i,j,k) \in I \times I \times I$, and where a map

$$(\beta_{i,j})_{(i,j)\in I\times I} \longrightarrow (\alpha_{i,j})_{(i,j)\in I\times I}$$

is a family $(\lambda_i)_{i\in I}$ with $\lambda_i \in \mathcal{O}_X(U_i)^\times$ such that the identity

$$\lambda_i|_{U_i\cap U_j} \cdot \beta_{i,j} = \alpha_{i,j} \cdot \lambda_j|_{U_i\cap U_j}$$

holds in $\mathcal{O}_X(U_i \cap U_j)$ for all $(i,j) \in I \times I$.

Given the data of a line bundle $\mathcal{L}$ on X and a family $(\varphi_i)_{i\in I}$ of trivializations

$$\mathcal{L}|_{U_i} \xrightarrow{\varphi_i} \mathcal{O}_X|_{U_i},$$

we obtain the descent datum $(\alpha_{i,j})_{(i,j)\in I\times I}$ with $\alpha_{i,j} \in \mathcal{O}_X(U_i \cap U_j)^\times$ given by

$$\alpha_{i,j} = \varphi_i|_{U_i\cap U_j} \cdot (\varphi_j|_{U_i\cap U_j})^{-1}.$$

This descent datum defines a globally trivial line bundle if and only if there exists a family of units $(\lambda_i)_{i\in I}$ with $\mathcal{O}_X(U_i)^\times$ such that

$$\alpha_{i,j} = \lambda_i|_{U_i\cap U_j} \cdot (\lambda_j|_{U_i\cap U_j})^{-1}$$

for all $(i,j) \in I \times I$.

The set of isomorphism classes of line bundles on X which can be trivialized on $(U_i)_{i\in I}$ forms an abelian group under tensor product. Later we will see that the above discussion identifies this group with the Čech cohomology group

$$\check{H}^1((U_i)_{i\in I}, \mathbb{G}_m)$$

with respect to the given covering.

Example 8.14. We construct a line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1_{\mathbb{Z}}$ for all $n \in \mathbb{Z}$. Recall that $\mathbb{P}^1_{\mathbb{Z}}$ is defined as the pushout of schemes

$$\begin{tikzcd}
\mathrm{Spec}(\mathbb{Z}[x^{\pm 1}]) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbb{Z}[x]) \arrow[d] \\
\mathrm{Spec}(\mathbb{Z}[y]) \arrow[r] & \mathbb{P}^1_{\mathbb{Z}}
\end{tikzcd}$$

with the left-hand vertical map induced by the unique ring homorphism

$$\mathbb{Z}[y] \longrightarrow \mathbb{Z}[x^{\pm 1}]$$

that sends y to x^{-1}. So by the above discussion, we can specify a line bundle on $\mathbb{P}^1_{\mathbb{Z}}$ which is trivial on each the open subsets $\mathrm{Spec}(\mathbb{Z}[x])$ and $\mathrm{Spec}(\mathbb{Z}[y])$ by specifying a single unit $u \in \mathbb{Z}[x, x^{-1}]^{\times}$. (There is no cocycle condition to verify, since there are no non-degenerate triple intersections.) The $\mathbb{Z}$-module $\mathbb{P}^1_{\mathbb{Z}}(\mathcal{L})$ of global sections of the corresponding line bundle $\mathcal{L}$ is given by the pullback

$$\begin{tikzcd} \mathbb{P}^1_{\mathbb{Z}}(\mathcal{L}) \arrow[r] \arrow[d] & \mathbb{Z}[y] \arrow[d] \\ \mathbb{Z}[x] \arrow[r] & \mathbb{Z}[x^{\pm 1}], \end{tikzcd}$$

where the right-hand vertical map is the unique $\mathbb{Z}[y]$-linear map that to 1 assigns u, or equivalently, the map that to the polynomial $p(y) \in \mathbb{Z}[y]$ assigns the Laurent polynomial $u \cdot p(x^{-1}) \in \mathbb{Z}[x^{\pm 1}]$. Choosing $u = x^n \in \mathbb{Z}[x^{\pm 1}]^{\times}$, we obtain a line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1_{\mathbb{Z}}$. If $n \geq 0$, then its $\mathbb{Z}$-module of global sections is the free $\mathbb{Z}$-module

$$\mathbb{P}^1_{\mathbb{Z}}(\mathcal{O}(n)) = \mathbb{Z}\{(x^i, y^{n-i}) \mid 0 \leq i \leq n\},$$

which has rank $n+1$, and if $n < 0$, then its $\mathbb{Z}$-module of global sections is zero.

We recall that, in general, we have

$$\mathbb{P}^1_{\mathbb{Z}}(\mathcal{L}) \simeq \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{L}).$$

Hence, if $n < 0$, then the only map of quasicoherent $\mathcal{O}$-modules $h \colon \mathcal{O} \to \mathcal{O}(n)$ is the zero map, so $\mathcal{O}(n)$ is not a direct summand of a free $\mathcal{O}$-module. It follows that the characterization of vector bundles on an affine scheme that we established in Lecture 7 does not extend to non-affine schemes. By constrast, every real vector bundle on a compact Hausdorff space is a summand of a trivial vector bundle, so here the distinction between geometry and topology begins to show!

Remark 8.15. In the pullback square that calculates $\mathbb{P}^1_{\mathbb{Z}}(\mathcal{O}(n))$, the map

$$\mathbb{Z}[x] \oplus \mathbb{Z}[y] \longrightarrow \mathbb{Z}[x^{\pm 1}]$$

is surjective for $n \geq -1$, but for $n \leq -2$, its cokernel is a free $\mathbb{Z}$-module of rank $1-n$ generated by the family consisting of the classes of x^i with $n < i < 0$. Later, we will recognize this cokernel as the coherent cohomology group $H^1(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(n))$.

9. Invertible $\mathcal{O}_X$-modules and effective Cartier divisors

A line bundle on a scheme X is a quasicoherent $\mathcal{O}_X$-module $\mathcal{L}$ such that, locally on X, there exists an isomorphism $\varphi\colon \mathcal{L} \to \mathcal{O}_X$. Last time, we used descent to show that if such isomorphisms exist on the elements of a cover $(U_i)_{i\in I}$ of X by open subsets, then we can describe $\mathcal{L}$ in terms of the transition functions

$$\alpha_{i,j} \in \mathcal{O}_X(U_i \cap U_j)^\times,$$

describing the passage from the trivialization $\varphi_j\colon \mathcal{L}|_{U_j} \to \mathcal{O}_X|_{U_j}$ to the trivialization of $\varphi_i\colon \mathcal{L}|_{U_i} \to \mathcal{O}_X|_{U_i}$ over $U_i \cap U_j$. Today, we continue the discussion of line bundles. As a preliminary, we introduce tensor product and inverse image of quasicoherent $\mathcal{O}_X$-modules. The constructions of both are a bit complicated, but their properties are simple, which is better than the other way around.

Given $\mathcal{O}_X$-modules $\mathcal{M}$ and $\mathcal{N}$, their tensor product is an $\mathcal{O}_X$-bilinear map

$$\mathcal{M} \times \mathcal{N} \longrightarrow \mathcal{M} \otimes \mathcal{N}$$

with the "universal" property that composition with this map induces a bijection

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M} \otimes \mathcal{N}, \mathcal{P}) \longrightarrow \mathrm{Bil}_{\mathcal{O}_X}(\mathcal{M} \times \mathcal{N}, \mathcal{P})$$

for all $\mathcal{O}_X$-modules $\mathcal{P}$. This property determines the map $\mathcal{M} \times \mathcal{N} \to \mathcal{M} \otimes \mathcal{N}$ uniquely, up to unique isomorphism under $\mathcal{M} \times \mathcal{N}$. Comparing universal properties, we see that this map is the sheafification of the map of presheaves that to $U \subset |X|$ open assigns the tensor product of $\mathcal{O}_X(U)$-modules $\mathcal{M}(U) \times \mathcal{N}(U) \to \mathcal{M}(U) \otimes \mathcal{N}(U)$. We will often abuse language and say that the $\mathcal{O}_X$-module $\mathcal{M} \otimes \mathcal{N}$ is the tensor product of $\mathcal{M}$ and $\mathcal{N}$. We enumerate some proporties of the tensor product.

(1) The functors $\mathcal{M} \otimes -$ and $- \otimes \mathcal{N}$ preserve colimits, or equivalently, preserve sums and cokernels.

(2) The projection $\mathcal{M} \times \mathcal{O}_X \to \mathcal{M}$ induces a bijection

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{P}) \longrightarrow \mathrm{Bil}_{\mathcal{O}_X}(\mathcal{M} \times \mathcal{O}_X, \mathcal{P}),$$

which gives a canonical isomorphism $\mathcal{M} \otimes \mathcal{O}_X \simeq \mathcal{M}$. Similarly, there is a canonical isomorphism $\mathcal{O}_X \otimes \mathcal{N} \simeq \mathcal{N}$.

(3) If $U \subset |X|$ is open, then there is a canonical isomorphism

$$(\mathcal{M} \otimes \mathcal{N})|_U \simeq (\mathcal{M}|_U) \otimes (\mathcal{N}|_U),$$

since sheafification and restriction to open subsets commute.

(4) If $\mathcal{M}, \mathcal{N} \in \mathrm{QCoh}(X)$, then $\mathcal{M} \otimes \mathcal{N} \in \mathrm{QCoh}(X)$. Indeed, by (3), we can localize and assume that $\mathcal{M}$ and $\mathcal{N}$ admits presentations, in which case, we conclude from (1) and (2) that also $\mathcal{M} \otimes \mathcal{N}$ admits a presentation.

(5) If $\mathcal{M}, \mathcal{N} \in \mathrm{Vect}(X)$, then $\mathcal{M} \otimes \mathcal{N} \in \mathrm{Vect}(X)$ by proof as in (4). Also,

$$\dim(\mathcal{M} \otimes \mathcal{N}) = \dim(\mathcal{M}) \cdot \dim(\mathcal{N})$$

as functions from X to $\mathbb{Z}_{\geq 0}$. So if $\mathcal{M}, \mathcal{N} \in \mathrm{Line}(X)$, then $\mathcal{M} \otimes \mathcal{N} \in \mathrm{Line}(X)$. Moreover, a trivialization of $\mathcal{M}$ and $\mathcal{N}$ on an open cover $(U_i)_{i\in I}$ of $|X|$ with transition functions $\alpha_{i,j}, \beta_{i,j} \in \mathcal{O}_X(U_i \cap U_j)^\times$ determines a trivialization of $\mathcal{M} \otimes \mathcal{N}$ on $(U_i)_{i\in I}$ with transition functions $\alpha_{i,j} \cdot \beta_{i,j} \in \mathcal{O}_X(U_i \cap U_j)^\times$.

We now characterize line bundles in terms of tensor products:

Proposition 9.1. *Let X be a scheme, and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. The following are equivalent:*

(1) *The $\mathcal{O}_X$-module $\mathcal{M}$ is a line bundle.*
(2) *There exists an $\mathcal{O}_X$-module $\mathcal{N}$ such that $\mathcal{M} \otimes \mathcal{N} \simeq \mathcal{O}_X$.*

In particular, any such $\mathcal{M}$ is quasicoherent (as is its inverse).

Before we prove the proposition, we need to understand (2) better.

Definition 9.2. Let X be a scheme. An inverse for an $\mathcal{O}_X$-module $\mathcal{M}$ is a pair $(\mathcal{N}, \iota)$ of an $\mathcal{O}_X$-module $\mathcal{N}$ and an isomorphism $\iota\colon \mathcal{M} \otimes \mathcal{N} \to \mathcal{O}_X$.

Lemma 9.3. *Let X be a scheme, and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. If $(\mathcal{N}, \iota)$ and $(\mathcal{N}', \iota')$ are inverses of $\mathcal{M}$, then there exists a unique isomorphism $\alpha\colon \mathcal{N} \to \mathcal{N}'$ such that*

$$\begin{tikzcd} \mathcal{M} \otimes \mathcal{N} \arrow[r, "\iota"] \arrow[d, "\mathcal{M}\otimes\alpha"] & \mathcal{O}_X \arrow[d, equal] \\ \mathcal{M} \otimes \mathcal{N}' \arrow[r, "\iota'"] & \mathcal{O}_X \end{tikzcd}$$

commutes.

Proof. This is a purely category theoretical statement, valid in any symmetric monoidal category. The isomorphism α is the composite isomorphism

$$\mathcal{N} \simeq \mathcal{O}_X \otimes \mathcal{N} \xleftarrow{\iota' \otimes \mathcal{N}} \mathcal{M} \otimes \mathcal{N}' \otimes \mathcal{N} \simeq \mathcal{M} \otimes \mathcal{N} \otimes \mathcal{N}' \xrightarrow{\iota \otimes \mathcal{N}'} \mathcal{O}_X \otimes \mathcal{N}' \simeq \mathcal{N}'.$$

In fact, the statement that $(\mathcal{N}, \iota)$ is an inverse of $\mathcal{M}$ is equivalent to the statement that for all $\mathcal{O}_X$-modules $\mathcal{P}$, the composition

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{N}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M} \otimes \mathcal{P}, \mathcal{M} \otimes \mathcal{N}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M} \otimes \mathcal{P}, \mathcal{O}_X)$$

of the map given by tensoring with $\mathcal{M}$ and the map given by composition with ι is a bijection. This latter statement characterizes $(\mathcal{N}, \iota)$, up to unique isomorphism. $\square$

Corollary 9.4. *Let X be a scheme, and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. The existence of an inverse of $\mathcal{M}$ can be checked locally: If $(U_i)_{i \in I}$ is an open cover of $|X|$, and if $\mathcal{M}|_{U_i}$ admits an inverse for all $i \in I$, then $\mathcal{M}$ admits an inverse.*

Proof. For every $i \in I$, we choose an inverse $(\mathcal{N}_i, \iota_i)$ of $\mathcal{M}|_{U_i}$. We apply Lemma 9.3 to obtain, for every $(i, j) \in I \times I$, a unique isomorphism

$$\mathcal{N}_j|_{U_i \cap U_j} \xrightarrow{\alpha_{i,j}} \mathcal{N}_i|_{U_i \cap U_j}$$

such that for every $(i, j) \in I \times I$, the diagram

$$\begin{tikzcd} \mathcal{M}|_{U_i \cap U_j} \otimes \mathcal{N}_j|_{U_i \cap U_j} \arrow[r, "\iota_j"] \arrow[d, "\mathcal{M}|_{U_i \cap U_j} \otimes \alpha_{i,j}"] & \mathcal{O}_X|_{U_i \cap U_j} \arrow[d, equal] \\ \mathcal{M}|_{U_i \cap U_j} \otimes \mathcal{N}_i|_{U_i \cap U_j} \arrow[r, "\iota_i|_{U_i \cap U_j}"] & \mathcal{O}_X|_{U_i \cap U_j} \end{tikzcd}$$

commutes. The uniqueness of the isomorphisms $\alpha_{i,j}$ implies that they satisfy the cocycle condition. So we get a global inverse $(\mathcal{N}, \iota)$ by descent. $\square$

Proof of Proposition 9.1. If (1) holds, then $\mathcal{M}$ is locally isomorphic to $\mathcal{O}_X$, and since (2) can be checked locally by Lemma 9.3, we are reduced to proving (2) in the case $\mathcal{M} = \mathcal{O}_X$. But in this case, the pair $(\mathcal{N}, \iota)$ consisting of $\mathcal{N} = \mathcal{O}_X$ and the canonical isomorphism $\iota\colon \mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X$ is an inverse. To show that (2) implies (1), we need a preliminary.

Lemma 9.5. *Let R be a local ring. If an R-module M is invertible, then it is isomorphic to R.*

Proof. Since M is invertible, the functor

$$\mathrm{Mod}_R \xrightarrow{M \otimes -} \mathrm{Mod}_R$$

is an equivalence of categories. Indeed, an inverse (N, i) provides a quasi-inverse of this functor. Now, the properties of an R-module M to be finitely presented and to be projective are purely category theoretical properties. Indeed, the statement that M is finitely presented is equivalent to the statement that the functor

$$\mathrm{Mod}_R \xrightarrow{\mathrm{Hom}_R(M,-)} \mathsf{Set}$$

preserves filtered colimits, and the statement that M is projective is equivalent to the statement that this functor preserves reflexsive coequalizers, that is, if it takes a colimit diagrams of R-modules of the form

$$\begin{tikzcd} M_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & M_0 \arrow[l, "s_0"'] \arrow[r, "d_0"] & M \end{tikzcd}$$

to a colimit diagram of sets.[5] At any rate, the only thing that is important for us is that, because the properties of being finitely presented and of being projective are expressable in terms of category theoretic terms, the are preserved by an equivalence of categories. In the case at hand, where we assume M to invertible, so that the functor $M \otimes -$ to be an equivalence of categories, we conclude that since R is finitely presented and projective, so is $M \simeq M \otimes R$. As we used last time, it follows from Nakayama's lemma that a finitely presented and projective module over a local ring is free of finite rank. So we conclude that $M \simeq R^{\oplus d}$ for some $d \geq 0$. If (N, i) is an inverse of M, then similarly $N \simeq R^{\oplus d'}$ for some $d' \geq 0$. It follows that

$$R \simeq M \otimes N \simeq (R^{\oplus d}) \otimes (R^{\oplus d'}) \simeq R^{\oplus dd'},$$

which shows that $dd' = 1$, and hence, $d = d' = 1$, as desired.[6] $\square$

We fix $x \in |X|$ and proceed to show that there exists $x \in U \subset |X|$ open and an isomorphism $\varphi\colon \mathcal{M}|_U \to \mathcal{O}_X|_U$. First, if $\mathcal{M}$ and $\mathcal{N}$ are any $\mathcal{O}_X$-modules, then

$$(\mathcal{M} \otimes \mathcal{N})_x \longrightarrow \mathcal{M}_x \otimes \mathcal{N}_x$$

is an isomorphism with the tensor product on the left-hand side that of $\mathcal{O}_X$-modules and the tensor product on the right-hand side that $\mathcal{O}_{X,x}$-modules. Therefore, for

[5] One can combine these two properties and show that an R-module M is finitely presented and projective if and only if the functor $\mathrm{Hom}_R(M, -)$ preserves *sifted* colimits.

[6] Here we use that the rank of a finitely generated free R-module is well-defined. This is true for all commutative rings, and more generally, for not necessarily commutative rings that admit a map to a field. But it fails for general non-commutative rings.

the invertible $\mathcal{O}_X$-module $\mathcal{M}$ in question, Lemma 9.5 shows that the stalk $\mathcal{M}_x$ is a free $\mathcal{O}_{X,x}$-module of rank 1. Thus, if $(\mathcal{N}, \iota)$ is an inverse of $\mathcal{M}$, then there exists $s_x \in \mathcal{M}_x \simeq \mathcal{O}_{X,x}$ and $t_x \in \mathcal{N}_x \simeq \mathcal{O}_{X,x}$ such that $\iota_x(s_x \otimes t_x) = 1 \in \mathcal{O}_{X,x}$. We can find $x \in U \subset |X|$ open and local sections $s \in \mathcal{M}(U)$ and $t \in \mathcal{N}(U)$ such that s and t represent s_x and t_x and such that $\iota_U(s \otimes t) = 1 \in \mathcal{O}_X(U)$. We claim that $s\colon \mathcal{O}_X|_U \to \mathcal{M}|_U$ is an isomorphism. Indeed, for all $y \in U$, the composite map

$$\mathcal{O}_{X,y} \xrightarrow{s_y} \mathcal{M}_y \xrightarrow{\mathcal{M}_y \otimes t_y} \mathcal{M}_y \otimes \mathcal{N}_y \simeq (\mathcal{M} \otimes \mathcal{N})_y \xrightarrow{\iota_y} \mathcal{O}_{X,y}$$

is the identity map, so the map s_y is a split injection between free $\mathcal{O}_{X,y}$-modules of rank 1, and hence, an isomorphism. This show that $s\colon \mathcal{O}_X|_U \to \mathcal{M}|_U$ is an isomorphism, so (1) holds. $\square$

In particular, if $\mathcal{M}$ is a line bundle on X, then so is any inverse $\mathcal{N}$ of $\mathcal{M}$. We will write $\mathcal{M}^{-1}$ for "the" inverse of $\mathcal{M}$, which is justified by Lemma 9.3. Moreover, if $\mathcal{M}$ is trivialized over the open cover $(U_i)_{i \in I}$ of $|X|$ with transitions functions

$$\alpha_{i,j} \in \mathcal{O}_X(U_i \cap U_j)^{\times},$$

then $\mathcal{M}^{-1}$ is also trivialized over $(U_i)_{i \in I}$ with transition functions

$$\alpha_{i,j}^{-1} \in \mathcal{O}_X(U_i \cap U_j)^{\times}.$$

So much for tensor products.

We next consider inverse image functors. In general, if $f = (p, \phi)\colon Y \to X$ is a map of ringed spaces, then we have an adjunction

$$\begin{tikzcd} \mathrm{Mod}_{\mathcal{O}_X}(\mathcal{P}(|X|)) \arrow[r, shift left, "f^p"] & \mathrm{Mod}_{\mathcal{O}_Y}(\mathcal{P}(|Y|)) \arrow[l, shift left, "f_p"] \end{tikzcd}$$

between the category of presheaves of $\mathcal{O}_X$-modules and the category of presheaves of $\mathcal{O}_Y$-modules. The right adjoint functor f_p is easy to define, namely,

$$f_p(\mathcal{N})(U) = \mathcal{N}(p^{-1}(U)),$$

and it preserves sheaves in the sense that it restricts to a functor

$$\begin{tikzcd} \mathrm{Mod}_{\mathcal{O}_X}(\mathcal{P}(|X|)) & \mathrm{Mod}_{\mathcal{O}_Y}(\mathcal{P}(|Y|)) \arrow[l, "f_p"'] \\ \mathrm{Mod}_{\mathcal{O}_X}(\mathrm{Sh}(|X|)) \arrow[u, "\iota_X"'] & \mathrm{Mod}_{\mathcal{O}_Y}(\mathrm{Sh}(|Y|)) \arrow[l, "f_*"'] \arrow[u, "\iota_Y"'] \end{tikzcd}$$

between the respective categories of sheaves. The left adjoint f^p is more complicated to define, but it has better properties. It is given by

$$f^p(\mathcal{M})(V) \simeq \mathrm{colim}_{p(V) \subset U \subset |X|} \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V),$$

where the colimit ranges over open subsets $p(V) \subset U \subset |X|$, and where the Uth term in the colimit is the cobase-change of the $\mathcal{O}_X(U)$-module $\mathcal{M}(U)$ along the composite ring homomorphism

$$\mathcal{O}_X(U) \xrightarrow{\phi_U} \mathcal{O}_Y(p^{-1}(U)) \longrightarrow \mathcal{O}_Y(V),$$

where the right-hand map is the restriction along $p^{-1}(U) \subset V$. It has the key property that if $y \in |Y|$ with image $x = p(y) \in |X|$, then the canonical map

$$f^p(\mathcal{M})_y \longrightarrow \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y}$$

is an isomorphism. Hence, if a map $h\colon \mathcal{M} \to \mathcal{M}'$ of presheaves of $\mathcal{O}_X$-modules becomes an isomorphism after sheafification, then so does the induced map of presheaves of $\mathcal{O}_Y$-modules $f^p(h)\colon f^p(\mathcal{M}) \to f^p(\mathcal{M}')$. Now, the functor

$$\mathrm{Mod}_{\mathcal{O}_X}(\mathrm{Sh}(|X|) \xrightarrow{f^*} \mathrm{Mod}_{\mathcal{O}_Y}(\mathrm{Sh}(|Y|)$$

defined by $f^* \simeq \mathrm{ass}_Y \circ f^p \circ \iota_X$ is left adjoint to the functor f_*, and in the diagram

$$\begin{tikzcd}
\mathrm{Mod}_{\mathcal{O}_X}(\mathcal{P}(|X|)) \arrow[r, shift left, "f^p"] \arrow[d, shift right, "\mathrm{ass}_X"'] & \mathrm{Mod}_{\mathcal{O}_Y}(\mathcal{P}(|Y|)) \arrow[l, shift left, "f_p"] \arrow[d, shift right, "\mathrm{ass}_Y"'] \\
\mathrm{Mod}_{\mathcal{O}_X}(\mathrm{Sh}(|X|)) \arrow[r, shift left, "f^*"] \arrow[u, shift right, "\iota_X"'] & \mathrm{Mod}_{\mathcal{O}_Y}(\mathrm{Sh}(|Y|)), \arrow[l, shift left, "f_*"] \arrow[u, shift right, "\iota_Y"']
\end{tikzcd}$$

the identity natural isomorphism $f_p \circ \iota_Y \to \iota_X \circ f_*$ induces a natural isomorphism $f^* \circ \mathrm{ass}_X \to \mathrm{ass}_Y \circ f^p$. Moreover, if $j\colon U \to X$ is an open immersion, then

$$j^*(\mathcal{M}) \simeq \mathcal{M}|_U,$$

since, in this case, the category that indexes the colimit that defines $j^p(\mathcal{M})(V)$ has the terminal object $j(V) = j(V) \subset |X|$.

We enumerate the some properties of the inverse image functor f^* associated with a map of schemes $f\colon Y \to X$.

(1) The canonical map $f^*(\mathcal{O}_X) \to \mathcal{O}_Y$ is an isomorphism.
(2) Being a left adjoint, the functor f^* preserves colimits.
(3) The functor f^* restricts to a functor $f^*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$. Indeed, we can work locally, where this follows from (1) and (2).
(4) If $f\colon Y \to X$ is the map of prime spectra corresponding to a map of rings $\phi\colon A \to B$, then we have a diagram of adjunctions

$$\begin{tikzcd}
\mathrm{Mod}_A \arrow[r, shift left, "\phi^*"] \arrow[d, shift right, "p^*"'] & \mathrm{Mod}_B \arrow[l, shift left, "\phi_*"] \arrow[d, shift right, "q^*"'] \\
\mathrm{QCoh}(X) \arrow[r, shift left, "f^*"] \arrow[u, shift right, "p_*"'] & \mathrm{QCoh}(Y), \arrow[l, shift left, "f_*"] \arrow[u, shift right, "q_*"']
\end{tikzcd}$$

where p_* and q_* are the respective global section functors, and where ϕ_* is the restriction of scalars functor. Again, the identity natural isomorphism $\phi_* \circ q_* \to p_* \circ f_*$ induces a natural isomorphism $f^* \circ p^* \to q^* \circ \phi^*$.
(5) The canonical natural map

$$f^*(\mathcal{M} \otimes \mathcal{M}') \longrightarrow f^*(\mathcal{M}) \otimes f^*(\mathcal{M}')$$

is a natural isomorphism.
(6) The functor f^* restricts to a functor $f^*\colon \mathrm{Vect}(X) \to \mathrm{Vect}(Y)$.
(7) The functor f^* restricts to a functor $f^*\colon \mathrm{Line}(X) \to \mathrm{Line}(Y)$.

If $g\colon Z \to Y$ and $f\colon Y \to X$ are composable maps of schemes, then the identity natural isomorphism $f_* \circ g_* \to (f \circ g)_*$ induces a natural isomorphism

$$(f \circ g)^* \longrightarrow g^* \circ f^*.$$

This natural isomorphism is also an instance of the fact that a left adjoint functor of a given functor, if it exists, is unique, up to unique isomorphism. We apply this to another special case, namely, the inclusion of a point map

$$\mathrm{Spec}(k(x)) \xrightarrow{i_x} X,$$

the inverse image functor of which is given by

$$i_x^*(\mathcal{M}) \simeq \mathcal{M}(x).$$

Let $f\colon Y \to X$ be a map of schemes, and let $y \in |Y|$ with image $x = f(y) \in |X|$. From the commutative diagram of schemes

$$\begin{tikzcd}
\mathrm{Spec}(k(y)) \arrow[r, "f(x)"] \arrow[d, "i_y"] & \mathrm{Spec}(k(x)) \arrow[d, "i_x"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

and the natural isomorphism above, we find that

$$\begin{aligned}
f^*(\mathcal{M})(y) &\simeq (i_y^* \circ f^*)(\mathcal{M}) \simeq (f \circ i_y)^*(\mathcal{M}) \simeq (i_x \circ f(x))^*(\mathcal{M}) \\
&\simeq (f(x)^* \circ i_x^*)(\mathcal{M}) \simeq \mathcal{M}(x) \otimes_{k(x)} k(y).
\end{aligned}$$

So if we think of a line bundle $\mathcal{L}$ on X as encoding the family of 1-dimensional vector spaces $\mathcal{L}(x)$ over $k(x)$, then the line bundle $f^*(\mathcal{M})$ on Y encodes the family of 1-dimensional vector spaces $\mathcal{L}(x) \otimes_{k(x)} k(y)$ over $k(y)$, where $x = f(y)$.

We next define a generalization of distinguished open subsets. Let X be a scheme, let $\mathcal{L}$ be a line bundle on X, let $U \subset |X|$ be an open subsets, and let $s \in \mathcal{L}(U)$ be a local section. In this situation, we define

$$|X_s| = \{x \in U \mid s(x) \neq 0 \text{ in } \mathcal{L}(x)\} \subset |X|.$$

Proposition 9.6. *Let X be a scheme, let $\mathcal{L}$ be a line bundle on X, let $U \subset |X|$ be an open subset, and let $s \in \mathcal{L}(U)$ be a local section.*

(1) *The subset $|X_s| \subset |X|$ is open.*
(2) *If $V \subset U$ is open, then $V \subset |X_s|$ if and only if the restriction*

$$\mathcal{O}_X|_V \xrightarrow{s|_V} \mathcal{L}_V$$

is an isomorphism.

Proof. We can work locally on X, and therefore, we may assume that there exists an isomorphism $\varphi\colon \mathcal{L} \to \mathcal{O}_X$. But in this case, the proposition reduces to what we have already proved earlier. ∎

Informally, the proposition states that $|X_s| \subset U$ is the largest open subset over which the map $s\colon \mathcal{O}_X|_U \to \mathcal{L}|_U$ becomes an isomorphism. Let $f\colon Y \to X$ is a map of schemes, let $\mathcal{L}$ be a line bundle on X, let $U \subset |X|$ be an open subset, and let $V = f^{-1}(U) \subset |Y|$. We recall that a local section $s \in \mathcal{L}_X(U)$ determines and is determined by a map of $\mathcal{O}_X|_U$-modules $s\colon \mathcal{O}_X|_U \to \mathcal{L}|_U$, so by applying the inverse image functor $(f|_V)^*$, it determines a map of $\mathcal{O}_Y|_V$-modules

$$\mathcal{O}_Y|_V \simeq f^*(\mathcal{O}_X)|_V \simeq (f|_V)^*(\mathcal{O}_X|_U) \xrightarrow{f^*(s)} (f|_V)^*(\mathcal{L}|_U) \simeq f^*(\mathcal{L})|_V,$$

and hence, a local section $f^*(s) \in f^*(\mathcal{L})(V)$. In this situation, we have

$$|Y_{f^*(s)}| = f^{-1}(|X_s|) \subset |Y|,$$

which follows from the definition of the distinguished open subsets and from the isomorphism $f^*(\mathcal{L})(y) \simeq \mathcal{L}(x) \otimes_{k(x)} k(y)$ for $y \in V$ with image $x = f(y) \in U$.

Remark 9.7. Let $\mathcal{L}$ be a line bundle on an affine scheme X, let $U \subset |X|$ be an open subset, and let $s \in \mathcal{L}(U)$ be a local section. The distinguished open subset

$$|X_s| \subset |X|$$

is an affine open subset, although it is not necessarily equal to a distinguished open subset of the form $|X_f| \subset |X|$ with $f \in \mathcal{O}_X(U)$, due to the non-triviality of $\mathcal{L}$. Why is it affine? Well, if $\mathcal{L} \simeq \mathcal{O}_X$ is trivial, then $|X_s| \subset |X|$ is a standard distinguished open subset, and hence, an affine open subset. Since $\mathcal{L}$ is locally trivial, we conclude that the open immersion $j\colon X_s \to X$ is an affine map, and because X is affine, so is X_s. More directly, the affine map $j\colon X_s \to X$ is classified by the quasicoherent $\mathcal{O}_X$-algebra $\mathcal{A} \simeq \operatorname{colim}_n \mathcal{L}^{\otimes n}$ with the maps in the diagram given by

$$\mathcal{L}^{\otimes n} \simeq \mathcal{L}^{\otimes n} \otimes \mathcal{O}_X \xrightarrow{\mathcal{L}^{\otimes n} \otimes s} \mathcal{L}^{\otimes n+1}.$$

Finally, we will give an example of a geometrically relevant source of line bundles, namely, that of effective Cartier divisors.

Definition 9.8. An effective Cartier divisor on a scheme X is a closed subscheme

$$D \subset X$$

that, locally on X, is defined by a single non-zero-divisor.

That $D \subset X$ is defined, locally on X, by a single non-zero-divisor means that there exists a cover $(U_i)_{i \in I}$ of X by affine open subschemes and a family of local sections $(f_i)_{i \in I}$ with $f_i \in \mathcal{O}_X(U_i)$ a non-zero-divisor such that

$$D \cap U_i \simeq \operatorname{Spec}(\mathcal{O}_X(U_i)/f_i)$$

as schemes over $U_i \simeq \operatorname{Spec}(\mathcal{O}_X(U_i)$. Morally, an effective Cartier divisor $D \subset X$ is of pure codimension 1.

Lemma 9.9. *Let X be a scheme. If $D \subset X$ is an effective Cartier divisor, then the corresponding quasicoherent ideal, denoted $\mathcal{O}_X(-D) \subset \mathcal{O}_X$, is a line bundle.*

Proof. If $f \in R$ is a non-zero-divisor, then $f\colon R \to (f) \subset R$ is an isomorphism. $\square$

Definition 9.10. If $D \subset X$ is an effective Cartier divisor on a scheme, then the twist of $\mathcal{O}_X$ along D is the line bundle $\mathcal{O}_X(D)$ on X given by the inverse of $\mathcal{O}_X(-D)$.

Why do we take the inverse? Because, by convention in algebraic geometry, we prefer line bundles that admit nonzero global sections, as opposed to line bundles whose inverses admit nonzero global sections. We note that the canonical inclusion

$$\mathcal{O}_X(-D) \xrightarrow{i} \mathcal{O}_X$$

gives, by taking inverse line bundles, a canonical map of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \xrightarrow{s_D} \mathcal{O}_X(D).$$

This defines a canonical global section $s_D \in \mathcal{O}_X(D)(X)$. Explicitly, the restriction

$$(\mathcal{O}_X(D))(U) \longrightarrow (\mathcal{O}_X(D))(V) \simeq \mathcal{O}_X(V)$$

along the open inclusion $V = U \smallsetminus (U \cap D) \to U$ is injective, and its image consists of the local sections that, locally, is of the form $f = g/\pi$, where g and π are the restrictions of an element of $\mathcal{O}_X(U)$ and a generator of $\mathcal{O}_X(-D)(U) \subset \mathcal{O}_X(U)$. So informally, we think of sections of $\mathcal{O}_X(D)$ as "meromorphic functions on X with at most a simple pole along D.". In this picture, the canonical section

$$\mathcal{O}_X \xrightarrow{s_D} \mathcal{O}_X(D)$$

is the inclusion of the functions without poles along D.

Remark 9.11. If $D \subset X$ is an effective Cartier divisor, then the distinguished open subscheme defined by the canonical section of the line bundle $\mathcal{O}_X(D)$ is given by

$$X_{s_D} = X \smallsetminus D \subset X.$$

Also, let $f\colon Y \to X$ is a map of schemes, let $i\colon D \to X$ be an effective Cartier divisor, and let $i'\colon D' \to Y$ be the base-change given by the fiber product[7]

$$\begin{tikzcd} D' \arrow[r, "i'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ D \arrow[r, "i"] & X. \end{tikzcd}$$

It is common to write $f^{-1}(D) \subset Y$ for the closed subscheme $D' \subset Y$, although this is not literally an inverse image in a set-theoretic sense. In this situation, if we further assume that $f^{-1}D \subset Y$ is also a Cartier divisor, then we have

$$f^*(\mathcal{O}_X(D)) \simeq \mathcal{O}_Y(f^{-1}(D)).$$

Both claims can be checked locally, picking a generator of $\mathcal{O}_X(-D)$ to trivialize everything, then keeping track of transition functions.

[7] We have not yet properly proved that the category of schemes admits finite limits, and hence, fiber products, but we will make sense of this later.

10. Projective space and its functor of points

What is the projective space $\mathbb{P}^n$ supposed to be? In the standard description, it is the space of lines (through the origin) in a fixed $(n+1)$-dimensional vector space $F^{\oplus(n+1)}$. In particular, there should be a "tautological" line bundle $\mathcal{L}$ over $\mathbb{P}^n_{\mathbb{Z}}$, whose fiber $\mathcal{L}_x$ at the point $x \in \mathbb{P}^n_{\mathbb{Z}}$ is the corresponding line in $k(x)^{\oplus(n+1)}$. However, on $\mathbb{P}^1_{\mathbb{Z}}$, this line bundle $\mathcal{L}$ is the line bundle $\mathcal{O}(-1)$, which has no nonzero global sections. This corresponds to the fact that there is no consistent algebraic way to pick a point in a line in $F^{\oplus(n+1)}$, expect to always pick the origin. But the situation improves if we pass to the inverse line bundle $\mathcal{O}(1)$. In general, on $\mathbb{P}^n_{\mathbb{Z}}$, this corresponds to the dual vector space of the tautological line, that is, the linear functionals on the tautological line. And the very embedding of the tautological line in $F^{\oplus(n+1)}$ is encoded in the $n+1$ linear functionals on the tautological line, the $n+1$ coordinate projections: Given a 1-dimensional F-vector space L and a family $(s_i)_{0\leq i\leq n}$ of elements of $L^\vee \simeq \operatorname{Hom}_F(L, F)$, the map

$$L \xrightarrow{(s_i)} \textstyle\prod_{0\leq i\leq n} F$$

is injective if and only if the family $(s_i)_{0\leq i\leq n}$ spans the dual line $L^\vee \simeq L^{-1}$.

This suggests that, for a field F, the set of maps $\eta\colon \operatorname{Spec}(F) \to \mathbb{P}^n_{\mathbb{Z}}$ should be the set of isomorphism classes of pairs (L, s) of a 1-dimensional F-vector space L and a surjective F-linear map $s\colon F^{\oplus(n+1)} \to L$, or equivalently, the set of codimension 1 subspaces of $F^{\oplus(n+1)}$. More generally, for a scheme X, the set of maps $\eta\colon X \to \mathbb{P}^n_{\mathbb{Z}}$ should be the set of isomorphism classes of pairs $(\mathcal{L}, s)$ of a line bundle $\mathcal{L}$ on X and a surjective map of $\mathcal{O}_X$-modules

$$\mathcal{O}_X^{\oplus(n+1)} \xrightarrow{s} \mathcal{L},$$

where we define an isomorphism of pairs $\alpha\colon (\mathcal{L}, s) \to (\mathcal{L}', s')$ to be an isomorphism of $\mathcal{O}_X$-modules $\alpha\colon \mathcal{L} \to \mathcal{L}'$ that makes the diagram

$$\begin{tikzcd} \mathcal{O}_X^{\oplus(n+1)} \arrow[r, "s"] \arrow[d, equal] & \mathcal{L} \arrow[d, "\alpha"] \\ \mathcal{O}_X^{\oplus(n+1)} \arrow[r, "s'"] & \mathcal{L}' \end{tikzcd}$$

commute. Since s and s' are surjective, such an isomorphism α is unique, if it exists. In other words, the category of such pairs $(\mathcal{L}, s)$ and isomorphisms between them is a discrete groupoid, which is just as good as a set. Again, we could also say that the set of maps $\eta\colon X \to \mathbb{P}^n_{\mathbb{Z}}$ should be the set of sub-$\mathcal{O}_X$-modules

$$\mathcal{K} \subset \mathcal{O}_X^{\oplus(n+1)}$$

with the property that the quotient $\mathcal{O}_X^{\oplus(n+1)}/\mathcal{K}$ is a line bundle on X, which would eliminate the need to speak about isomorphisms.

We claim that there exists a scheme $\mathbb{P}^n_{\mathbb{Z}}$ with this property. Let us first justify this somewhat loosely and then provide the proper theoretical context for the argument.

Given $\mathcal{L} \in \mathrm{Line}(X)$, how to understand surjections $s\colon \mathcal{O}_X^{\oplus(n+1)} \to \mathcal{L}$? Such a map s determines and is determined by a family of global sections

$$(\mathcal{O}_X \xrightarrow{s_i} \mathcal{L})_{0 \leq i \leq n}$$

of $\mathcal{L}$, and the map s is surjective if and only if the family $(s_{i,x})_{0\leq i\leq n}$ generates the stalk $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module for all $x \in |X|$. By Nakayama's lemma, this happens if and only if the family $(s_i(x))_{0\leq i\leq n}$ spans the $k(x)$-vector space $\mathcal{L}(x)$ for all $x \in |X|$. But the $k(x)$-vector space $\mathcal{L}(x)$ is 1-dimensional, so the family $(s_i(x))_{0\leq i\leq n}$ spans it if and only if there exists some $0 \leq i \leq n$ such that $s_i(x) \neq 0$. Hence, in terms of the distinguished open subschemes $X_{s_i} \subset X$ from last time, we see that

$$\textstyle\bigcup_{0\leq i\leq n} X_{s_i} = X$$

if and only if the map of $\mathcal{O}_X$-modules $s\colon \mathcal{O}_X^{\oplus(n+1)} \to \mathcal{L}$ is surjective. The restriction of $s_i\colon \mathcal{O}_X \to \mathcal{L}$ along the open immersion $j_i\colon X_{s_i} \to X$ is an isomorphism, so it gives a trivialization of $\mathcal{L}|_{X_{s_i}}$. Using this trivialization, we can transport the n remaining local sections $s_j|_{X_{s_i}}$ over to local sections of $\mathcal{O}_X|_{X_{s_i}}$, namely,

$$g_{i,j} = (s_i|_{X_{s_i}})^{-1} s_j|_{X_{s_i}} \in \mathcal{O}_X(X_{s_i}),$$

or equivalently, maps of schemes

$$X_{s_i} \xrightarrow{g_{i,j}} \mathrm{Spec}(\mathbb{Z}[x_j])$$

with $0 \leq j \leq n$ and $j \neq i$. The family of maps $(g_{i,j})$, in turn, determines and is determined by the single map of schemes

$$X_{s_i} \xrightarrow{g_i} \mathrm{Spec}(\mathbb{Z}[x_0,\dots,\widehat{x}_i,\dots,x_n])$$

classified by the unique ring homomorphism

$$\mathbb{Z}[x_0,\dots,\widehat{x}_i,\dots,x_n] \xrightarrow{\phi_i} \mathcal{O}_X(X_{s_i})$$

with $\phi_i(x_j) = g_{i,j}$. Moreover, if $0 \leq i \neq j \leq n$, then $X_{s_i} \cap X_{s_j}$ is equal to both

$$g_i^{-1}(\mathrm{Spec}(\mathbb{Z}[x_0,\dots,\widehat{x}_i,\dots,x_n][x_j^{-1}])) \subset X_{s_i}$$

and

$$g_j^{-1}(\mathrm{Spec}(\mathbb{Z}[x_0,\dots,\widehat{x}_j,\dots,x_n][x_i^{-1}])) \subset X_{s_j},$$

because, as we showed last time, the inverse image of generalized distinguished open subsets are again generalized distinguished open subsets. In fact, it is also clear that we have a canonical isomorphism of rings

$$\mathbb{Z}[x_0,\dots,\widehat{x}_i,\dots,x_n][x_j^{-1}] \simeq \mathbb{Z}[x_0,\dots,\widehat{x}_j,\dots,x_n][x_i^{-1}],$$

since both rings are canonically isomorphic to

$$\mathbb{Z}[x_0,\dots,x_n]/(x_ix_j - 1).$$

So this tells us how to build the scheme $\mathbb{P}^n_{\mathbb{Z}}$. Namely, glue the $n+1$ copies

$$\mathrm{Spec}(\mathbb{Z}[x_0,\dots,\widehat{x}_i,\dots,x_n])$$

with $0 \leq i \leq n$ of the affine space $\mathbb{A}^n_{\mathbb{Z}}$ along the canonical isomorphisms

$$\mathrm{Spec}(\mathbb{Z}[x_0,\dots,\widehat{x}_i,\dots,x_n])_{x_j} \simeq \mathrm{Spec}(\mathbb{Z}[x_0,\dots,\widehat{x}_j,\dots,x_n])_{x_i}$$

for all $0 \leq i \neq j \leq n$. We can also build a line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n_{\mathbb{Z}}$ by descent: We take the trivial line bundle on $\mathrm{Spec}(\mathbb{Z}[x_0,\dots,\widehat{x}_i,\dots,x_n])$ for all $0 \leq i \leq n$ with the transition function $\alpha_{i,j}$ given by multiplication by x_i. We further get $n+1$ global sections $s_0, \dots, s_n$ of $\mathcal{O}(1)$ such that

$$(\mathbb{P}^n_{\mathbb{Z}})_{s_i} = \operatorname{Spec}(\mathbb{Z}[x_0, \dots, \widehat{x}_i, \dots, x_n]) \subset \mathbb{P}^n_{\mathbb{Z}}$$

for all $0 \leq i \leq n$, and such that for any scheme X, the set of maps of schemes

$$X \xrightarrow{f} \mathbb{P}^n_{\mathbb{Z}}$$

is canonically bijective to the set of isomorphism classes of pairs $(\mathcal{L}, s)$ of a line bundle $\mathcal{L}$ on X and a surjective map of $\mathcal{O}_X$-modules

$$\mathcal{O}_X^{\oplus(n+1)} \xrightarrow{t} \mathcal{L}.$$

Indeed, to a map of schemes $f \colon X \to \mathbb{P}^n_{\mathbb{Z}}$ we assign the isomorphism class of the pair $(f^*(\mathcal{O}(1)), f^*(s_0), \dots, f^*(s_n))$, and to a pair $(\mathcal{L}, t_0, \dots, t_n)$, we assign the unique map of schemes $f \colon X \to \mathbb{P}^n_{\mathbb{Z}}$ that, on $X_{t_i} \subset X$, is given by the map

$$X_{t_i} \xrightarrow{h_i} \operatorname{Spec}(\mathbb{Z}[x_0, \dots, \widehat{x}_i, \dots, x_n])$$

classified by the unique ring homomorphism

$$\mathbb{Z}[x_0, \dots, \widehat{x}_i, \dots, x_n] \xrightarrow{\phi_i} \mathcal{O}_X(X_{t_i})$$

that to x_j assigns $(t_i|_{X_{t_i}})^{-1}(t_j|_{X_{t_i}})$.

The theoretical context for the above is the "functor of points" perspective or the "moduli" perspective or the "Yoneda" perspective. We consider the functor

$$\mathsf{Sch} \xrightarrow{h} \operatorname{Fun}(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Set})$$

that to a scheme X assigns its functor of points $h_X \colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ defined by

$$h_X(T) = \operatorname{Map}(T, X).$$

By the Yoneda lemma, this functor is fully faithful,[8] so knowing a scheme X is equivalent to knowing the set of maps $\eta \colon T \to X$ into X for all schemes T, and how these relate under pullback along maps $g \colon T' \to T$.

Example 10.1. (1) If $X = \mathbb{A}^1_{\mathbb{Z}}$ is the affine line, then the functor of points h_X takes a scheme T to the set of global sections $\mathcal{O}_T(T)$, forgetting its ring structure.
(2) If $X = \operatorname{Spec}(\mathbb{Z}[x,y]/(y^2 - x^3 + 1))$, then the functor of points h_X takes a scheme T to the set of solutions to the equation "$y^2 - x^3 + 1 = 0$" in the ring of global sections $\mathcal{O}_T(T)$. So $h_X(T) = \{(f,g) \in \mathcal{O}_T(T) \times \mathcal{O}_T(T) \mid f^2 - g^3 + 1 = 0\}$.

We say that a functor $F \colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ is representable if it is in the essential image of the Yoneda embedding. While the Yoneda embedding is fully faithful, it is far from being essentially surjective. So not every functor $F \colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ is representable. However, note that if we have isomorphisms $\varphi \colon h_X \to F$ and $\psi \colon h_Y \to F$, then the composite isomorphism $\varphi \circ \psi^{-1} \colon h_Y \to h_X$ is induced by a unique isomorphism of schemes $f \colon Y \to X$. Hence, if F is representable, then it determines the scheme that represents it uniquely, up to unique isomorphism.

So what special properties do the representable functors $h_X \colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ satisfy among all functors $F \colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$?

[8] Since the category Sch of schemes is large, the functor category $\operatorname{Fun}(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Set})$ only exists in a larger universe. We ignore this is issue here.

First, the functor $F = h_X$ is a Zariski sheaf in the sense that if U is a scheme, and if $(U_i)_{i\in I}$ is a cover of U by open subschemes, then the diagram

$$F(U) \xrightarrow{\gamma} \prod_{i\in I} F(U_i) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{(i,j)\in I\times I} F(U_i \cap U_j)$$

with α, β, and γ defined as usual is a limit diagram. Since $F(-) = \mathrm{Map}(-, X)$, this follows from the statement that that the diagram

$$\coprod_{(i,j)\in I\times I} U_i \cap U_j \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \coprod_{i\in I} U_i \xrightarrow{\gamma} U$$

is a colimit diagram in the categories of schemes, which we proved in Theorem 1 in Lecture 7. This sheaf property captures the locality property of representable sheaves. But the most important part of being a scheme is the property of being locally affine. How to express this for a general functor $F\colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$?

Definition 10.2. A subobject of an object F of a category $\mathcal{C}$ is an isomorphism class in $\mathcal{C}_{/F}$ of monomorphisms $j\colon G \to F$.

Remark 10.3. (1) Let $\mathcal{C}$ be a category. An isomorphism in $\mathcal{C}_{/F}$ from $j\colon G \to F$ to $j'\colon G' \to F$ is an isomorphism $\alpha\colon G \to G'$ in $\mathcal{C}$ with the property that the diagram

$$\begin{tikzcd} G \arrow[rr, "\alpha"] \arrow[dr, "j"'] & & G' \arrow[dl, "j'"] \\ & F & \end{tikzcd}$$

commutes. If j and j' are monomorphisms, then such an isomorphism is unique, if it exists. Thus, the (non-full) subcategory of $\mathcal{C}_{/F}$ spanned by the monomorphisms $j\colon G \to F$ and the isomorphisms between these is a discrete groupoid.

(2) If $\mathcal{C} = \mathrm{Fun}(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Set})$, then every isomorphism class in $\mathcal{C}_{/F}$ of monomorphisms $j\colon G \to F$ has a unique representative with the property that for every $T \in \mathsf{Sch}$, the injective map $j_T\colon G(T) \to F(T)$ is equal to the canonical inclusion of a subset. Thus, a subobject of F is uniquely specified by giving a subset $G(T) \subset F(T)$ for every $T \in \mathsf{Sch}$ with the property that for every map $f\colon T \to S$ in Sch, the induced map $F(f)\colon F(S) \to F(T)$ restricts to a map $G(f)\colon G(S) \to G(T)$. In this case, we write $G \subset F$ and say that G is a subfunctor of F.

Definition 10.4. Let $F\colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ be a functor. A subfunctor $G \subset F$ is open if its base-change $j'\colon G' \to h_T$ along any map $f\colon h_T \to G$ from a representable functor is representable by an open subscheme $U \subset T$.

We spell out the definition. First, given the maps $j\colon G \to F$ and $f\colon h_T \to F$, we can form the pullback diagram

$$\begin{tikzcd} G' \arrow[r, "f'"] \arrow[d, "j'"] & G \arrow[d, "j"] \\ h_T \arrow[r, "f"] & F, \end{tikzcd}$$

and in this situation, we say that j' is the base-change of j along f and that f' is the base-change of f along j. Moreover, the statement that an open subscheme

$U \subset T$ represents the map $j' \colon G' \to h_T$ translates to the statement that a map $g \colon T' \to T$ in Sch factors through $U \subset T$ if and only if the composite map

$$h_{T'} \xrightarrow{h_g} h_T \xrightarrow{f} F$$

in $\mathrm{Fun}(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Set})$ factors through $G \subset F$.

We can interpret this further. By the Yoneda lemma, the map

$$\mathrm{Map}(h_T, F) \longrightarrow F(T)$$

that to $f \colon h_T \to F$ assigns $f_T(\mathrm{id}_T) \in F(T)$ is a bijection. So $G \subset F$ is an open subfunctor if and only if for all $T \in \mathsf{Sch}$ and for all $s \in F(T)$, there exists an open subset $U \subset T$ with the property that a map $g \colon T' \to T$ factors through $U \subset T$ if and only if $F(g)(s) \in F(T')$ belongs to the subset $G(T') \subset F(T')$. We refer to this statement by saying that the locus of those $s \in F(T)$ which lie in $G(T) \subset F(T)$ is open in T.

Example 10.5. If $U \subset X$ is open, then $h_U \subset h_X$ is open. Indeed, if $f \colon T \to X$, then the base-change of $h_U \to h_X$ along $h_f \colon h_T \to h_X$ is $h_{f^{-1}(U)} \to h_T$. In fact, the Yoneda embedding restricts to a bijection

$$\{U \subset X \mid \text{open subset}\} \longrightarrow \{G \subset h_X \mid \text{open subfunctor}\}.$$

The inverse map is obtained by applying the definition to $\mathrm{id}_{h_X} \colon h_X \to h_X$.

Theorem 10.6. *A functor $F \colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ is representable if and only if it is a sheaf for the Zariski topology and it is locally affine in the sense that there exists a family $(F_i)_{i \in I}$ of open subfunctors $F_i \subset F$ such that $F \simeq \bigcup_{i \in I} F_i$ as Zariski sheaves and such that each F_i is representable by an affine scheme.*

That $F \simeq \bigcup_{i \in I} F_i$ as Zariski sheaves means that the canonical map $\bigcup_{i \in I} F_i \to F$ becomes an isomorphism after sheafification, or equivalently, that it becomes a surjection after sheafification. This, in turn, is equivalent to the statement that for every scheme T and every map $f \colon h_T \to F$, there exists an open cover $(U_i)_{i \in I}$ of T such that the restriction $f|_{U_i} \colon h_{U_i} \to F$ factors through $F_i \to F$.

Proof. Suppose first that $F \simeq h_X$ is representable. We proved earlier that F is a Zariski sheaf, so we must prove that F is locally affine. In fact, if $(U_i)_{i \in I}$ is an affine open cover of X, then $h_X = \bigcup_{i \in I} h_{U_i}$ as Zariski sheaves. Indeed, given $T \in \mathsf{Sch}$ and $f \colon h_T \to h_X$, there is a unique map $g \colon T \to X$ with $f = h_g$, so $(g^{-1}(U_i))_{i \in I}$ is the required open cover of T.

Conversely, if F is both a sheaf for the Zariski topology and locally affine, then we may write $F = \bigcup_{i \in I} F_i$ as a Zariski sheaf with F_i representable by an affine scheme Y_i. Moreover, the intersection $F_i \cap F_j$ is representable both by an open subscheme $U_{i,j} \subset Y_i$ and by an open subscheme $U_{j,i} \subset Y_j$. So by the Yoneda lemma, we get an isomorphism $\varphi_{i,j} \colon U_{i,j} \to U_{j,i}$ between these open subschemes. This gives us the gluing data to build a scheme X using Theorem 1 of Lecture 7. Indeed, we let $Y = \coprod_{i \in I} Y_i$ and $R = \coprod_{i,j} U_{i,j}$ and define $(s,t) \colon R \to Y \times Y$ to be the map, where $s|_{U_{i,j}}$ is the composite map

$$U_{i,j} \longrightarrow Y_i \longrightarrow Y,$$

and where $t|_{U_{i,j}}$ is the composite map

$$U_{i,j} \xrightarrow{\varphi_{i,j}} U_{j,i} \longrightarrow Y_j \longrightarrow Y.$$

We must argue that $(s,t)\colon R \to Y \times Y$ is an equivalence relation, or equivalently, that the family of isomorphisms $(\varphi_{i,j})_{(i,j)\in I\times I}$ satisfies the cocycle condition. Now, for the family consisting of the corresponding isomorphisms between the functors $F_i \cap F_j$ represented by the $U_{i,j}$'s, the cocycle condition is tautologically satisfied. But then it is also satisfied for the family $(\varphi_{i,j})_{(i,j)\in I\times I}$ by the Yoneda lemma. Hence, it follows from Theorem 1 of Lecture 7 that the quotient scheme $X \simeq Y/R$ exists. It remains to show that the scheme X represents F. But, as Zariski sheaves, we have $F \simeq \bigcup_{i\in I} F_i$ and $h_X \simeq \bigcup_{i\in I} h_{Y_i}$ and the isomorphisms $F_i \simeq h_{Y_i}$ glue to give the desired isomorphism. $\square$

Let us now revisit the construction of $\mathbb{P}^n_{\mathbb{Z}}$. We define a functor

$$\mathsf{Sch}^{\mathrm{op}} \xrightarrow{F} \mathsf{Set}$$

as follows. It takes a scheme T to the set $F(T)$ of isomorphism classes of tuples

$$(\mathcal{L}, s_0, \dots, s_n),$$

where $\mathcal{L}$ a line bundle on T, and where $s_0, \dots, s_n \in \mathcal{L}(T)$ are global sections with the property that $T = \bigcup_{0\leq i\leq n} T_{s_i}$. And it takes a map of schemes $f\colon T' \to T$ to the map $F(f)\colon F(T) \to F(T')$ that to the isomorphism class of $(\mathcal{L}, s_0, \dots, s_n)$ assigns the isomorphism class of $(f^*(\mathcal{L}), f^*(s_0), \dots, f^*(s_n))$. It is well-defined, because

$$T'_{f^*(s_i)} = f^{-1}(T_{s_i}),$$

as we proved last time. We wish to show that F is representable, so we must check that it satisfies the hypotheses of Theorem 10.6.

Lemma 10.7. *The functor $F\colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ is a Zariski sheaf.*

Proof. Given a scheme U and a covering $(U_i)_{i\in I}$ of U by open subschemes, we must show that the diagram

$$F(U) \xrightarrow{\gamma} \prod_{i\in I} F(U_i) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{(i,j)\in I\times I} F(U_i \cap U_j)$$

is a limit diagram. So let $(x_i)_{i\in I}$ be a family with $x_i \in F(U_i)$ such that

$$\mathrm{res}^{U_i}_{U_i\cap U_j}(x_i) = \mathrm{res}^{U_j}_{U_i\cap U_j}(x_j)$$

for all $(i,j) \in I \times I$. If we choose a representative $(\mathcal{L}_i, s_{0,i}, \dots, s_{n,i})$ of x_i for all $i \in I$, then for all $(i,j) \in I \times I$, there is a *unique* isomorphism

$$\mathcal{L}_j|_{U_i\cap U_j} \xrightarrow{\alpha_{i,j}} \mathcal{L}_i|_{U_i\cap U_j}$$

with the property that $\alpha_{i,j}(s_{h,j}|_{U_i\cap U_j}) = s_{h,i}|_{U_i\cap U_j}$ for all $0 \leq h \leq n$. Indeed, as we noted earlier, if such an isomorphism exists, then it is unique. Therefore, these isomorphisms necessarily satisfy the cocycle condition, so we can glue to get

$$(\mathcal{L}, s_0, \dots, s_n),$$

which represents an element $x \in F(U)$ with $\gamma(x) = (x_i)_{i\in I}$. A similar argument shows that γ is injective, so we are done. $\square$

Lemma 10.8. *The functor $F\colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ is locally affine.*

Proof. We define $F_i \subset F$ to be the subfunctor such that $F_i(T) \subset F(T)$ consists of the $x \in F(T)$ that are represented by tuples $(\mathcal{L}, s_0, \dots, s_n)$ such that $T_{s_i} = T$.

It is an open subfunctor. Indeed, if T is a scheme and if $(\mathcal{L}, s_0, \dots, s_n)$ represents an element $x \in F(T)$, then for $f\colon T' \to T$, the tuple $f^*(\mathcal{L}, s_0, \dots, s_n)$ represents an element of $F_i(T')$ if and only if $f^{-1}(T_{s_i}) = T'$ if and only if $f\colon T' \to T$ factors through the open subscheme $T_{s_i} \subset T$.

Moreover, as Zariski sheaves $F \simeq \bigcup_{0 \leq i \leq n} F_i$. Indeed, if T is a scheme and if the tuple $(\mathcal{L}, s_0, \dots, s_n)$ represents an element $x \in F(T)$, then $T = \bigcup_{0 \leq i \leq n} T_{s_i}$, and therefore, the restriction $x|_{T_{s_i}} \in F(T_{s_i})$ belongs to $F_i(T_{s_i})$.

The subfunctor $F_i \subset F$ is represented by an affine scheme, namely,

$$F_i \simeq h_{X_i}$$

with $X_i = \operatorname{Spec}(\mathbb{Z}[x_0, \dots, \widehat{x}_i, \dots, x_n])$. Indeed, the map

$$F_i(T) \longrightarrow F_i'(T) = \{(g_0, \dots, \widehat{g}_i, \dots, g_n) \mid g_i \in \mathcal{O}_T(T)\}$$

that to the class of $(\mathcal{L}, s_0, \dots, s_n)$ assigns $(g_0, \dots, \widehat{g}_i, \dots, g_n)$ with $g_j = s_i^{-1} s_j$ is a bijection, whose inverse is the map that to $(g_0, \dots, \widehat{g}_i, \dots, g_n)$ assigns the class of $(\mathcal{O}_T, g_0, \dots, 1, \dots, g_n)$, and the natural transformation

$$h_{X_i} \longrightarrow F_i'$$

corresponding to $(x_0, \dots, \widehat{x}_i, \dots, x_n) \in F_i'(X_i)$ is a natural isomorphism. $\square$

Given Lemmas 10.7 and 10.8, we conclude from Theorem 10.6 that the functor

$$\mathsf{Sch}^{\mathrm{op}} \xrightarrow{F} \mathsf{Set}$$

is representable. We define projective n-space over $\mathbb{Z}$ to be a pair

$$(X, \varphi)$$

of a scheme X and a natural isomorphism $\varphi\colon h_X \to F$. Such a pair exists and is unique, up to unique isomorphism over F, which justifies that we write $\mathbb{P}^n_{\mathbb{Z}}$ for any such X.

By the Yoneda lemma, to give a natural transformation

$$h_X \xrightarrow{\varphi} F$$

amounts to giving a tuple $(\mathcal{O}(1), s_0, \dots, s_n)$, where $\mathcal{O}(1)$ is a line bundle on X, and where $s_0, \dots, s_n \in \mathcal{O}(1)(X)$ are global sections such that

$$X = \bigcup_{0 \leq i \leq n} X_{s_i}.$$

Moreover, for φ to be an isomorphism, it must have the additional property that for every scheme T and every tuple $(\mathcal{L}, t_0, \dots, t_n)$, where $\mathcal{L}$ is a line bundle on T, and where $t_0, \dots, t_n \in \mathcal{L}(T)$ are global sections such that $T = \bigcup_{0 \leq i \leq n} T_{t_i}$, there exists a unique map of scheme $f\colon T \to X$ such that

$$(\mathcal{L}, t_0, \dots, t_n) \simeq f^*(\mathcal{O}(1), s_0, \dots, s_n).$$

This is a very abstract description of $\mathbb{P}^n_{\mathbb{Z}}$, but it really matches the intuitive idea!

A fiber product in a category $\mathcal{C}$ is a limit diagram of the form

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X. \end{tikzcd}$$

We also express this by saying that the diagram is cartesian. Spelling out the definition of a limit diagram, this means that, given any commutative diagram

$$\begin{tikzcd} Z \arrow[r, "b"] \arrow[d, "a"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X, \end{tikzcd}$$

there exists a *unique* map $h\colon Z \to Y'$ such that $a = f' \circ h$ and $b = g' \circ h$. In this situation, we also say that f' is the base-change of f along g and that g' is the base-change of g along f. We also write $(a, b)\colon Z \to Y'$ for the unique map.

Example 11.1. In the category of sets, a square diagram

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

is cartesian if and only if the map $Y' \to X' \times Y$ that to y' assigns $(f'(y'), g'(y')$ is injective and its image consists of the pairs (x', y) such that $f(y) = g(x')$.

Example 11.2. In the functor category $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$, a diagram

$$\begin{tikzcd} G' \arrow[r] \arrow[d] & G \arrow[d] \\ F' \arrow[r] & F \end{tikzcd}$$

is cartesian if and only if the induced diagram of sets

$$\begin{tikzcd} G'(S) \arrow[r] \arrow[d] & G(S) \arrow[d] \\ F'(S) \arrow[r] & F(S) \end{tikzcd}$$

is cartesian for all objects S in $\mathcal{C}$. This is a special case of the general fact that, in a functor category, limits and colimits are calculated pointwise.

The definition of a fiber product and the description of fiber products in the category of sets shows immediately that the Yoneda embedding

$$\mathcal{C} \xrightarrow{h} \mathcal{P}(\mathcal{C})$$

preserves and reflects fiber products. This means that a diagram

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

in $\mathcal{C}$ is cartesian if and only if its image diagram

$$\begin{tikzcd} h_{Y'} \arrow[r, "h_{g'}"] \arrow[d, "h_{f'}"] & h_Y \arrow[d, "h_f"] \\ h_{X'} \arrow[r, "h_g"] & h_X \end{tikzcd}$$

in $\mathcal{P}(\mathcal{C})$ is cartesian. Now, for every diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X, \end{tikzcd}$$

the induced diagram in $\mathcal{P}(\mathcal{C})$ can be completed to a cartesian diagram

$$\begin{tikzcd} G' \arrow[r, "(h_g)'"] \arrow[d, "(h_f)'"] & h_Y \arrow[d, "h_f"] \\ h_{X'} \arrow[r, "h_g"] & h_X. \end{tikzcd}$$

Hence, the original diagram in $\mathcal{C}$ can be completed to a fiber product

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

if and only if the functor G' is representable. In this case, we say that $\mathcal{C}$ admits fiber products. It is also common to write $Y \times_X X'$ for Y' with the understanding that this object, if it exists, is well-defined, up to unique isomorphism.

Theorem 11.3. *The category of schemes admits fiber products.*

Proof. We have already translated the problem into the problem of showing that in every cartesian diagram in $\mathcal{P}(\mathsf{Sch})$ of the form

$$\begin{tikzcd} G' \arrow[r, "(h_g)'"] \arrow[d, "(h_f)'"] & h_Y \arrow[d, "h_f"] \\ h_{X'} \arrow[r, "h_g"] & h_X, \end{tikzcd}$$

the functor $G' \colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ is representable, and to prove this, we apply Theorem 6 from Lecture 10. Since the sheaf property is expressed in terms of limits, the full subcategory spanned by the Zariski sheaves

$$\mathrm{Sh}(\mathsf{Sch}) \subset \mathcal{P}(\mathsf{Sch})$$

is closed under limits. In particular, the functor G' is a Zariski sheaf. So it remains to prove that it is locally affine.

In Theorem 15 of Lecture 5, we proved that there is an adjunction

$$\begin{tikzcd} \mathsf{Sch} \arrow[r, shift left, "\mathcal{O}(-)"] & \mathrm{CAlg}(\mathsf{Ab})^{\mathrm{op}}. \arrow[l, shift left, "\mathrm{Spec}"] \end{tikzcd}$$

The opposite of the category of commutative rings admits fiber products

$$\begin{tikzcd} B' & B \arrow[l, "\psi'"'] \\ A' \arrow[u, "\phi'"] & A, \arrow[l, "\psi"] \arrow[u, "\phi"'] \end{tikzcd}$$

which are given by the tensor product $B' \simeq B \otimes_A A'$. Being a right adjoint, the functor Spec preserves all limits that exist in its domain, so we conclude that the functor $G' \colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ is representable, if X, Y, and X' are affine schemes.

Lemma 11.4. *If $U \subset X$, $V \subset Y$, and $U' \subset X'$ are open subschemes with $f(V) \subset U$ and $g(U') \subset U$, then $h_V \times_{h_U} h_{U'} \subset h_Y \times_{h_X} h_{X'}$ is an open subfunctor.*

Proof. A map $\alpha \colon h_T \to h_Y \times_{h_X} h_{X'}$ determines and is determined by a diagram

$$\begin{tikzcd} T \arrow[r, "b"] \arrow[d, "a"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

of schemes. Moreover, given a map of schemes $q \colon T' \to T$, the composite map

$$h_{T'} \xrightarrow{h_q} h_T \xrightarrow{\alpha} h_Y \times_{h_X} h_{X'}$$

factors through the subfunctor $h_V \times_{h_U} h_{U'} \subset h_Y \times_{h_X} h_{X'}$ if and only if q factors through $a^{-1}(U') \cap b^{-1}(V) \subset T$, which is an open subscheme. So the subfunctor in question is open as stated. ☐

Hence, to show that $h_Y \times_{h_X} h_{X'}$ is locally affine, it suffices to show that

$$h_Y \times_{h_X} h_{X'} \simeq \textstyle\bigcup_{(V,U,U')} h_V \times_{h_U} h_{U'}$$

as Zariski sheaves, where the union ranges over triples (V, U, U') of affine open subschemes $V \subset Y$, $U \subset X$, and $U' \subset X'$ with $f(V) \subset U$ and $g(U') \subset U$. Indeed, we proved in Lemma 11.4 that $h_V \times_{h_U} h_{U'}$ is an open subfunctor of $h_Y \times_{h_X} h_{X'}$, and we proved earlier that it representable by an affine scheme. To this end, we first choose a cover $(U_i)_{i \in I}$ of X by affine open subschemes. Next, for every $i \in I$, we choose affine open covers $(V_{i,j})_{j \in J_i}$ of $f^{-1}(U_i)$ and $(U'_{i,k})_{k \in K_i}$ of $g^{-1}(U_i)$. Now, given a scheme T and a map $\alpha\colon h_T \to h_Y \times_{h_X} h_{X'}$ corresponding to a diagram

$$\begin{tikzcd} T \arrow[r, "b"] \arrow[d, "a"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

of schemes, the family $(W_{i,j,k})_{i\in I, j\in J_i, k\in K_i}$ consisting of the of open subschemes

$$W_{i,j,k} = b^{-1}(V_{i,j}) \cap a^{-1}(U'_{i,k}) \subset T$$

covers T, and the restriction

$$h_{W_{i,j,k}} \xrightarrow{\alpha|_{W_{i,j,k}}} h_Y \times_{h_X} h_{X'}$$

factors through $h_{V_{i,j}} \times_{h_{U_i}} h_{U'_{i,k}} \subset h_Y \times_{h_X} h_{X'}$. This completes the proof. □

Example 11.5. Suppose that $j\colon U \to X$ is the open immersion of an open subscheme and that $g\colon X' \to X$ is any map of schemes. In this situation, the diagram

$$\begin{tikzcd} U' \arrow[r, "g'"] \arrow[d, "j'"] & U \arrow[d, "j"] \\ X' \arrow[r, "g"] & X, \end{tikzcd}$$

where $j'\colon U' \to X'$ is the open immersion of the open subscheme, whose underlying topological space is the open subset $g^{-1}(|U|) \subset |X'|$ is cartesian. Indeed, we have proved that for every scheme Z, composition with j defines an injective map

$$\mathrm{Map}(Z, U) \longrightarrow \mathrm{Map}(Z, X)$$

whose image consists of the set of maps of schemes $f = (p, \phi)\colon Z \to X$ with the property that $p(|Z|) \subset |U| \subset |X|$.

Why do we care about fiber products? As we will see below, they perform many important roles. The first role is that of intersections of subschemes.

Definition 11.6. A subscheme of a scheme X is an isomorphism class in $\mathsf{Sch}_{/X}$ of maps $f\colon Y \to X$ with the property that $h_f\colon h_Y \to h_X$ is a monomorphism.

That $h_f\colon h_Y \to h_X$ is a monomorphism means that for every factorization

$$\begin{tikzcd} & W \arrow[dl, "\tilde{g}"'] \arrow[dr, "g"] & \\ Y \arrow[rr, "f"] & & X \end{tikzcd}$$

of a map $g\colon W \to X$ through $f\colon Y \to X$, the map $\tilde{g}\colon W \to Y$ is uniquely determined by g. Thus, the existence of such a factorization is a condition on the map g.

Example 11.7. (1) An open subscheme $U \subset X$ and a closed subscheme $Z \subset X$ are both subschemes of X.

(2) If $f\colon Y = \mathrm{Spec}(k[t]) \to X = \mathrm{Spec}(k)$ is the affine line, then $h_f\colon h_Y \to h_X$ is not a monomorphism. Indeed, a factorization of $\mathrm{id}_X\colon X \to X$ through $f\colon Y \to X$ is a section $s\colon X \to Y$ of $f\colon Y \to X$. Given $a \in k$, the unique k-algebra homomorphism

$\phi_a\colon k[t] \to k$ with $\phi_a(t) = a$ defines a section $s_a\colon X \to Y$, and conversely, every section is of this form. In particular, sections s of f are not unique.

Definition 11.8. If $f\colon Y \to X$ and $g\colon X' \to X$ represent subschemes, then the scheme-theoretic intersection of the two subschemes in question is the subscheme represented by the composite map $fg' = gf'\colon Y' \to X$, where

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

is a fiber product.

Example 11.9. (1) If $U, V \subset X$ are open subschemes, then

$$U \times_X V \simeq U \cap V \subset X.$$

So their scheme-theoretic intersection and their set-theoretic intersections agree and form an open subscheme of X.

(2) Suppose that $f\colon Y \to X$ and $g\colon X' \to X$ are closed immersions, so that their isomorphism classes in $\mathsf{Sch}_{/X}$ are closed subschemes of X. So they correspond to quasicoherent ideals $\mathcal{J} \subset \mathcal{O}_X$ and $\mathcal{I}' \subset \mathcal{O}_X$, and we claim that the square

$$\begin{tikzcd} Y' \simeq \operatorname{Spec}(\mathcal{O}_X/\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}') \arrow[r, "g'"] \arrow[d, "f'"] & Y \simeq \operatorname{Spec}(\mathcal{O}_X/\mathcal{J}) \arrow[d, "f"] \\ X' \simeq \operatorname{Spec}(\mathcal{O}_X/\mathcal{I}') \arrow[r, "g"] & X \simeq \operatorname{Spec}(\mathcal{O}_X) \end{tikzcd}$$

is cartesian. Indeed, this follows from the proof of Theorem 11.3: If X, X' and Y are all affine, then so is Y' and the statement follows from the first part of the proof. In general, we choose a covering $(U_i)_{i\in I}$ of X by affine open subschemes. The maps f and g are, in particular, affine, so the families $(f^{-1}(U_i))_{i\in I}$ and $(g^{-1}(U_i))_{i\in I}$ are affine open covers of Y and X', respectively. Since f and g are closed immersions, if $U_i \simeq \operatorname{Spec}(R_i)$, then $f^{-1}(U_i) \simeq \operatorname{Spec}(R_i/J_i)$ and $g^{-1}(U_i) \simeq \operatorname{Spec}(R_i/I_i')$, so

$$f^{-1}(U_i) \times_{U_i} g^{-1}(U_i) \simeq \operatorname{Spec}(R_i/J_i \otimes_{R_i} R_i/I_i'),$$

which proves the claim. We note that $\mathcal{O}_X/\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}'$ is a quotient of $\mathcal{O}_X$ by a quasicoherent ideal.

Example 11.10. Let us now calculate the scheme-theoretic intersection of the closed subschemes "$y = x^2$" and "$y = 0$" in $X = \operatorname{Spec}(k[x,y])$ that we first considered in Lecture 1. By definition, it is given by the fiber product

$$\begin{aligned} &\operatorname{Spec}(k[x,y]/(y-x^2)) \times_{\operatorname{Spec}(k[x,y])} \operatorname{Spec}(k[x,y]/(y) \\ &\simeq \operatorname{Spec}(k[x,y]/(y-x^2,y)) \simeq \operatorname{Spec}(k[x,y]/(x^2,y)) \simeq \operatorname{Spec}(k[x]/(x^2)), \end{aligned}$$

so the desired multiplicity 2 of this intersection is encoded by the scheme-theoretic intersection.

The second role of the fiber product is that of product of varieties. If k is a field, then we (re)define a variety over k to be a map of schemes

$$Y \xrightarrow{f} X \simeq \operatorname{Spec}(k)$$

that is quasicompact and, locally on Y, isomorphic in $\mathsf{Sch}_{/X}$ to the map of prime spectra induced by a finitely generated and reduced k-algebra $\phi\colon k \to A$. The product of varieties of k is (re)defined to be the fiber product of schemes over $X \simeq \operatorname{Spec}(k)$. For example, we have

$$\mathbb{A}^1_k \times_{\operatorname{Spec}(k)} \mathbb{A}^1_k \simeq \mathbb{A}^2_k,$$

because $k[x] \otimes_k k[y] \simeq k[x,y]$.

The third role is that of fibers of maps. If $f\colon Y \to X$ is a map of schemes, then we define its fiber over $x \in |X|$ to be the fiber product

$$\begin{tikzcd} Y_x \arrow[r, "i'_x"] \arrow[d, "f_x"] & Y \arrow[d, "f"] \\ \operatorname{Spec}(k(x)) \arrow[r, "i_x"] & X. \end{tikzcd}$$

So we may think of the map f as a "family of varieties" parametrized by the base scheme X. This may not be literally true, since the map f_x may not be a variety over $k(x)$ — it is just a general $k(x)$-scheme. Still, it's important to note that there are two very different ways to think of a map of schemes: We may either think of it as a family of schemes and write it vertically as

$$\begin{tikzcd} Y \arrow[d, "f"] \\ X \end{tikzcd}$$

or we may think of it as a Y-valued point of X and write it horizontally as

$$Y \xrightarrow{f} X.$$

Both points of view are due to Grothendieck, and he was the first person to write a map vertically!

Example 11.11. The map of schemes

$$\begin{tikzcd} Y \simeq \operatorname{Spec}(\mathbb{Z}[x,y]/(y^2 - x^3 + 7x + 2)) \arrow[d, "f"] \\ X \simeq \operatorname{Spec}(\mathbb{Z}) \end{tikzcd}$$

is a family of plane curves

$$\begin{tikzcd} Y' \simeq \operatorname{Spec}(k[x,y]/(y^2 - x^3 + 7x + 2)) \arrow[r, "i'"] \arrow[d, "f'"] & Y \simeq \operatorname{Spec}(\mathbb{Z}[x,y]/(y^2 - x^3 + 7x + 2)) \arrow[d, "f"] \\ X' \simeq \operatorname{Spec}(k) \arrow[r, "i"] & X \simeq \operatorname{Spec}(\mathbb{Z}). \end{tikzcd}$$

We note that given $f\colon Y \to X$ and $g\colon Z \to X$, the diagram

$$\begin{tikzcd} Y_x \times_{\operatorname{Spec}(k(x))} Z_x \arrow[r] \arrow[d] & Y \times_X Z \arrow[d] \\ \operatorname{Spec}(k(x)) \arrow[r] & X \end{tikzcd}$$

is cartesian. This tells us that the fiber of $Y \times_X Z \to X$ over $x \in |X|$ is the product of varieties $Y_x \times_{\mathrm{Spec}(k(x))} Z_x$ over $k(x)$.

Warning 11.12. If $f\colon Y \to X$ and $g\colon X' \to X$ are maps of schemes, then there is a continuous surjective map of topological spaces

$$|Y \times_X X'| \xrightarrow{p} |Y| \times_{|X|} |X'|$$

and one can show that if $f(y) = x = g(x')$, then

$$p^{-1}(y, x') \simeq |\operatorname{Spec}(k(y) \otimes_{k(x)} k(x'))|.$$

So the map p is typically not injective!

Remark 11.13. Grothendieck formulated the principle that for a property of maps of schemes to be considered a geometric property, it must be stable under base-change. This means that if $f\colon Y \to X$ has the property in question, then the base-change

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

of f along *any* map g must again have the same property. The properties of being an open immersion and a closed immersion are both geometric properties.

Finally, we will define a new geometric property of maps of schemes, namely, that of being proper. A map $f\colon Y \to X$ is defined to be proper, if it satisfies the following conditions, to be explained:

(1) The map f is locally of finite type and qcqs ("quasicompact quasiseparated").
(2) The map f satisfies the valuative criterion for properness.

Definition 11.14. A scheme X is qcqs if it is a finite union of affine open subschemes U_i such that each intersection $U_i \cap U_j$ also is a finite union of affine open subschemes. A map of schemes $f\colon Y \to X$ is qcqs if $f^{-1}(U) \subset Y$ is qcqs for every affine open $U \subset X$.

A scheme X is qcqs if and only if its underlying space $|X|$ is quasicompact and the intersection of every pair of quasicompact open subsets of $|X|$ is quasicompact.

Definition 11.15. A map of schemes $f\colon Y \to X$ is locally of finite type, if there exists a cover $(U_i)_{i \in I}$ of X by affine open subschemes such that $f^{-1}(U_i)$ is locally isomorphic in $\mathsf{Sch}_{/U_i}$ to the map of prime spectra induced by a finitely generated $\mathcal{O}_X(U_i)$-algebra.

Remark 11.16. In Definition 11.15, we only require that for some cover $(U_i)_{i \in I}$ of X by affine open subschemes, the base-change of $f\colon Y \to X$ along the open immersions $j_i\colon U_i \to X$ is, locally on $Y \times_X U_i$, isomorphic to the map of prime spectra induced by a finitely generated $\mathcal{O}_X(U_i)$-algebra. However, this implies that for any map $g\colon X' \to X$ from an affine scheme, the base-change $f'\colon Y' \to X'$ of f along g is, locally on Y', Y', isomorphic to the map of prime spectra induced by a finitely generated $\mathcal{O}_{X'}(X')$-algebra.

Being qcqs and locally of finite type are just reasonable finiteness conditions. The key to properness lies in the valuative criterion. For intuition, let us first think of the case $X \simeq \operatorname{Spec}(k)$ and work in the context of varieties, but using the scheme-theoretic language. The idea is that a variety $f\colon Y \to X \simeq \operatorname{Spec}(k)$ is proper, if it has no missing point. For example, projective space

$$Y \simeq \mathbb{P}^n_k \xrightarrow{\ f\ } X$$

is proper, but if $Z \subset Y$ is a closed subscheme with $Z \neq \emptyset$ and $Z \neq Y$, then the open complement $U = Y \smallsetminus Z \to X$ is not proper. How can we "see" that the points in Z are missing? There are still generic points in U, which "touch" the missing Z in that they specialize into Z. More precisely, if we let $z \in |Z|$ and let $\mathcal{O}_{Y,z}$ be the local ring of Y at z, then under the canonical map

$$\operatorname{Spec}(\mathcal{O}_{Y,z}) \longrightarrow Y,$$

a generic point $\eta \in \operatorname{Spec}(\mathcal{O}_{Y,z})$ is mapped to a point in U, whereas the closed point $z \in \operatorname{Spec}(\mathcal{O}_{Y,z})$ is mapped to $z \in Z \subset Y$. We can simplify a bit and replace the local ring $\mathcal{O}_{Y,z}$ by its image in the residue field of the generic point.

$$\begin{tikzcd}
\mathcal{O}_{Y,z} \arrow[rr] \arrow[dr, two heads] & & k(\eta) \\
& \widetilde{\mathcal{O}}_{Y,z} \arrow[ur, hook] &
\end{tikzcd}$$

The ring $\widetilde{\mathcal{O}}_{Y,z}$ is still local, and the quotient map is a local map. So we can "see" missing points as maps from prime spectra of fields to U, which do not extend to the prime spectra of certain local rings contained in the field. This leads to:

Definition 11.17. A map of schemes $f\colon Y \to X$ satisfies the valuative criterion for properness if for every diagram

$$\begin{tikzcd}
\operatorname{Spec}(F) \arrow[r, "\eta"] \arrow[d] & Y \arrow[d, "f"] \\
\operatorname{Spec}(R) \arrow[r, "\bar{\eta}_0"] & X
\end{tikzcd}$$

with F a field and $R \subset F$ a local subring, there exists $R \subset R' \subset F$ with R' a local ring and $R \to R'$ a local map and a unique map $\bar{\eta}'\colon \operatorname{Spec}(R') \to Y$ such that

$$\begin{tikzcd}
\operatorname{Spec}(F) \arrow[r, "\eta"] \arrow[d] & Y \arrow[d, "f"] \\
\operatorname{Spec}(R') \arrow[r, "\bar{\eta}'_0"] \arrow[ur, "\bar{\eta}'"] & X
\end{tikzcd}$$

commutes.

We can simplify the definition to eliminate the enlargement $R \subset R' \subset F$.

Definition 11.18. A valuation ring is an integral domain R with the property that for every nonzero element $x \in F$ in its quotient field, $x \in R$ or $x^{-1} \in R$.

Lemma 11.19. (1) *If R is an integral domain with quotient field F, then R is a valuation ring if and only if R is local and if the only $R \subset R' \subset F$ with R' a local ring and the inclusion $R \to R'$ being a local map is $R' = R$.*

(2) *If F is a field and if $R \subset F$ is a local ring, then there exists a valuation ring $R \subset R' \subset F$ such that $R \to R'$ is local.*

Proof. See e.g. [5]. □

With this simplification, we can restate the definition of the valuative criterion for properness as follows:

Definition 11.20. A map of schemes $f\colon Y \to X$ satisfies the valuative criterion for properness if for every solid diagram

$$\begin{tikzcd} \mathrm{Spec}(F) \arrow[r, "\eta"] \arrow[d] & Y \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r, "\bar{\eta}_0"] \arrow[ur, dashed, "\bar{\eta}"] & X \end{tikzcd}$$

in which F is a field and $R \subset F$ is a valuation ring, there exists a unique dotted map $\bar{\eta}\colon \mathrm{Spec}(R) \to Y$ making the diagram commute.

To first approximation, you can think of the case where R has two points, one closed and one generic. The map $\bar{\eta}_0$ then picks out a specialization of points in X, and the commutative square gives a lift of the more generic point to Y. Then the dashed arrow says the whole specialization lifts.

We recall that a valuation ring is an integral domain R with the property that for every nonzero element $x \in F$ in its quotient field, $x \in R$ or $x^{-1} \in R$ or both. Equivalently, an integral domain R with quotient field F is a valuation ring if it is maximal among local rings contained in F in the sense that

(1) the ring R is local, and
(2) if $R \subset R' \subset F$ is a local ring and $\mathfrak{m}' \cap R = \mathfrak{m}$, then $R = R'$.

Moreover, given any local integral domain R_0 with quotient field F, there exists a valuation ring $R \subset F$ such that $R_0 \subset R$ and $\mathfrak{m} \cap R_0 = \mathfrak{m}_0$.

Definition 12.1. A map of schemes $f\colon Y \to X$ satisfies the valuative criterion for properness if for every solid diagram

$$\begin{tikzcd} \operatorname{Spec}(F) \arrow[r, "\eta"] \arrow[d] & Y \arrow[d, "f"] \\ \operatorname{Spec}(R) \arrow[r, "\bar{\eta}_0"] \arrow[ur, dashed, "\exists ! \bar{\eta}"] & X \end{tikzcd}$$

in which F is a field and $R \subset F$ is a valuation ring, there exists a unique dotted map $\bar{\eta}\colon \operatorname{Spec}(R) \to Y$ making the diagram commute.

An advantage of this kind of "lifting property" definition is that it makes it easy to prove that it is geometric.

Lemma 12.2. *In the category of schemes, the following holds:*

(1) *If $f\colon Y \to X$ satisfies the valuative criterion for properness, then so does the base-change $f'\colon Y' \to X'$ along any map $g\colon X' \to X$.*
(2) *If $f\colon Y \to X$ and $g\colon Z \to Y$ satisfy the valuative criterion for properness, then so does their composition $f \circ g\colon Z \to X$.*
(3) *If $f\colon Y \to X$ has the property that there exists a cover $(U_i)_{i\in I}$ of X by open subschemes such that $f|_{f^{-1}(U_i)}\colon f^{-1}(U_i) \to U_i$ satisfies the valuative criterion for properness for all $i \in I$, then f satisfies the valuative criterion for properness.*
(4) *Given composable maps $f\colon Y \to X$ and $g\colon Z \to Y$, if $f \circ g$ and f satisfy the valuative criterion for properness, then so does g.*

Proof. (1) By the universal property of fiber products, given a diagram of schemes

$$\begin{tikzcd} \operatorname{Spec}(F) \arrow[r, "\eta"] \arrow[d] & Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ \operatorname{Spec}(R) \arrow[r, "\bar{\eta}_0"] & X' \arrow[r, "g"] & X \end{tikzcd}$$

with F a field and $R \subset F$ a valuation ring, a lift in the right-hand square determines and is determined a lift in the outer square.

(2) Given a diagram

$$\begin{tikzcd}
\mathrm{Spec}(F) \arrow[r, "\eta"] \arrow[dd] & Z \arrow[d, "g"] \\
& Y \arrow[d, "f"] \\
\mathrm{Spec}(R) \arrow[r, "\bar{\eta}_0"] \arrow[uur, dashed, "\bar{\eta}"] \arrow[ur, dashed, "\overline{g\eta}"] & X
\end{tikzcd}$$

with F a field and $R \subset F$ a valuation ring, we first find the lift $\overline{g\eta}$ using that f satisfies the valuative criterion for properness, and then find the lift $\bar{\eta}$ using that g satisfies the valuative criterion for properness.

(3) Suppose that we are given a diagram

$$\begin{tikzcd}
\mathrm{Spec}(F) \arrow[r, "\eta"] \arrow[d] & Y \arrow[d, "f"] \\
\mathrm{Spec}(R) \arrow[r, "\bar{\eta}_0"] & X
\end{tikzcd}$$

with F a field and $R \subset F$ a valuation ring. Since R is a local ring, we know from Problem set 8 that the image of $\bar{\eta}_0$ is fully contained in $U_i \subset X$ for some $i \in I$, and hence, the image of η is fully contained in $V_i = f^{-1}(U_i) \subset Y$. In the diagram

$$\begin{tikzcd}
\mathrm{Spec}(F) \arrow[r, "\eta"] \arrow[d] & V_i \arrow[r] \arrow[d, "f|_{V_i}"] & Y \arrow[d, "f"] \\
\mathrm{Spec}(R) \arrow[r, "\bar{\eta}_0"] \arrow[ur, dashed, "\bar{\eta}"] & U_i \arrow[r] & X,
\end{tikzcd}$$

the right-hand square is cartesian, so the indicated lift in the left-hand square, which exists and is unique by our assumption that $f|_{V_i}$ satisfies the valuative criterion for properness, gives the desired unique lift in the outer square.

(4) Given a diagram

$$\begin{tikzcd}
\mathrm{Spec}(F) \arrow[r, "\eta"] \arrow[d] & Z \arrow[d, "g"] \\
\mathrm{Spec}(R) \arrow[r, "\bar{\eta}_0"] \arrow[ur, dashed, "\bar{\eta}"] & Y \arrow[d, "f"] \\
& X
\end{tikzcd}$$

with F a field and $R \subset F$ a valuation ring, we first find a unique lift $\bar{\eta}$ with the property that $fg\bar{\eta} = f\bar{\eta}_0$ using that fg satisfies the valuative criterion of properness, and then use that f satisfies (the uniqueness part of) the valuative criterion for properness to conclude that $g\bar{\eta} = \bar{\eta}_0$. □

Remark 12.3. In Lemma 12.2, we refer to (1)–(3) by saying that the property of satisfying the valuative criterion for properness is stable under arbitrary base-change, is stable under composition, and is local on the base, or equivalently, on the target. We note that, in (4), we only used that $f\colon Y \to X$ satisfies the uniqueness part of the valuative criterion for properness. This part is called the valuative criterion for separatedness.

Now let us give the definition of quasicompact and quasiseparated. The condition of being both quasicompact and quasiseparated is equivalent to the condition of being qcqs mentioned in the previous lecture.

Definition 12.4. A map of schemes $f\colon Y \to X$ is quasicompact if for every quasicompact open subscheme $U \subset X$, the inverse image $f^{-1}(U) \subset Y$ is a quasicompact open subscheme, and it is quasiseparated if the diagonal map $\Delta_f\colon Y \to Y \times_X Y$ is quasicompact.

A map of schemes $f = (p, \phi)\colon : Y \to X$ is quasiseparated if and only if it has the following property: If $U \subset |X|$ and $V, W \subset |Y|$ are quasicompact open subsets with $p(V), p(W) \subset U$, then the open subset $V \cap W \subset |Y|$ is quasicompact.

Lemma 12.5. *The properties of maps of schemes of being quasicompact and of being quasiseparated are both stable under base-change along arbitrary maps and under composition. Moreover, they are local on the base.*

Proof. We skip the proof. This is stated in [3, Propositions 1.1.2 and 1.2.2], but the proofs are partly given in [2]. The proofs are not difficult and ultimately rest on the facts that a scheme, by definition, admits a cover by affine open subschemes and that affine schemes are quasicompact. ☐

The assumption that a map of schemes be quasicompact and quasiseparated is a very mild assumption.

Remark 12.6. You have proved on Problem set 6 that if $f\colon Y \to X$ is quasicompact and quasiseparated, then the direct image functor restricts to a functor

$$\begin{tikzcd} \mathrm{QCoh}(\mathcal{O}_Y) \arrow[r, "f_*"] \arrow[d] & \mathrm{QCoh}(\mathcal{O}_X) \arrow[d] \\ \mathrm{Mod}_{\mathcal{O}_Y} \arrow[r, "f_*"] & \mathrm{Mod}_{\mathcal{O}_X} \end{tikzcd}$$

between the respective full subcategories spanned by the quasicoherent modules.

A map of rings $\phi\colon A \to B$ is defined to be of finite type if it exhibits B as a finitely generated A-algebra, or equivalently, if it admits a factorization

$$A \longrightarrow A[x_1, \dots, x_n] \xrightarrow{\psi} B$$

with ψ surjective.

Definition 12.7. A map of schemes $f = (p, \phi)\colon Y \to X$ is locally of finite type if for all affine open subsets $V \subset |Y|$ and $U \subset |X|$ with $p(V) \subset U$, the composite map

$$\mathcal{O}_X(U) \xrightarrow{\phi_U} \mathcal{O}_Y(p^{-1}(U)) \longrightarrow \mathcal{O}_Y(V)$$

is ring homomorphism of finite type.

Lemma 12.8. *The property of a map of schemes of being locally of finite type is stable under base-change along arbitrary maps and under composition, and it is local on the base.*

Proof. We skip the proof. This is stated in [3, Proposition 1.3.4], but the proof is given in [2, Proposition 6.6.6]. □

The assumption that a map of schemes be of locally of finite type is one that permeates algebraic geometry. In arithmetic geometry, there are very reasonable situations, where this assumption is not satisfied.

Definition 12.9. A map of schemes is proper if it is quasicompact, quasiseparated, locally of finite type, and satisfies the valuation criterion for properness.

This definition is often stated differently, but equivalently, as a map being proper if it is separated of finite type and universally closed.

Corollary 12.10. *The property of a map of schemes of being proper is stable under base-change along arbitrary maps and under composition. Moreover, it is local on the base.*

We remark that the assumption that a map of schemes satisfy the valuative criterion of properness is a very strong assumption. So we will encounter many maps that are not proper, for instance $f\colon \mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec}(\mathbb{Z})$ and $f\colon \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$. But let us now show that there are some non-trivial examples.

Proposition 12.11. *A closed immersion of schemes is a proper map.*

Proof. Since the property of a map of schemes of being proper is local on the base, and since closed immersions are affine, it suffices to consider the case of the map of prime spectrum $i\colon Y \to X$ induced by the canonical projection $\pi\colon A \to A/I$ for some ideal $I \subset A$. It is clear that the map is quasicompact, quasiseparated, and locally of finite type, but we must prove that it satisfies the valuative criterion for properness. So we let F be a field and $R \subset F$ a valuation ring, and let

$$\begin{tikzcd} \operatorname{Spec}(F) \arrow[r, "\eta"] \arrow[d] & Y \arrow[d, "i"] \\ \operatorname{Spec}(R) \arrow[r, "\bar{\eta}_0"] \arrow[ur, dashed, "\bar{\eta}"] & X \end{tikzcd}$$

be a diagram of schemes, which determines and is determined by a diagram

$$\begin{tikzcd} A \arrow[r, "\bar{\phi}_0"] \arrow[d, "\pi"] & R \arrow[d, "\iota"] \\ A/I \arrow[r, "\phi"] \arrow[ur, dashed, "\bar{\phi}"] & F \end{tikzcd}$$

of rings. We have $(\iota \circ \bar{\phi}_0)(I) = (\phi \circ \pi)(I) = \{0\} \subset F$, and since ι is injective, we conclude that $\bar{\phi}_0(I) = \{0\} \subset R$. So $\bar{\phi}_0 = \bar{\phi} \circ \pi$ for a unique map $\bar{\phi}\colon A/I \to R$, as we wanted to show. □

We conclude from Proposition 12.11 that if $f = (p, \phi)\colon Y \to X$ is a proper map of schemes, then the map $p\colon |Y| \to |X|$ need not be surjective.

Proposition 12.12. *The unique map $\pi\colon \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}(\mathbb{Z})$ is proper for all $n \geq 0$.*

Proof. It is clear that the map is quasicompact, quasiseparated, and locally of finite type, so we must show that it satisfies the valuative criterion of properness. So given a diagram of schemes of the form

$$\begin{tikzcd} \mathrm{Spec}(F) \arrow[r, "\eta"] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d, "\pi"] \\ \mathrm{Spec}(R) \arrow[r, "\bar{\eta}_0"] \arrow[ur, dashed, "\bar{\eta}"] & \mathrm{Spec}(\mathbb{Z}) \end{tikzcd}$$

with F a field and $R \subset F$ a valuation ring, we must show that a unique lift exists. The lower triangular diagram automatically commutes, since $\mathrm{Spec}(\mathbb{Z})$ is a final object in the category of schemes, but we must prove that the restriction

$$\mathbb{P}^n_{\mathbb{Z}}(\mathrm{Spec}(R)) \longrightarrow \mathbb{P}^n_{\mathbb{Z}}(\mathrm{Spec}(F))$$

along $R \subset F$ is a bijection. Since R and F are local, every line bundle on their prime spectra is trivial, so we can identify the map in question with the map

$$(R^{n+1} \smallsetminus \mathfrak{m}^{n+1})/R^{\times} \longrightarrow (F^{n+1} \smallsetminus \{0\})/F^{\times}$$

induced by the inclusion $R \to F$.

We first show that the map is injective. So let $(f_0, \dots, f_n)$ and $(f_0', \dots, f_n')$ be in $R^{n+1} \smallsetminus \mathfrak{m}^{n+1}$, and suppose that there exists $\lambda \in F^{\times}$ such that $f_i' = \lambda f_i$ for all $0 \leq i \leq n$. Since some $f_i \in R^{\times} = R \smallsetminus \mathfrak{m}$, we conclude that $\lambda \in R$, and since some $f_i' \in R^{\times}$, we further conclude that $\lambda \in R^{\times}$. This proves injectivity.

It remains to prove surjectivity. Given $(f_0, \dots, f_n) \in F^{n+1} \smallsetminus \{0\}$, we can assume that $f_i \in R$ for all $0 \leq i \leq n$ by clearing the denominators. We claim that

$$f_0 \mid f_1 \mid \dots \mid f_n,$$

up to reordering. Indeed, by the definition of a valuation ring, given $f, g \in R$, we have $f|g$ or $g|f$ or both. In particular, we have $f_0 \in R \smallsetminus \{0\}$, so we can divide by it to get $f_0 = 1 \in R \smallsetminus \mathfrak{m}$, which shows that the class of $(f_0, \dots, f_n)$ is in the image. This proves surjectivity. ☐

Definition 12.13. A map of schemes $f \colon Y \to X$ is projective if, locally on X, it can be factored as

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_X \arrow[d, "\pi'"] \\ & X \end{tikzcd}$$

with π' the base-change of the unique map $\pi \colon \mathbb{P}^n_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$ along the unique map $g \colon X \to \mathrm{Spec}(\mathbb{Z})$ and with i a closed immersion.

Theorem 12.14. *Every projective map of schemes is proper.*

Proof. Since $\pi \colon \mathbb{P}^n_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$ is proper, so is $\pi' \colon \mathbb{P}^n_X \to X$, because proper maps are stable under base-change. So both i and π' are proper, and therefore, so is $f = \pi' \circ i$, because proper maps are stable under composition. Since the condition of being proper is also local on the base, we conclude the desired claim. ☐

There do exists non-projective proper maps, even if $X = \operatorname{Spec}(k)$ with k a field and even with $k = \mathbb{C}$. (Hironaka: Certain "blow-up" of a 3-dimensional projective variety.) Another hint that proper maps are better than projective maps is the following result, which is proved in Serre's GAGA paper [7].

Theorem 12.15. *A map of finite type $f\colon X \to \operatorname{Spec}(\mathbb{C})$ is proper if and only if the space of closed points $X(\mathbb{C})$ with the euclidean topology is compact Hausdorff.*

For example, the space $\mathbb{A}^n_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}^n$ is Hausdorff, but not compact.

Nevertheless, one can often reduce the study of proper maps to the study of projective maps thanks to the following result, whose proof we omit.

Theorem 12.16 (Chow's lemma)**.** *Suppose that $f\colon Y \to X$ is proper with X locally noetherian. In this situation, there exists a diagram of schemes*

$$\begin{tikzcd} Y' \arrow[r, "\pi"] \arrow[dr, "f'"'] & Y \arrow[d, "f"] \\ & X \end{tikzcd}$$

with π and f' projective and a dense open subscheme $U \subset Y$ such that

$$U' \simeq \pi^{-1}(U) \xrightarrow{\pi'} U$$

is an isomorphism.[9]

Let us also give examples of non-proper maps. We let k be a field, and let $f\colon X \to \operatorname{Spec}(k)$ be any proper map. Let us assume that X is irreducible. So for example, $X \simeq \mathbb{P}^n_k$ will do.

Lemma 12.17. *Let k be a field, and let $f\colon X \to \operatorname{Spec}(k)$ be a proper map with X irreducible. If $Z \subset |X|$ be a closed subset with $Z \neq \emptyset$ and $Z \neq X$, then*

$$U \simeq X \smallsetminus Z \xrightarrow{f|_U} \operatorname{Spec}(k)$$

is not proper.

Proof. Let $\eta \in |X|$ be the point, and let $F = k(\eta)$. The generic point is unique, because we assume X to be irreducible. We let $x \in Z$ and factor $\mathcal{O}_{X,x} \to F$ as

$$\mathcal{O}_{X,x} \twoheadrightarrow \overline{\mathcal{O}}_{X,x} \hookrightarrow F.$$

The middle ring is again local, as is the left-hand map. We choose a factorization of the right-hand map as a composition

$$\overline{\mathcal{O}}_{X,x} \hookrightarrow R \hookrightarrow F$$

with R a valuation ring and with the left-hand map local, and consider the diagram

$$\begin{tikzcd} \operatorname{Spec}(F) \arrow[r, "\eta"] \arrow[d] & U \arrow[r] \arrow[d, "f|_U"] & X \arrow[d, "f"] \\ \operatorname{Spec}(R) \arrow[r, "\bar{\eta}_0"] & \operatorname{Spec}(k) \arrow[r, equal] & \operatorname{Spec}(k). \end{tikzcd}$$

[9] We refer to the latter property by saying that π is birational.

Since f is proper, there exists a unique lift $\bar{\eta}\colon \operatorname{Spec}(R) \to X$ in the outer square. Let $s \in |\operatorname{Spec}(R)|$ be the unique closed point. Since $\mathcal{O}_{X,x} \to R$ is local, we have

$$\bar{\eta}(s) = x \in Z \subset |X|.$$

Therefore, there cannot exist a lift $\bar{\eta}'\colon \operatorname{Spec}(R) \to U$ in the left-hand square. For this would give a lift in the outer square, and this lift would be distinct from the lift $\bar{\eta}\colon \operatorname{Spec}(R) \to X$, since $\bar{\eta}'(s) \in U \subset |X|$, whereas $\bar{\eta}(s) \in Z \subset |X|$. This shows that $f|_U \colon U \to \operatorname{Spec}(k)$ is not proper, as stated. $\square$

We will now consider cohomology. We recall that if

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is a short exact sequence of sheaves of abelian groups on a topological space X, then the sequence of abelian groups of global sections

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X)$$

is exact, but, in general, the right-hand map is not surjective. Let us give an example with quasicoherent $\mathcal{O}_X$-modules on $X = \mathbb{P}^1_k$.

First, a non-example. Take an arbitrary point on X, say, $0 \in \operatorname{Spec}(k[t]) \subset X$ with associated quasicoherent ideal $\mathcal{I}_0 \subset \mathcal{O}_X$ given by the functions that vanish at 0. So we have a short exact sequence of quasicoherent $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{I}_0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{I}_0 \longrightarrow 0.$$

We calculated in Lecture 8 that $\mathcal{O}_X(X) \simeq k$. Similarly, since

$$\begin{aligned} (\mathcal{O}_X/\mathcal{I}_0)(\operatorname{Spec}(k[t])) &\simeq k[t]/t \simeq k \\ (\mathcal{O}_X/\mathcal{I}_0)(\operatorname{Spec}(k[t^{-1}])) &\simeq 0, \end{aligned}$$

we also have $(\mathcal{O}_X/\mathcal{I}_0)(X) \simeq k$. The map $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_0$ is given by restriction of functions, so the induced map on global sections is given by the composite map

$$k \longrightarrow k[t] \longrightarrow k[t]/t \simeq k,$$

which is an isomorphism, and hence, in particular surjective.

Now, an example: use $\mathcal{I}_0^{\otimes 2} \simeq \mathcal{O}_X(-2)$ instead of $\mathcal{I}_0 \simeq \mathcal{O}_X(-1)$. We again have a short exact sequence of quasicoherent $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{I}_0^{\otimes 2} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{I}_0^{\otimes 2} \longrightarrow 0,$$

and taking global sections, we get an exact sequence

$$0 \longrightarrow (\mathcal{I}_0^{\otimes 2})(X) \longrightarrow k \longrightarrow k[t]/t^2,$$

where, this time, the right-hand map is not surjective. The explanation is that, in the former case, $H^1(X, \mathcal{O}_X(-1)) = 0$, but in the latter case, $H^1(X, \mathcal{O}_X(-2)) \neq 0$.

In general, the abelian category $\operatorname{Sh}(X, \mathsf{Ab})$ of sheaves of abelian groups on a topological space X has enough injectives. So to the left exact functor

$$\operatorname{Sh}(X, \mathsf{Ab}) \xrightarrow{p_*} \mathsf{Ab}$$

that to a sheaf $\mathcal{F}$ assigns the abelian group $p_*(\mathcal{F}) = \mathcal{F}(X)$ of global sections, we can assign a family of right derived functors $R^i p_*(-)$, one for all $i \geq 0$, together with "boundary maps" that to every short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

assigns a long exact sequence

$$\begin{aligned} 0 \longrightarrow R^0 p_*(\mathcal{F}') &\longrightarrow R^0 p_*(\mathcal{F}) \longrightarrow R^0 p_*(\mathcal{F}'') \\ \longrightarrow R^1 p_*(\mathcal{F}') &\longrightarrow R^1 p_*(\mathcal{F}) \longrightarrow R^1 p_*(\mathcal{F}'') \\ \longrightarrow R^2 p_*(\mathcal{F}') &\longrightarrow R^2 p_*(\mathcal{F}) \longrightarrow \quad \cdots \end{aligned}$$

We usually write $H^i(X, -) = R^i p_*(-)$, so that the long exact sequence becomes

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{F}') &\longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{F}'') \\ \longrightarrow H^1(X, \mathcal{F}') &\longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{F}'') \\ \longrightarrow H^2(X, \mathcal{F}') &\longrightarrow H^2(X, \mathcal{F}) \longrightarrow \quad \cdots \end{aligned}$$

In principle, we calculate these by taking choosing an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \cdots$$

and calculating $H^i(X, \mathcal{F})$ as the ith cohomology group of the complex

$$\mathcal{I}^0(X) \longrightarrow \mathcal{I}^1(X) \longrightarrow \mathcal{I}^2(X) \longrightarrow \cdots,$$

albeit this is really not practical. We are mainly interested in the quasicoherent $\mathcal{O}_X$-modules on a scheme X. There are three possible variants of cohomology, gotten by restricting the source category for the derived functors:

$$\begin{tikzcd} \text{(1)} & \text{(2)} & \text{(3)} \\ \mathrm{QCoh}(X) \arrow[r] & \mathrm{Mod}_{\mathcal{O}_X}(\mathrm{Sh}(X, \mathsf{Ab})) \arrow[r] & \mathrm{Sh}(X, \mathsf{Ab}) \arrow[d, "p_*"] \\ & & \mathsf{Ab} \end{tikzcd}$$

Each of (1)–(3) is an abelian category with enough injectives, so we could consider the right derived functors of $\mathcal{F} \mapsto p_*(\mathcal{F}) \simeq \mathcal{F}(X)$ with each of (1)–(3) as its source category. The horizontal functors above are all exact, but this is no guarantee that all these right derived functors agree! However:

(a) For every scheme X, every $\mathcal{M} \in \mathrm{Mod}_{\mathcal{O}_X}(\mathrm{Sh}(X, \mathsf{Ab}))$ admits an injective resolution such that each term in the resolution also is injective as a sheaf of abelian group on X. This implies that

$$H^i_{(2)}(X, \mathcal{M}) \simeq H^i_{(3)}(X, \mathcal{M}).$$

(b) For every locally noetherian scheme X, every $\mathcal{M} \in \mathrm{QCoh}(X)$ admits an injective resolution such that each term in the resolution is also injective as an $\mathcal{O}_X$-module in sheaves of abelian groups. So in this situation,

$$H^i_{(1)}(X, \mathcal{M}) \simeq H^i_{(2)}(X, \mathcal{M}).$$

So for locally noetherian schemes X, all three options agree. The "official" definition is (3): The cohomology of $\mathcal{M} \in \mathrm{QCoh}(X)$ is the cohomology of the underlying sheaf of abelian groups. Why?

(1) There are non-quasicoherent sheaves of abelian groups on a scheme X, whose cohomology is interesting, e.g.
$$H^1(X, \mathcal{O}_X^\times) \simeq \mathrm{Pic}(X).$$

(2) There are maps of quasicoherent $\mathcal{O}_X$-modules, which are not $\mathcal{O}_X$-linear, but where we still want to have an induced map on $H^i(X, -)$, e.g. the de Rham complex
$$\mathcal{O}_X \xrightarrow{\ d\ } \Omega_X^1 \xrightarrow{\ d\ } \Omega_X^2 \longrightarrow \cdots$$
whose differential is not $\mathcal{O}_X$-linear.

The most important basic result on the cohomology of quasicoherent $\mathcal{O}_X$-modules is the following:

Theorem 12.18. *If X is an affine scheme, then*
$$H^i(X, \mathcal{F}) = 0$$
for all $\mathcal{F} \in \mathrm{QCoh}(X)$ and all $i > 0$.

Proof. We cheat and assume that $X \simeq \mathrm{Spec}(R)$ is noetherian. In this case, we can calculate the cohomology groups in question as the right derived functors of the functor $\mathrm{QCoh}(X) \to \mathsf{Ab}$ that to $\mathcal{M}$ assigns the abelian group $\mathcal{M}(X)$ of global sections. But the functor $\mathrm{QCoh}(X) \to \mathrm{Mod}_R(\mathsf{Ab})$ that to $\mathcal{M}$ assigns the R-module $\mathcal{M}(X)$ of global sections is an equivalence of categories. So we want to calculate the right derived functors of the functor $\mathrm{Mod}_R(\mathsf{Ab}) \to \mathsf{Ab}$ that to an R-module assigns its underlying abelian group. But this functor is exact, so its (higher) right derived functors are all zero. $\square$

Remark 12.19. The conclusion of Theorem 12.18 does not hold for general sheaves of abelian groups. For example, $H^1(X, \mathcal{O}_X^\times) \simeq \mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles, and we saw that for $X \simeq \mathrm{Spec}(R)$ with R a Dedekind domain, this group is typically nonzero.

We explain how to really calculate cohomology of quasicoherent $\mathcal{O}_X$-modules.

Lemma 12.20 (Mayer–Vietoris)**.** *Let X be a topological space, let $\mathcal{F}$ be a sheaf of abelian groups on X, and for $U \subset X$ open, let $H^i(U, \mathcal{F}) = H^i(U, \mathcal{F}|_U)$. Suppose that $U, V \subset X$ are open and that $X = U \cup V$. There is a long exact sequence*
$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{F}) &\longrightarrow H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \longrightarrow H^0(U \cap V, \mathcal{F}) \\ \xrightarrow{\ \partial\ } H^1(X, \mathcal{F}) &\longrightarrow H^1(U, \mathcal{F}) \oplus H^1(V, \mathcal{F}) \longrightarrow H^1(U \cap V, \mathcal{F}) \\ \xrightarrow{\ \partial\ } H^2(X, \mathcal{F}) &\longrightarrow H^2(U, \mathcal{F}) \oplus H^2(V, \mathcal{F}) \longrightarrow \quad \cdots, \end{aligned}$$
where the left-hand maps are given by $(\mathrm{res}_U^X, \mathrm{res}_V^X)$, and where the right-hand maps are given by $\mathrm{res}_{U\cap V}^U + (-\mathrm{res}_{U\cap V}^V)$.[10]

[10] The minus sign is necessary for the composition of the two maps to be zero, but we could also have used e.g. $(\mathrm{res}_U^X, -\mathrm{res}_V^X)$ and $\mathrm{res}_{U\cap V}^U + \mathrm{res}_{U\cap V}^V$ instead. So this choice is non-canonical. A consistent choice of signs is made in [4, Definition 1.1.2.11].

Proof. First, we claim that if $\mathcal{I}$ is an injective sheaf of abelian groups, then the map $\mathcal{I}(W) \to \mathcal{I}(W')$ associated to any inclusion $W' \subset W$ of open subsets is surjective. Indeed, this follows by considering the inclusion $h_{W'} \to h_W$ of representable sheaves of sets, the induced inclusion $\mathbb{Z}[h_{W'}] \to \mathbb{Z}[h_W]$ of free abelian group presheaves, and then the sheafification of these, which is still injective, because sheafification preserves limits. Mapping out to $\mathcal{I}$ and using the definition of injectivity, we deduce the claim.

Given this, it follows that there is a Mayer–Vieteris short exact sequence

$$0 \longrightarrow \mathcal{I}(X) \longrightarrow \mathcal{I}(U) \oplus \mathcal{I}(V) \longrightarrow \mathcal{I}(U \cap V) \longrightarrow 0.$$

The result follows by taking an injective resolution of $\mathcal{F}$, and using the fact from homological algebra that a short exact sequence of complexes gives rise to a long exact sequence of cohomology groups. □

Corollary 12.21. *Let X be a scheme and suppose that $U, V \subset X$ are affine open subschemes with $X = U \cup V$ and with $U \cap V$ affine. In this situation,*

$$H^i(X, \mathcal{M}) \simeq \begin{cases} \ker(\mathcal{M}(U) \oplus \mathcal{M}(V) \to \mathcal{M}(U \cap V)) & \text{if } i = 0, \\ \operatorname{coker}(\mathcal{M}(U) \oplus \mathcal{M}(V) \to \mathcal{M}(U \cap V)) & \text{if } i = 1, \\ 0 & \text{if } i \geq 2, \end{cases}$$

for all $\mathcal{M} \in \mathrm{QCoh}(X)$.

Proof. This follows immediately from Mayer–Vietoris and vanishing of higher cohomology of quasicoherent sheaves on affines. □

The corollary applies $X = \mathbb{P}^1_k$ with $U = \mathrm{Spec}(k[t])$ and $V = \mathrm{Spec}(k[t^{-1}])$ and with intersection $U \cap V = \mathrm{Spec}(k[t^{\pm 1}])$, for any field (or indeed any ring) k. So we can easily calculate

$$H^i(X, \mathcal{O}_X(n))$$

for all integers n and $i \geq 0$. We note that $\mathcal{O}_X(-1)$ is the only line bundle all of whose cohomology groups, including H^0, vanish.

What if it takes more than two open affine subschemes to cover X? Two options:

(1) Use induction on the number of affines and Mayer–Vietoris.
(2) Direct generalization of Mayer–Vietoris: Čech cohomology.

Theorem 12.22. *Let X be a scheme, and let $(U_i)_{i \in I}$ be a family of open subschemes that cover X. Suppose that for all $k \geq 0$ and all $(i_0, \dots, i_k) \in I^{k+1}$, the open subscheme $U_{i_0} \cap \cdots \cap U_{i_k} \subset X$ is affine. The cohomology of $\mathcal{M} \in \mathrm{QCoh}(X)$ is given by the cohomology of the Čech complex $\check{C}((U_i)_{i \in I}, \mathcal{M})$ with kth term*

$$\check{C}^k((U_i)_{i \in I}, \mathcal{M}) = \textstyle\prod_{(i_0, \dots, i_k) \in I^{k+1}} \mathcal{M}(U_{i_0} \cap \cdots \cap U_{i_k}).$$

Proof. Omitted. □

Remark 12.23. If X is separated in the sense that $\Delta \colon X \to X \times X$ is a closed immersion, then the intersection of a finite number of affine open subsets of X is again affine.

Remark 12.24. The Čech complex can be made smaller: If we choose a total order on the index set I, then complex with kth term

$$\prod_{i_0 < \cdots < i_k} \mathcal{M}(U_{i_0} \cap \cdots \cap U_{i_k})$$

also calculates the cohomology of $\mathcal{M} \in \mathrm{QCoh}(X)$. If I has two elements, then this recovers Mayer–Vietoris.

Given a scheme X, we defined the ith cohomology functor $H^i(X,-)$ for $i \geq 0$ to be the ith right derived functor of the left exact functor

$$\mathrm{Mod}_{\mathcal{O}_X}(\mathrm{Sh}(X,\mathsf{Ab})) \xrightarrow{p_*} \mathsf{Ab}$$

that to a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ assigns the abelian group $\mathcal{F}(X)$ of global sections. More generally, given a map of schemes $f\colon Y \to X$, we have the left exact functor

$$\mathrm{Mod}_{\mathcal{O}_Y}(\mathrm{Sh}(Y,\mathsf{Ab})) \xrightarrow{f_*} \mathrm{Mod}_{\mathcal{O}_X}(\mathrm{Sh}(X,\mathsf{Ab}))$$

so we may consider its ith right derived functor $R^i f_*$ for $i \geq 0$.[11] There are three fundamental theorems concerning these cohomology functors.

First, we have "Acyclicity of affines with quasicoherent coefficients."

Theorem 13.1. *If $f\colon Y \to X$ is affine, then*

$$R^i f_*(\mathcal{F}) \simeq \begin{cases} f_*(\mathcal{F}) & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

for every quasicoherent $\mathcal{O}_Y$-module $\mathcal{F}$.

Corollary 13.2. *Let $f\colon Y \to X$ and $g\colon Z \to Y$ be composable maps of schemes. If the map g is affine, then there is a canonical natural isomorphism*

$$R^i(f \circ g)_*(\mathcal{G}) \simeq (R^i f_* \circ g_*)(\mathcal{G})$$

for all $i \geq 0$ and all quasicoherent $\mathcal{O}_Z$-modules $\mathcal{G}$.

Proof. It follows from Theorem 13.1 that if

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{J}^0 \longrightarrow \mathcal{J}^1 \longrightarrow \mathcal{J}^2 \longrightarrow \cdots$$

is an injective resolution of $\mathcal{G}$, then

$$0 \longrightarrow g_*(\mathcal{G}) \longrightarrow g_*(\mathcal{J}^0) \longrightarrow g_*(\mathcal{J}^1) \longrightarrow g_*(\mathcal{J}^2) \longrightarrow \cdots$$

is a resolution of $g_*(\mathcal{G})$. But g_* admits the exact left adjoint functor g^*, so it preserves injective objects. Hence, the complex

$$(f_* \circ g_*)(\mathcal{J}^0) \longrightarrow (f_* \circ g_*)(\mathcal{J}^1) \longrightarrow (f_* \circ g_*)(\mathcal{J}^2) \longrightarrow \cdots$$

calculates both the left-hand side and the right-hand side in the statement, up to unique natural isomorphism. □

Corollary 13.3. *If $f\colon Y \to X$ is an affine map of schemes, then there is a canonical natural isomorphism $H^i(Y,\mathcal{F}) \simeq H^i(X, f_*(\mathcal{F}))$ for all $i \geq 0$ and all $\mathcal{F} \in \mathrm{QCoh}(Y)$.*

Proof. Same proof. □

[11] If $X \simeq \mathrm{Spec}(\mathbb{Z})$, then the category $\mathrm{Mod}_{\mathcal{O}_X}(\mathrm{Sh}(X,\mathsf{Ab}))$ is not equivalent to Ab, but its full subcategory $\mathrm{QCoh}(X)$ is so. We return to this in Remark 13.7.

Corollary 13.4. *Let X be a scheme, and let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Suppose that X admits a covering $(U_i)_{i\in I}$ by open subschemes such that for all $k \geq 0$ and all $(i_0, \dots, i_k) \in I^{k+1}$, $U_{i_0} \cap \dots \cap U_{i_k}$ is affine. There is a canonical natural isomorphism between $H^i(X, \mathcal{F})$ and the ith cohomology of the Čech complex*

$$\check{C}^0((U_i)_{i\in I}, \mathcal{F}) \xrightarrow{d} \check{C}^1((U_i)_{i\in I}, \mathcal{F}) \xrightarrow{d} \check{C}^2((U_i)_{i\in I}, \mathcal{F}) \longrightarrow \cdots,$$

whose kth term is given by the product

$$\check{C}^k((U_i)_{i\in I}, \mathcal{F}) = \textstyle\prod_{(i_0,\dots,i_k)\in I^{k+1}} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_k}),$$

and whose differential is the alternating sum

$$d = \textstyle\sum_{0\leq s\leq k} (-1)^s d^s$$

of the "coface" maps, which are the unique maps that make the diagrams

$$\begin{tikzcd}
\check{C}^{k-1}((U_i)_{i\in I}, \mathcal{F}) \arrow[r, "d^s"] \arrow[d, "\mathrm{pr}_{(i_0,\dots,i_{s-1},i_{s+1},\dots,i_k)}"] & \check{C}((U_i)_{i\in I}, \mathcal{F}) \arrow[d, "\mathrm{pr}_{(i_0,\dots,i_k)}"] \\
\mathcal{F}(U_{i_0} \cap \dots \cap U_{i_{s-1}} \cap U_{i_{s+1}} \dots \cap U_{i_k}) \arrow[r, "\mathrm{res}"] & \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_k})
\end{tikzcd}$$

commute for all $(i_0, \dots, i_k) \in I^{k+1}$.

We recall from Problem set 6 that if the map $f \colon Y \to X$ is quasicompact and quasiseparated, then the functor f_* restricts to a functor

$$\mathrm{QCoh}(Y) \xrightarrow{f_*} \mathrm{QCoh}(X)$$

between the respective full subcategories of quasicoherent modules. In this situation, a relative version of the Čech complex shows:

Corollary 13.5. *If $f \colon Y \to X$ is a quasicompact and quasiseparated, then*

$$R^i f_*(\mathcal{F}) \in \mathrm{QCoh}(X)$$

for all $i \geq 0$ and all $\mathcal{F} \in \mathrm{QCoh}(Y)$.

Proof. We omit it. To properly organize this proof, one first generalizes the Čech complex to allow for "hypercoverings." □

Corollary 13.6. *If $f \colon X \to \mathrm{Spec}(R)$ is quasicompact and quasiseparated, then*

$$R^i f_*(\mathcal{F}) \simeq \widetilde{H^i(X, \mathcal{F})}$$

for all $i \geq 0$ and all $\mathcal{F} \in \mathrm{QCoh}(X)$.

Proof. Let $S \simeq \mathrm{Spec}(R)$. If we choose an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \cdots$$

of $\mathcal{F}$ as an $\mathcal{O}_X$-module in $\mathrm{Sh}(X, \mathsf{Ab})$, then

$$0 \longrightarrow f_*(\mathcal{I}^0) \longrightarrow f_*(\mathcal{I}^1) \longrightarrow f_*(\mathcal{I}^2) \longrightarrow \cdots$$

is a complex of injective $\mathcal{O}_S$-modules in $\mathrm{Sh}(S, \mathsf{Ab})$, and Corollary 13.5 shows that its ith cohomology $R^i f_*(\mathcal{F})$ is a quasicoherent $\mathcal{O}_S$-module for all $i \geq 0$. We claim that for this specific complex, taking global sections commutes with taking cohomology (i.e. kernel modulo cokernel of the differentials in the complex). Granting this for the moment, we conclude that

$$H^i(X, \mathcal{F}) \simeq (R^i f_*(\mathcal{F}))(S),$$

and since the $\mathcal{O}_S$-module $R^i f_*(\mathcal{F})$ is quasicoherent, we are done.

To prove the claim, let us show more generally that if $\mathcal{F}^\bullet$ is a complex of sheaves of abelian groups on a topological space such that for all $n \geq 0$, both sheaves $\mathcal{F}^n$ and $H^n(\mathcal{F}^\bullet)$ have vanishing sheaf cohomology (i.e. right derived functors of global sections) in degrees > 0, then taking global sections commutes with taking cohomology (i.e. kernel modulo cokernel). Indeed, consider the list of sheaves

$$\ker(d^0), \operatorname{im}(d^0), \ker(d^1), \operatorname{im}(d^1), \ldots$$

Inductively, working our way along this list, and using the appropriate short exact sequences, we deduce that all these sheaves all have vanishing sheaf cohomology in degrees > 0. Finally, using the long exact sequence in sheaf cohomology associated to the short exact sequence giving $\ker(d^k)/\operatorname{im}(d^{k-1})$ yields the result. □

Remark 13.7. In particular, if $f\colon X \to \operatorname{Spec}(\mathbb{Z})$ is quasicompact and quasiseparated, then for all $\mathcal{F} \in \operatorname{QCoh}(X)$, the equivalence of categories

$$\operatorname{QCoh}(\operatorname{Spec}(\mathbb{Z})) \longrightarrow \mathsf{Ab}$$

takes $R^i f_*(\mathcal{F})$ to $H^i(X, \mathcal{F})$. So in this situation, the relative sheaf cohomology agrees with the absolute sheaf cohomology.

Remark 13.8. Let $f\colon Y \to X$ be a map of schemes, and let $U \subset X$ be an open subscheme with inverse image $V = f^{-1}(U) \subset Y$. For all $i \geq 0$ and for all sheaves $\mathcal{F}$ of abelian groups on Y, there is a canonical natural isomorphism

$$(R^i f_*(\mathcal{F}))|_U \longrightarrow R^i (f|_V)_*(\mathcal{F}|_V).$$

Hence, if $f\colon Y \to X$ is quasicompact and quasiseparated and $\mathcal{F} \in \operatorname{QCoh}(Y)$, then by letting U range over an affine open cover of X and using Corollary 13.6, we can reduce the calculation of the relative cohomology $R^i f_*(-)$ to the calculation of absolute cohomology $H^i(U, -)$.

Second, we have "Finite dimensionality."

Theorem 13.9. *If X is a scheme of Krull dimension $\leq d$, then*

$$H^i(X, \mathcal{F}) = 0$$

for all $i > d$ and all sheaves $\mathcal{F}$ of abelian groups on X.

Proof. For simplicity, we assume that $d = 1$ and that X is irreducible with unique generic point $\eta \in |X|$. There are two ingredients in the proof:

First, for all $x \in |X|$ and $i > 0$, we have

$$\operatorname{colim}_{x \in U \subset |X|} H^i(U, \mathcal{F}) = 0.$$

Indeed, this filtered colimit is the ith right derived functor of the exact functor $\mathcal{F} \mapsto \mathcal{F}_x$, and hence, is zero for $i > 0$.

Second, if $U \subset |X|$ is a non-empty open subset, then $|X| \smallsetminus U = \{x_1, \dots, x_n\}$ is a finite set of closed points, because $d = 1$ and X is irreducible. In this situation, there exists a "Mayer–Vietoris" sequence of the form

$$\begin{aligned}
0 \longrightarrow H^0(X, \mathcal{F}) &\longrightarrow H^0(U, \mathcal{F}) \oplus \textstyle\bigoplus_{1 \leq i \leq n} \mathcal{F}_{x_i} \longrightarrow \mathcal{F}_\eta \\
\xrightarrow{\partial} H^1(X, \mathcal{F}) &\longrightarrow H^1(U, \mathcal{F}) \longrightarrow 0 \\
\xrightarrow{\partial} H^2(X, \mathcal{F}) &\longrightarrow H^2(U, \mathcal{F}) \longrightarrow 0 \\
\xrightarrow{\partial} H^3(X, \mathcal{F}) &\longrightarrow \quad \cdots
\end{aligned}$$

To produce this, given an open cover of X of the form $(U, U_1, \dots, U_n)$ with $x_i \in U_i$, we an associated Čech complex with coefficients in $\mathcal{F}$. Now take the filtered colimit of these Čech complexes as $(U_1, \dots, U_n)$ varies over all such neighborhoods.

Now, suppose that $s \in H^i(X, \mathcal{F})$ with $i > 1$. By the first ingredient, there exists a non-empty open subset $U \subset |X|$ such that $s|_U = 0$. (This only uses that $i > 0$.) Hence, by the second ingredient, we conclude that $s = 0$. $\square$

To state the third fundamental theorem, we define the full subcategory

$$\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$$

of coherent $\mathcal{O}_X$-modules.

Definition 13.10. (1) Let R be a ring. An R-module M is coherent if it is finitely generated and if every finitely generated submodule $N \subset M$ is finitely presented.
(2) Let X be a scheme. A quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ is coherent if there exists a cover $(U_i)_{i \in I}$ of X by affine open subschemes such that $\mathcal{M}(U_i)$ is a coherent $\mathcal{O}_X(U_i)$-module for all $i \in I$.

The full subcategory $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$ is closed under the kernel, cokernels, and extensions, and hence, is abelian. Moreover, the inclusion functor is exact.

Remark 13.11. If X is a locally noetherian scheme, then, in Definition 13.10 (2), the requirement that the $\mathcal{O}_X(U_i)$-modules $\mathcal{M}(U_i)$ be finitely presented is equivalent to the (otherwise weaker) requirement that they be finitely generated. In particular, if X is locally noetherian, then $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module, and therefore, so is every locally free $\mathcal{O}_X$-module of finite rank.

Now, the third fundamental theorem concerning coherent cohomology is the following "Finiteness of proper maps" theorem.

Theorem 13.12. *Let $f\colon Y \to X$ be a proper map of schemes with X locally noetherian. If $\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module, then $R^i f_*(\mathcal{F})$ is a coherent $\mathcal{O}_X$-module for all $i \geq 0$.*

Before we prove the theorem, we discuss a few consequences.

Corollary 13.13. *Let $f\colon X \to \mathrm{Spec}(k)$ be a proper map with k a field. If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space.*

So for $f\colon X \to \mathrm{Spec}(k)$ proper with k a field, the Euler characteristic

$$\chi(X, \mathcal{F}) = \textstyle\sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F})$$

is well-defined. Indeed, by "Finiteness of proper maps," all summands are finite, and the Krull dimension d of X is finite, so by "Finite dimensionality," only the summands $0 \leq i \leq d$ are nonzero.

The Riemann–Roch problem consists in giving a nice formula for $\chi(X, \mathcal{F})$. This is especially useful in combination with vanishing theorems, which say that, for particular X and $\mathcal{F}$, the cohomology groups $H^i(X, \mathcal{F})$ with $i > 0$ vanish. Indeed, in this case, the Riemann–Roch formula determines the dimension

$$\dim_k \mathcal{F}(X) = \dim_k H^0(X, \mathcal{F})$$

of the k-vector space of global sections of $\mathcal{F}$. We will see examples of this later, when we discuss curves.

Proof of Theorem. We divide the proof in a number of steps.

Step 0: The properties of being coherent and being proper are both local on the base, so we can assume that $X \simeq \operatorname{Spec}(R)$.

Step 1: The theorem holds for $\pi\colon Y \simeq \mathbb{P}^n_X \to X$. We will prove this case by using Hilbert syzygy: Every coherent $\mathcal{O}_Y$-modules admits a finite resolution, where each term is a finite sum of coherent $\mathcal{O}_Y$-modules of the form

$$\pi^*(\mathcal{G}) \otimes \mathcal{O}_Y(m)$$

for some coherent $\mathcal{O}_X$-module $\mathcal{G}$ and $m \in \mathbb{Z}$. Using the long exact sequence for $R^i\pi_*$, we reduce, by induction on the length of the resolution, to showing that

$$R^i\pi_*(\pi^*(\mathcal{G}) \otimes \mathcal{O}_Y(m))$$

is a coherent $\mathcal{O}_X$-module. For this, we calculate using the Čech complex for the standard affine open cover of $\mathbb{P}^n_X$. We find that the "projection formula"

$$R^i\pi_*(\pi^*(\mathcal{G}) \otimes \mathcal{O}_Y(m)) \simeq \mathcal{G} \otimes R^i\pi_*(\mathcal{O}_Y(m))$$

holds and that $R^i\pi_*(\mathcal{O}_Y(m))$ is a coherent $\mathcal{O}_X$-module given by a very explicit formula; see e.g. [6, §8, Lemma 1.1].

Step 2: The theorem holds for $f\colon Y \to X$ projective. Working locally, we choose a factorization

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_X \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

with i a closed immersion and use that, since i is affine, we have

$$R^i f_*(\mathcal{F}) \simeq R\pi_*(i_*(\mathcal{F}))$$

by Corollary 13.2. But i_* preserves coherent modules, because i is a closed immersion. Indeed, if M is a finitely generated R/I-module, then the R-module obtained from M by restriction of scalars along $R \to R/I$ is also finitely generated.

Step 3: General case. Since $f\colon Y \to X$ is proper and $X \simeq \operatorname{Spec}(R)$ is noetherian, it follows that also Y is noetherian. Since the topological space $|Y|$ is noetherian, we can apply noetherian induction. So to prove the statement for $(Y, \mathcal{F})$, we can assume that it has been proved for $(Z, \mathcal{G})$, where $i\colon Z \to Y$ is a closed immersion, whose image is not all of Y, and $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module. This implies that the statement holds for $(Y, i_*(\mathcal{G}))$. Now, by Chow's lemma, there exists

$$\begin{tikzcd} Y' \arrow[r, "p"] \arrow[dr, "q"'] & Y \arrow[d, "f"] \\ & X \end{tikzcd}$$

with p and q projective and a dense open subscheme $U \subset Y$ such that

$$U' \simeq \pi^{-1}(U) \xrightarrow{p|_{U'}} U$$

is an isomorphism. The idea is to relate $R^i f_*(\mathcal{F})$ to $R^i q_*(p^*(\mathcal{F}))$. The latter is coherent. For $p^*(\mathcal{F})$ is coherent, because $\mathcal{F}$ is coherent and Y' is noetherian, so the claim follows form Step 2. We claim that the "difference" between $R^i f_*(\mathcal{F})$ and $R^i q_*(p^*(\mathcal{F}))$ is built from $R^i f_*(i_*(\mathcal{G}))$ as above, and hence, is controlled by noetherian induction. To see this, we choose an injective resolution

$$0 \longrightarrow p^*(\mathcal{F}) \longrightarrow \mathcal{J}^0 \longrightarrow \mathcal{J}^1 \longrightarrow \mathcal{J}^2 \longrightarrow \cdots$$

and look at the adjunct complex

$$0 \longrightarrow \mathcal{F} \longrightarrow p_*(\mathcal{J}^0) \longrightarrow p_*(\mathcal{J}^1) \longrightarrow p_*(\mathcal{J}^2) \longrightarrow \cdots$$

in which each $p_*(\mathcal{J}^i)$ is injective. Let us write $\mathcal{F}'^{\cdot}$ for the complex. We have

$$H^i(\mathcal{F}'^{\cdot}) \simeq \begin{cases} \ker(\mathcal{F} \to p_*p^*(\mathcal{F})) & \text{if } i = -1, \\ \operatorname{coker}(\mathcal{F} \to p_*p^*(\mathcal{F})) & \text{if } i = 0, \\ R^i p_* p^*(\mathcal{F}) & \text{if } i \geq 1, \end{cases}$$

which, by Step 2, are coherent $\mathcal{O}_Y$-modules. But we also have $H^i(\mathcal{F}'^{\cdot})|_U \simeq 0$, because the map $p|_{U'} \colon U' \to U$ is an isomorphism. Now we prove:

Lemma 13.14. *If $\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module such that $\mathcal{F}|_U \simeq 0$, then there exists a closed immersion $i \colon Z \to Y$ with $Y \smallsetminus Z = U$ and a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ with*

$$\mathcal{F} \simeq i_*(\mathcal{G}).$$

Proof. Suppose for simplicity that $Y \simeq \operatorname{Spec}(A)$ is affine. By shrinking U, we can assume that $U = Y_f$ is a distinguished open subscheme. In this case, the statement is that if M is an A-module such that $M_f \simeq 0$, then there exists $N \geq 0$ such that the map $f^N \colon M \to M$ given by multiplication by $f^N \in R$ is the zero map, or equivalently, such that the R-module M is obtained from an A/f^N-module N by restriction of scalars along $A \to A/f^N$. But M is generated as an A-module by a finite family $(x_i)_{i \in I}$, and the image of each x_i in the filtered colimit M_f is zero. So for all $i \in I$, there exists $N_i \geq 0$ such that $f^{N_i} x_i = 0$, and since I is finite, the maximum N of the the N_i will do. □

We can now argue by "dévissage" to complete the proof. Let us state this in the language of stable ∞-categories, which makes the argument much clearer. (It is possible to argue with injective resolutions and Grothendieck spectral sequences, but this obscures the simple idea.) We define $\mathcal{F}'$ to be the fiber

$$\mathcal{F}' \longrightarrow \mathcal{F} \xrightarrow{\eta} Rp_*p^*(\mathcal{F}).$$

of the unit map and apply Rf_* to this fiber sequence to obtain the fiber sequence

$$Rf_*(\mathcal{F}') \longrightarrow Rf_*(\mathcal{F}) \xrightarrow{\eta} Rf_*Rp_*p^*(\mathcal{F}) \simeq Rq_*(p^*(\mathcal{F})).$$

Hence, to prove that the cohomology of the middle term are coherent $\mathcal{O}_X$-modules, it will suffice to show that the cohomology of the remaining terms are coherent $\mathcal{O}_X$-modules. This is true for the right-hand term by Step 2, since q is projective and $p^*(\mathcal{F})$ a coherent $\mathcal{O}_{Y'}$-module. It is also true for the left-hand term. For we have proved that $\mathcal{F}'$ admits a finite filtration such that the graded pieces for the filtration all are of the form $i_*(\mathcal{G})$ for some closed immersion $i\colon Z \to Y$ with $Y \smallsetminus Z = U$ and some coherent $\mathcal{O}_Z$-module $\mathcal{G}$. It follows that, by applying Rf_*, we obtain a finite filtration of $Rf_*(\mathcal{F}')$, whose graded pieces are of the form

$$Rf_*(i_*(\mathcal{G})) \simeq R(f \circ i)_*(\mathcal{G}).$$

Hence, it suffices to show the cohomology of $R(f \circ i)_*(\mathcal{G})$ are coherent $\mathcal{O}_X$-modules. But this follows from the inductive hypothesis, since the dimension of Z is strictly smaller than the dimension of Y. The basic case $\dim(Y) = 0$ follows from Step 2, since $f\colon Y \to X$ is a closed immersion in this case. □

We now shift gear and consider curves over a field. To be a curve is a property of a map of schemes. However, the definition varies throughout the literature, and the precise definition is often unclear. An exception is anything written by Deligne, whom one can always trust to give precise (and reasonable) definitions. So we will follow [1, I.1.0], except that, at the outset, we will not assume curves to be proper.

Definition 13.15. Let k be a field. A map of schemes $f\colon X \to \operatorname{Spec}(k)$ is a curve if it is separated and of finite type and if X has Krull dimension 1.

Often is also assumed that the scheme X be reduced and irreducible.

Example 13.16. The curve $f\colon \operatorname{Spec}(k[x,y]/xy) \to \operatorname{Spec}(k)$ consists of the coordinate lines in the plane and is not irreducible. The curve $f\colon \operatorname{Spec}(k[x,y]/y^3) \to \operatorname{Spec}(k)$ is an infinitesimal thickening of the x-axes in the plane and is not reduced.

Lemma 13.17. *If a scheme X is reduced and irreducible, then the ring $\mathcal{O}_X(U)$ is an integral domain for every non-empty affine open subscheme $U \subset X$.*

Proof. If X is reduced and irreducible, then its unique generic point $\eta \in |X|$ belongs to every non-empty affine open subscheme $U \subset X$, and U is again reduced and irreducible. Therefore, it suffices to show that if R is a nonzero reduced ring such that $|\operatorname{Spec}(R)|$ is irreducible, then R is an integral domain. So suppose that $f, g \in R$ and $fg = 0$. We have $|\operatorname{Spec}(R)| = V(f) \cup V(g)$, and since $|\operatorname{Spec}(R)|$ is irreducible, we either have $|\operatorname{Spec}(R)| = V(f)$ or $|\operatorname{Spec}(R)| = V(g)$ or both. So at least one of $f, g \in R$ is nilpotent, and since R is reduced, at least one of $f, g \in R$ is zero. □

The assumption that a noetherian scheme X be reduced and irreducible is fairly harmless, since, in general, the reduced scheme $X_{\mathrm{red}} \subset X$ will be the finite union of its irreducible components.

However, curves $f\colon X \to \operatorname{Spec}(k)$ with X reduced and irreducible can still be complicated, because of two phenomena:

(1) Singularities: For example, the curve

$$X = \operatorname{Spec}(k[x,y]/y^2 - x^2(x+1) \xrightarrow{f} \operatorname{Spec}(k)$$

has a "nodal singularity" at the origin: Close to the origin (this can be made precise using the complete local ring), it looks like the coordinate axes. And the curve

$$X = \operatorname{Spec}(k[x,y]/y^2 - x^3) \xrightarrow{f} \operatorname{Spec}(k)$$

has a "cuspical singularity" at the origin: Close to the origin, it looks like a double line.

(2) Non-properness: For exampe, the curve

$$X = \mathbb{P}^1_k \smallsetminus \{0,1,\infty\} \xrightarrow{f} \operatorname{Spec}(k)$$

is not proper.

In general, the first phenomenon is much more complicated than the second. Let us now give some precise definitions.

Definition 13.18. If $f\colon X \to \operatorname{Spec}(k)$ is a curve with X reduced, then a closed point $x \in |X|$ is regular if the reduced closed subscheme $\{x\}_{\mathrm{red}} \subset X$ is an effective Cartier divisor. A generic point $\eta \in |X|$ is regular.

So a closed point $x \in |X|$ is regular if there exists $x \in U \subset |X|$ affine open such that $\{x\}_{\mathrm{red}} = \operatorname{Spec}(\mathcal{O}_X(U)/f) \subset U$ for some non-zero-divisor $f \in \mathcal{O}_X(U)$.

Proposition 13.19. *Let $f\colon X \to \operatorname{Spec}(k)$ be a curve with X reduced and irreducible, and let $x \in |X|$ be a closed point. The following are equivalent:*

(1) *The closed point $x \in |X|$ is regular.*
(2) *There exists a nonzero element $f \in \mathcal{O}_{X,x}$ such that $\mathfrak{m}_x = (f)$.*
(3) *The local ring $\mathcal{O}_{X,x}$ is a discrete valuation ring: There exists a function*

$$k(X) \simeq \operatorname{Frac}(\mathcal{O}_{X,x}) \xrightarrow{\operatorname{ord}_x} \mathbb{Z} \cup \{\infty\}$$

with the following properties:
   (i) *For all $f,g \in k(X)$, $\operatorname{ord}_x(fg) = \operatorname{ord}_x(f) + \operatorname{ord}_x(g)$.*
   (ii) *For all $f,g \in k(X)$, $\operatorname{ord}_x(f+g) \geq \min\{\operatorname{ord}_x(f), \operatorname{ord}_x(g)\}$.*
   (iii) *For all $f \in k(X)$, $\operatorname{ord}_x(f) = \infty$ if and only if $f = 0$.*
   (iv) *For all $f \in k(X)$, $\operatorname{ord}_x(f) \geq 0$ if and only if $f \in \mathcal{O}_{X,x}$.*
   (v) *There exists $\pi \in \mathcal{O}_{X,x}$ such that $\operatorname{ord}_x(\pi) = 1$.*

The intuition is that $k(X)$ is the field of meromorphic functions on X and that the function $\operatorname{ord}_x\colon k(X) \to \mathbb{Z} \cup \{\infty\}$ measures the order of vanishing at $x \in |X|$. Poles have finite depth, but the zero function has zeros of infinite order.

Proof. It is clear that (1) implies (2). Conversely, if (2) holds, then we can find a lift of f to $f \in \mathcal{O}_X(U)$ for some $x \in U \subset |X|$ affine open. Since X is reduced and $f \in \mathcal{O}_X(U)$ is nonzero, the closed subset $V(f) \subset |X|$ is not all of $|X|$. So $V(f)$ is a finite set of closed points, and $x \in V(f)$ by construction. Finally, by shrinking $x \in U \subset |X|$, we can remove the other points, which shows that (1) holds.

We now assume (2) and prove (3). We consider the sequence of ideals

$$\cdots \subset \mathfrak{m}_x^n \subset \cdots \subset \mathfrak{m}_x^2 \subset \mathfrak{m}_x^1 \subset \mathfrak{m}_x^0 = \mathcal{O}_{X,x}.$$

Their intersection is an ideal $I = \bigcap_{n \geq 0} \mathfrak{m}_x^n \subset \mathcal{O}_{X,x}$. It is finitely generated, because the ring $\mathcal{O}_{X,x}$ is noetherian, and by construction, it satisfies $\mathfrak{m}_x I = I$. So we conclude from Nakayama's lemma that $I = 0$. Hence, for every nonzero $f \in \mathcal{O}_{X,x}$, there exists a unique integer $n = \operatorname{ord}_x(f) \geq 0$ such that $f \in \mathfrak{m}_x^n \smallsetminus \mathfrak{m}_x^{n+1}$. More generally, for nonzero $f \in k(X)$, we write $f = g/h$ and define

$$\operatorname{ord}_x(f) = \operatorname{ord}_x(g) - \operatorname{ord}_x(h).$$

Let $\pi \in \mathfrak{m}_x$ be a generator, which exists by (2). Then, equivalently, for every nonzero $f \in k(X)$, there exists a unique integer $n = \operatorname{ord}_x(f)$ such that $f = \pi^n u$ with $u \in \mathcal{O}_{X,x}^\times$, and one readily verifies that (i)–(v) are satisfied. So (3) holds.

Finally, we show that (3) implies (2). We choose $\pi \in \mathcal{O}_{X,x}$ with $\operatorname{ord}_x(\pi) = 1$ and show that $\mathfrak{m}_x = (\pi)$. We note that $f \in \mathcal{O}_{X,x}^\times$ if and only if $\operatorname{ord}_x(f) = 0$. Hence, we have $f \in \mathfrak{m}_x$ if and only if $\operatorname{ord}_x(f) > 0$, since $\mathcal{O}_{X,x}$ is local. So $\pi \in \mathfrak{m}_x$, which implies that $(\pi) \subset \mathfrak{m}_x$. Conversely, if $f \in \mathfrak{m}_x$ is nonzero, then we can write

$$f = \pi^{\operatorname{ord}_x(f)} u$$

with $u \in k(X)^\times$. But then $\operatorname{ord}_x(u) = 0$, so $u \in \mathcal{O}_{X,x}^\times$, and since $\operatorname{ord}_x(f) \geq 1$, we conclude that $f \in (\pi)$. So (2) holds. □

Remark 13.20. There are many equivalent characterizations of discrete valuation rings, including (1)–(3) below, where (1) follows from the proof of Proposition 13.19.

(1) A ring R is a DVR if and only if R is a noetherian local ring, which is not a field, whose maximal ideal is principal.
(2) A ring R is a DVR if and only if R is an integrally closed[12] noetherian local ring of Krull dimension 1.
(3) A ring R is a DVR if and only if R is a noetherian valuation ring.

Suppose that R is a discrete valuation ring with quotient field F and with valuation $v\colon F \to \mathbb{Z} \cup \{\infty\}$. We should at least see that R is a valuation ring. So let $x \in F$ be a nonzero element. If $v(x) \geq 0$, then $x \in R$, and if $v(x) \leq 0$, then $v(x^{-1}) \geq 0$, so $x^{-1} \in R$. So a discrete valuation ring is indeed a valuation ring.

Theorem 13.21. *If $f\colon X \to \operatorname{Spec}(k)$ is a curve with X reduced, then the subset*

$$\operatorname{Reg}(X) \subset |X|$$

consisting of the regular points is open.

Proof. Omitted. □

Remark 13.22. In the situation of Theorem 13.21, the open subset $\operatorname{Reg}(X) \subset |X|$ contains the generic points of X, since generic points, by definition, are regular. It follows that there are only finitely many singular (= non-regular) points in $|X|$.

We define a map of scheme $f = (p, \phi)\colon Y \to X$ to be finite if it is affine and if for every affine open subset $U \subset |X|$ with inverse image $V = p^{-1}(U) \subset |Y|$, the map of rings $\phi_U\colon \mathcal{O}_X(U) \to \mathcal{O}_Y(V)$ exhibits $\mathcal{O}_Y(V)$ as a finitely generated $\mathcal{O}_X(U)$-module.

[12] An integral domain R with quotient field $R \subset F$ is integrally closed if it has the following property: If $a \in F$ is a root in a monic polynomial $p(x) \in R[x]$, then $a \in R$.

Theorem 13.23 (Resolution of singularities for curves). *If $f\colon X \to \mathrm{Spec}(k)$ is a curve with X reduced, then there exists a diagram of schemes*

$$\begin{tikzcd} X' \arrow[rr, "\pi"] \arrow[dr, "f'"'] & & X \arrow[dl, "f"] \\ & \mathrm{Spec}(k) & \end{tikzcd}$$

with the following properties:

(1) *The map $f'\colon X' \to \mathrm{Spec}(k)$ is a curve with X' reduced and $\mathrm{Reg}(X') = X'$.*
(2) *The map $\pi\colon X' \to X$ is finite and restricts to an isomorphism*

$$\pi^{-1}(\mathrm{Reg}(X)) \xrightarrow{\pi'} \mathrm{Reg}(X).$$

Moreover, the map $\pi\colon X' \to X$ is unique, up to unique isomorphism in $\mathsf{Sch}_{/X}$.

Remark 13.24. The uniqueness of $\pi\colon X' \to X$, up to unique isomorphism in $\mathsf{Sch}_{/X}$, is very specific to curves. In general, resolutions of singularities are not unique.

Proof. (Sketch) We assume for simplicity that X is irreducible and let $\eta \in |X|$ be the unique generic point. We take $\pi\colon X' \to X$ to be the normalization of X defined as follows: Let $F = \mathcal{O}_{X,\eta}$ be the function field of X. If $U \subset |X|$ is a non-empty affine open subset, then the canonical map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta}$ identifies F as the quotient field of $\mathcal{O}_X(U)$. It follows that the sheaf on X that to $U \subset |X|$ non-empty affine open assigns $\mathrm{Frac}(\mathcal{O}_X(U))$ is the constant sheaf $p^*(F)$ with value F. We define

$$\mathcal{A} \subset p^*(F)$$

to be the sub-$\mathcal{O}_X$-algebra sheaf such that for $U \subset |X|$ non-empty affine open,

$$\mathcal{A}(U) \subset F$$

is the subset consisting of the elements $a \in F$ that are roots of monic polynomials with coefficients in $\mathcal{O}_X(U)$. We claim that the $\mathcal{O}_X$-algebra $\mathcal{A}$ is quasicoherent and that $\phi_U\colon \mathcal{O}_X(U) \to \mathcal{A}(U)$ exhibits $\mathcal{A}(U)$ as a finitely generated $\mathcal{O}_X(U)$-module for every non-empty affine open subset $U \subset |X|$. Granting this, the map

$$X' \simeq \mathrm{Spec}(\mathcal{A}) \xrightarrow{\pi} X$$

induced by $\phi\colon \mathcal{O}_X \to \mathcal{A}$ satisfies (1)–(2) thanks to the characterization of discrete valuation rings as integrally closed noetherian local rings of Krull dimension 1. $\square$

Example 13.25. Let us do the example of the cuspidal cubic

$$X \simeq \mathrm{Spec}(R) \xrightarrow{f} \mathrm{Spec}(k)$$

with $R = k[x,y]/y^2 - x^3$. The origin is a singular point, since the corresponding maximal ideal $\mathfrak{m} = (x,y)$ needs two generators. In this case, we have $X' \simeq \mathrm{Spec}(R')$, where $R' \subset F = \mathrm{Frac}(R)$ is the subset of $a \in F$ that are roots in some monic $p(T) \in R[T]$. We should have $R' \neq R$. Why? We have $a = y/x \in F$ and

$$a^2 = y^2/x^2 = x^3/x^2 = x,$$

which shows that a is a root of $T^2 - x$. So $a \in R'$, but $a \notin R$, because x does not divide y in R. In fact, the unique $k[T]$-algebra map

$$k[T] \longrightarrow R[T]/(T^2 - x)$$

is an isomorphism, whose inverse maps x and y to T^2 and T^3, respectively, and the common ring is integrally closed. So the unique k-algebra map $k[T] \to F$ that to T assigns y/x is an isomorphism onto $R' \subset F$. Hence, in this case, we find that the regular curve $f' \colon X' \to \operatorname{Spec}(k)$ is the affine line. The map

$$X' \xrightarrow{\pi} X$$

is a homeomorphism of underlying spaces, but the fiber

$$X'_{(0,0)} \simeq \operatorname{Spec}(R' \otimes_R k) \simeq \operatorname{Spec}(k[T]/T^2)$$

is a non-reduced point.

Example 13.26. Let us also do the example of the nodal cubic

$$X \simeq \operatorname{Spec}(R) \xrightarrow{f} \operatorname{Spec}(k)$$

with $R = k[x,y]/y^2 - x^2(x+1)$. We assume the characteristic of k is $\neq 2$. Again, the origin is the only singular point, and we find as before that the unique $k[T]$-algebra map

$$k[T] \longrightarrow R[T]/(T^2 - (x+1))$$

is an isomorphism onto the integral closure $R' \subset F$ of R in its quotient field. So the regular curve $f' \colon X' \to \operatorname{Spec}(k)$ is again the affine line, but the map

$$X' \xrightarrow{\pi} X$$

is no longer a homeomorphism of underlying spaces, since the fiber

$$\begin{aligned} X'_{(0,0)} &\simeq \operatorname{Spec}(R' \otimes_R k) \simeq \operatorname{Spec}(k[T]/(T^2 - 1)) \\ &\simeq \operatorname{Spec}(k[T]/(T-1) \times k[T]/(T+1)) \\ &\simeq \operatorname{Spec}(k[T]/(T-1)) \sqcup \operatorname{Spec}(k[T]/(T+1)) \end{aligned}$$

consists of two reduced points.

In general, the structure of the fiber of $\pi \colon X' \to X$ is more complicated than these examples might indicate. For example, let (a,b) be a pair or relatively prime positive integers and assume without loss of generality that $a < b$. We can consider the generalized cuspidal curve of degree b given by

$$X \simeq \operatorname{Spec}(R) \xrightarrow{f} \operatorname{Spec}(k)$$

with $R = k[x,y]/(y^a - x^b)$. Its normalization is given by the map

$$X' \simeq \operatorname{Spec}(k[T]) \xrightarrow{\pi} X \simeq \operatorname{Spec}(R)$$

induced by the unique k-algebra map $R \to R' = k[T]$ that to x and y assign T^a and T^b, respectively. In this case, the fiber over the singular point is

$$X'_{(0,0)} \simeq \operatorname{Spec}(R' \otimes_R k) \simeq \operatorname{Spec}(k[T]/(T^a, T^b)).$$

The k-algebra $k[T]/(T^a, T^b)$ is finite, and its length is

$$\operatorname{length}_k(k[T]/(T^a, T^b)) = \tfrac{1}{2}(a-1)(b-1),$$

as was first proved by Sylvester [8]. For example, if $a = 3$ and $b = 5$, then

$$R' \otimes_R k = k[T]/(T^3) \times_k k[T^4]/(T^8).$$

Let k be a field and let $S = \operatorname{Spec}(k)$. Last time, we defined a map of schemes

$$X \xrightarrow{f} S$$

to be a curve over S, if the map f is separated and of finite type, and if the scheme X has Krull dimension 1. For any such curve, there exists a diagram of curves

$$\begin{tikzcd} X \arrow[d, "f"] & X_{\mathrm{red}} \arrow[l, "i"'] \arrow[d, "f_{\mathrm{red}}"] & X'_{\mathrm{red}} \arrow[l, "\pi"'] \arrow[r, "j"] \arrow[d, "f'_{\mathrm{red}}"] & \overline{X'_{\mathrm{red}}} \arrow[d, "\overline{f'_{\mathrm{red}}}"] \\ S \arrow[r, equal] & S \arrow[r, equal] & S \arrow[r, equal] & S, \end{tikzcd}$$

which is unique, up to unique isomorphism in $\mathsf{Sch}_{/S}$, where i is the closed immersion of the underlying reduced curve, and where π is the normalization of X_{red}, which is a finite birational map. The final map j is an open immersion of the regular curve $Y \simeq X'_{\mathrm{red}} \to S$ into a proper regular curve $\overline{Y} \to S$. In this way, the study of general curves begins with the study of proper regular curves.

Remark 14.1. If $f\colon X \to S$ is a regular curve, then the irreducible components of X do not intersect. Indeed, if $C, D \subset X$ are irreducible components and $x \in C \cap D$, then $x \in \operatorname{Spec}(\mathcal{O}_{X,x})$ has more than one generalization, so $\mathcal{O}_{X,x}$ is not a discrete valuation ring. So if $f\colon X \to S$ is a regular curve, then X is irreducible if and only if X is connected. Hence, a proper regular curve is the finite sum

$$X = \coprod_{i \in I} X_i \xrightarrow{f = \sum_{i \in I} f_i} S$$

of proper regular curves $f_i\colon X_i \to S$ with X_i connected, or equivalently, irreducible. So we focus on this case.

Proposition 14.2. *Let k be a field and $S = \operatorname{Spec}(k)$. Given a map*

$$\begin{tikzcd} Y \arrow[rr, "f=(p,\phi)"] \arrow[dr, "h"'] & & X \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

between proper regular curves with X and Y irreducible, either:

1. *the image of $p\colon |Y| \to |X|$ consists of a single closed point $x \in |X|$, or*
2. *the map $p\colon |Y| \to |X|$ is surjective, and for every $x \in |X|$, the fiber*

$$Y_x \xrightarrow{f_x} \operatorname{Spec}(k(x))$$

is finite.

Proof. Let $\eta \in |X|$ be the unique generic point. We classify $f\colon Y \to X$ according to whether or not $\eta \in p(|Y|)$.

Suppose first that $\eta \notin p(|Y|)$. In this case, the generic fiber

$$Y_\eta \xrightarrow{f_\eta} \operatorname{Spec}(k(\eta))$$

is empty. We choose finite covers $(U_i)_{i\in I}$ and $(V_i)_{i\in I}$ of X and Y by affine open subschemes such that $f\colon Y \to X$ restricts to maps $f_i = f|_{V_i}\colon V_i \to U_i$ for all $i \in I$. Let us write $A_i = \mathcal{O}_X(|U_i|)$ and $B_i = \mathcal{O}_Y(|V_i|)$. Since we assumed X to be irreducible, we have $\eta \in |U_i|$ for all $i \in I$. So the open immersion $V_i \to Y$ induces a map of fibers $V_{i,\eta} \to Y_\eta$. Since Y_η is empty, so is $V_{i,\eta}$, or equivalently,

$$B_i \otimes_{A_i} k(\eta) = 0.$$

We claim that $\phi_i\colon A_i \to B_i$ is finitely generated, or equivalently, that $f_i\colon V_i \to U_i$ is of finite type. Indeed, since the map $h_i\colon V_i \to S$ is both locally of finite type and quasicompact, it is of finite type, and since it factors as $h_i = f_i \circ g_i\colon V_i \to S$, it follows that f_i is of finite type as claimed. Hence, there exists $a_i \in A_i$ such that

$$B_i \otimes_{A_i} A_i[a_i^{-1}] = 0.$$

Moreover, since I is finite, the subset

$$|U| = \textstyle\bigcap_{i\in I} |\operatorname{Spec}(A_i[a_i^{-1}])| \subset |X|$$

is open, and since $U \times_X V_i$ is empty for all $i \in I$, we conclude that

$$U \times_X Y = \emptyset.$$

This shows that $p(|Y|) \subset |X| \smallsetminus |U|$. But $\eta \in |U|$, so $|U| \subset |X|$ is dense, and hence

$$|X| \smallsetminus |U| = \{x_1, \dots, x_n\}$$

is a finite set of closed points in $|X|$. Finally, since $|Y|$ is connected, so is

$$p(|Y|) \subset \{x_1, \dots, x_n\} \subset |X|,$$

so we have $p(|Y|) = \{x_j\}$ for some $1 \leq j \leq n$.[13]

Suppose next that $\eta \in p(|Y|)$, and let us write $\eta_X = \eta$ and η_Y for the generic points of X and Y, respectively. We claim that $\eta_X = p(\eta_Y)$. If this is not the case, then $\eta_X = p(y)$ for some closed point $y \in |Y|$, in which case, we get a map

$$k(\eta) \xrightarrow{f(\eta)} k(y)$$

of k-algebras. However:

Lemma 14.3. *Let $g\colon X \to S = \operatorname{Spec}(k)$ be a curve over a field k.*

(1) *If $\eta \in |X|$ is a generic point, then* $\operatorname{tr.deg}_k(k(\eta)) = 1$.
(2) *If $x \in |X|$ is a closed point, then* $\operatorname{tr.deg}_k(k(x)) = 0$.

Proof. We may assume that $X = \operatorname{Spec}(A)$ is affine. By Noether normalization, there exists a ring homomorphism $k[T] \to A$ that exhibits A as a finitely generated $k[T]$-module. It follows that, in case (1), the induced map

$$k(T) \longrightarrow \operatorname{Frac}(A) = k(\eta)$$

is finite as stated. In case (2), the closed point $x \in |X|$ lies above a closed point of $|\operatorname{Spec}(k[T])|$, so we are reduced to $X = \operatorname{Spec}(k[T])$. ∎

[13] This argument is an example of *spreading out*: if something happens at the generic point, then it happens in a neighborhood of the generic point. Also, note that so far we did not use the assumptions of regularity and properness.

So $p(\eta_Y) = \eta_X$. The induced map of residue fields

$$k(\eta_X) \xrightarrow{f(\eta_X)} k(\eta_Y)$$

is a map of k-algebras of transcendence degree 1, and hence, is finite extension of fields. Now, let $x \in |X|$ be a closed point. The integral closure

$$\widetilde{\mathcal{O}_{X,x}} \subset k(\eta_Y)$$

of the local ring $\mathcal{O}_{X,x}$ in $k(\eta_Y)$ is a discrete valuation ring. So by the valuative criterion for properness, there exists a unique lifting in the diagram

$$\begin{tikzcd}
\operatorname{Spec}(k(\eta_Y)) \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
\operatorname{Spec}(\widetilde{\mathcal{O}_{X,x}}) \arrow[r] \arrow[ur, dashed, "\exists!"'] & X,
\end{tikzcd}$$

This shows that there exists $y \in |Y|$ such that $x = p(y) \in |X|$, as stated. $\square$

Remark 14.4. In the case of 2, we can actually be much more precise about the fibers Y_x at points $x \in X$: namely they all have the same scheme-theoretic cardinality in the sense that the function $X \to \mathbb{N}$ defined by

$$x \mapsto \dim_{k(x)} \mathcal{O}_{Y_x}(Y_x)$$

is constant. Indeed, $f\colon Y \to X$, being a map between proper k-schemes, is itself proper; but 2) also says it has finite fibers. It is a theorem (called "Zariski's main theorem") that any such map is in fact affine. Let $\mathcal{A} = f_*\mathcal{O}_Y$ be the quasicoherent sheaf of algebras on X whose spectrum recovers f. Then, by base-change, $\mathcal{O}_{Y_x}(Y_x)$ identifies with $\mathcal{A}(x)$ as a $k(x)$-algebra, so to prove the desired constancy, it suffices to show that $\mathcal{A}$ is a vector bundle. Being a coherent sheaf, it corresponds locally to a finitely generated module over an affine, but by regularity every such affine is a Dedekind domain. Thus, by the characterization of flatness over a Dedekind domain, it suffices to show that $\mathcal{A}$ is $\mathcal{O}_X$-torsionfree. However, the map

$$\mathcal{O}_X(U) \longrightarrow \mathcal{A}(U) = \mathcal{O}_Y(f^{-1}(U))$$

is injective, because we can compare with the generic point, where we have a map of fields, hence injective. Thus, the statement follows because $\mathcal{O}_Y$ is torsionfree over itself, due to Y being integral.

In case (2) of Proposition 14.2, remarkably, the behavior at the generic point determines everything.

Proposition 14.5. *Let $S = \operatorname{Spec}(k)$ with k a field, and suppose that $g\colon X \to S$ and $h\colon Y \to S$ are proper, regular, connected curves. The map from the set of maps*

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "h"'] & & X \arrow[dl, "g"] \\
& S &
\end{tikzcd}$$

of schemes over S with f non-constant to the set of k-algebra maps $k(X) \to k(Y)$ that to f assigns the induced map $f(\eta)$ of residue fields at the generic point $\eta \in |X|$ is a bijection.

Proof. In fact, we will only use that Y is regular, that X is reduced and irreducible, and that $g\colon X \to S$ is proper.

To prove injectivity, let $f, f'\colon Y \to X$ be non-constant maps with $fg = f'g = h$ such that $f(\eta) = f'(\eta)\colon k(X) \to k(Y)$. We wish to prove that $f = f'$, and it suffices to prove that this equality holds locally on Y. So it suffices to show that given $U \subset X$ and $V \subset Y$ affine open with $f(V), f'(V) \subset U$, then $f|_V = f'|_V$. Thus, we are reduced to proving that if $\phi, \phi'\colon A \to B$ are maps between integral domains that induce the same map on quotient fields, then $\phi = \phi'$. But $B \to \mathrm{Frac}(B)$ is injective, so this is true.

To prove surjectivity, given a map $\phi\colon k(X) \to k(Y)$, we obtain, by spreading out, a map $f_V\colon V \to U$ from a dense open subscheme $V \subset Y$ to a dense open subscheme $U \subset X$ such that $f_V(\eta) = \phi$. So we wish to find a filler

$$\begin{tikzcd} V \arrow[r, "f_V"] \arrow[d] & U \arrow[d] \\ Y \arrow[r, dashed, "f"] & X. \end{tikzcd}$$

Since $|Y| \smallsetminus |V| = \{y_1, \dots, y_m\}$ is finite, it suffices, by induction on $m \geq 0$, to show that we can extend f_V over $y_1 \in |Y|$. Since $g\colon X \to S$ is proper, the valuative criterion for properness gives us a unique lift in the diagram

$$\begin{tikzcd} \mathrm{Spec}(k(Y)) \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ \mathrm{Spec}(\mathcal{O}_{Y,y_1}) \arrow[r] \arrow[ur, dashed, "\exists!"'] & S, \end{tikzcd}$$

where the top horizontal map is induced by $f_V\colon V \to X$. By spreading out, we obtain a map $f_W\colon W \to X$ from an open subscheme $W \subset Y$ with $y_1 \in |W|$, which generically agrees with $f_V\colon V \to X$. Hence, by shrinking $W \subset Y$, we can arrange that $f_W|_{V\cap W} = f_V|_{V\cap W}$ as desired. □

Theorem 14.6. *Let k be a field, and let $S = \mathrm{Spec}(k)$. The functor from the opposite of the non-full subcategory of $\mathsf{Sch}_{/S}$ spanned by the proper, regular, connected curves $g\colon X \to S$ and non-constant maps to the full subcategory of $\mathrm{CAlg}(\mathsf{Ab})_{k/}$ spanned by the finitely generated k-algebras $\phi\colon k \to F$, which are fields of transcendence degree 1, that to a curve $g\colon X \to S$ assigns its function field $k \to k(X)$,*

$$\begin{pmatrix} \textit{proper, irreducible, regular curves} \\ \textit{over } S\textit{, non-constant maps} \end{pmatrix}^{\mathrm{op}} \longrightarrow \begin{pmatrix} \textit{f.g. } k\textit{-algebras which are fields} \\ \textit{of tr. deg. } 1\textit{, } k\textit{-algebra maps} \end{pmatrix},$$

is an equivalence of categories.

Proof. Proposition 14.5 shows that the functor is fully faithful, so it remains to prove that it is essentially surjective. So let $\phi\colon k \to F$ be a finitely generated k-algebra which is a field extension of transcendence degree 1. We factor ϕ as the composition

$$k \longrightarrow k(T) \longrightarrow k(T, \alpha_1, \dots, \alpha_n) = F$$

of a purely transcendental extension of transcendence degree 1 over k and a finite algebraic field extension of $k(T)$. Let $R \subset F$ be the sub-k-algebra generated by

$(T, \alpha_1, \dots, \alpha_n)$. If $d \in k[T]$ is the product of the denominators of all coefficients of minimal polynomials of $\alpha_1, \dots, \alpha_n$ over $k(T)$, then

$$k[T][d^{-1}] \longrightarrow R$$

is finite, so by Noether normalization, the map

$$U = \operatorname{Spec}(R) \xrightarrow{g_0} S = \operatorname{Spec}(k)$$

is a reduced, irreducible curve with function field $k(U) = F$. Finally, we resolve singularites and add missing points to obtain a proper, regular, irreducible curve

$$X \xrightarrow{g} S$$

with the same function field $k(X) = F$. □

We now consider the Riemann–Roch problem for curves. So we let k be a field and $S = \operatorname{Spec}(k)$, let $f\colon X \to S$ be a proper, regular, irreducible curve, and let $\mathcal{L}$ be a line bundle on X.

Problem: How to calculate $h^0(\mathcal{L}) = \dim_k X(\mathcal{L}) = \dim_k H^0(X, \mathcal{L})$?

Answer: Try first to understand the Euler characteristic

$$\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) = \dim_k H^0(X, \mathcal{L}) - \dim_k H^1(X, \mathcal{L}).$$

Why is this easier?

Lemma 14.7. *Let $f\colon X \to S$ be as above. If $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is an exact sequence of coherent $\mathcal{O}_X$-modules, then*

$$\chi(\mathcal{M}') - \chi(\mathcal{M}) + \chi(\mathcal{M}'') = 0.$$

Proof. Since X has Krull dimension 1, it follows from "Finite dimensionality" that the associated long exact sequence in coherent cohomology takes the form

$$0 \longrightarrow H^0(X, \mathcal{M}') \longrightarrow H^0(X, \mathcal{M}) \longrightarrow H^0(X, \mathcal{M}'')$$

$$\xrightarrow{\partial} H^1(X, \mathcal{M}') \longrightarrow H^1(X, \mathcal{M}) \longrightarrow H^1(X, \mathcal{M}'') \longrightarrow 0.$$

Moreover, it follows from "Finiteness for proper maps" that the k-vector spaces in the sequence all are finite dimensional. The lemma now follows from the general fact that for a bounded cochain complex $C^\cdot$ of finite dimensional k-vector spaces,

$$\textstyle\sum_{n \in \mathbb{Z}} (-1)^n \dim_k(C^n) = \sum_{n \in \mathbb{Z}} (-1)^n \dim_k(H^n(C^\cdot)).$$

We apply this general fact to the bounded cochain complex given by the long exact sequence in of coherent cohomology. □

Before stating Riemann–Roch for curves, we need a definition.

Definition 14.8. Let $f\colon X \to S$ be as above. Given a line bundle $\mathcal{L}$ on X and a closed point $x \in |X|$, set $\mathcal{L}(x) = \mathcal{L} \otimes \mathcal{O}_X(x)$.

Remark 14.9. Beware that this is not the same as the fiber of $\mathcal{L}$ at x, though we sometimes use the same notation for that! The $\mathcal{L}(x)$ defined above is a line bundle on X, whereas the fiber of $\mathcal{L}$ at x is a one-dimensional $k(x)$-vector space.

The interpretation is that sections of the line bundle $\mathcal{L}(x)$ are meromorphic sections of the original line bundle $\mathcal{L}$ that are regular away from $x \in |X|$ and have at worst a simple pole at $x \in |X|$.

Definition 14.10. The Picard group of a scheme X is the abelian group $\mathrm{Pic}(X)$ given by the set of isomorphism classes of line bundles on X with the group structure given by tensor product of line bundles.

Theorem 14.11 (Riemann–Roch for curves)**.** *Let k be a field and $S = \mathrm{Spec}(k)$, and let $f\colon X \to S$ be a proper, regular, irreducible curve. The exists a unique function*

$$\mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z}$$

with the following properties:

(1) *If $\mathcal{L} \simeq \mathcal{O}_X$ is a trivial line bundle on X, then*

$$\deg(\mathcal{L}) = 0.$$

(2) *If $\mathcal{L}$ is any line bundle $\mathcal{L}$ on X and $x \in |X|$ a closed point, then*

$$\deg(\mathcal{L}(x)) = \deg(\mathcal{L}) + \dim_k k(x).$$

Moreover, the following hold:

(a) *The degree map* $\deg\colon \mathrm{Pic}(X) \to \mathbb{Z}$ *is a group homomorphism.*
(b) (Riemann–Roch formula). *If $\mathcal{L}$ is a line bundle on X, then*

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \deg(\mathcal{L}).$$

(c) *The degree of a line bundle $\mathcal{L}$ on X can be calculated easily in terms of local trivializations, as specified in the proof.*

Proof. To show existence (and the Riemann–Roch formula), it suffices to show that the map that to the class of $\mathcal{L}$ assigns $\chi(\mathcal{L}) - \chi(\mathcal{O}_X)$ satisfies (1) and (2). It is trivial that it satisfies (1), and to verify (2), we consider the short exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{J}_x \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{J}_x \longrightarrow 0,$$

where $\mathcal{J}_x \subset \mathcal{O}_X$ is the quasicoherent ideal that defines the closed immersion

$$\mathrm{Spec}(k(x)) \xrightarrow{i_x} X.$$

Being a closed immersion, it is affine, so

$$H^i(X, \mathcal{O}_X/\mathcal{J}_x) \simeq H^i(\mathrm{Spec}(k(x)), \mathcal{O}_{\mathrm{Spec}(k(x))}) \simeq \begin{cases} k(x) & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

from which we conclude that

$$\chi(\mathcal{O}_X/\mathcal{J}_x) = \dim_k(k(x)).$$

We now apply the exact functor $- \otimes \mathcal{L}(x)$ to the exact sequence of $\mathcal{O}_X$-modules and remember that, by definition, we have $\mathcal{O}_X(x) \simeq \mathcal{J}_x^{-1}$, to get

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(x) \longrightarrow \mathcal{O}_X/\mathcal{J}_x \otimes \mathcal{L}(x) \longrightarrow 0.$$

The right-hand term is

$$i_{x*}(k(x)) \otimes \mathcal{L}(x) \simeq i_{x*}(k(x) \otimes i_x^*(\mathcal{L}(x))) \simeq i_{x*}(k(x)),$$

where the right-hand isomorphism holds, because the $k(x)$-vector space $i_x^*(\mathcal{L}(x))$ is 1-dimensional, and hence, isomorphic to $k(x)$. From Lemma 14.7 and the above calculation of $\chi(\mathcal{O}_X/\mathcal{J}_x)$, we conclude that

$$\chi(\mathcal{L}) - \chi(\mathcal{L}(x)) + \dim_k(k(x)) = 0,$$

which shows that $\mathcal{L} \mapsto \chi(\mathcal{L}) - \chi(\mathcal{O}_X)$ satisfies (2).

To prove the uniqueness of the degree map and that it is a group homomorphism, we consider divisors on X, that is, formal sums

$$D = \textstyle\sum_{x \in |X|} n_x \cdot x,$$

where the sum ranges over the closed points of $|X|$, where $n_x \in \mathbb{Z}$, and where all but finitely many of the n_x are equal to zero. So the abelian group $\mathrm{Div}(X)$ of divisors on X is the free abelian group spanned by the set of closed points of $|X|$. To a divisor D on X, we assign the line bundle[14]

$$\mathcal{O}_X(D) = \textstyle\bigotimes_{x \in |X|} \mathcal{O}_X(x)^{\otimes n_x},$$

extending the definition for effective divisors that we made in Definition 9.10. The interpretation of this line bundle is that sections of $\mathcal{O}_X(D)$ are meromorphic functions on X, whose pole order at $x \in |X|$ is at most n_x. If n_x is negative, then this means that the function has a zero at $x \in |X|$ of order at least $-n_x$.

By the definition of $\mathcal{O}_X(D)$, the map

$$\mathrm{Div}(X) \xrightarrow{\partial} \mathrm{Pic}(X)$$

that to a divisor D assigns the isomorphism class of $\mathcal{O}_X(D)$ is a homomorphism of abelian groups. The key claim is that this map is surjective. Granting this claim, we complete the proof of the theorem as follows. By induction on $\sum_{x \in |X|} |n_x|$, we conclude from (1) and (2) that

$$\deg(\mathcal{O}_X(D)) = \textstyle\sum_{x \in |X|} n_x \dim_k(k(x)),$$

which proves the uniqueness of the degree map. More precisely, the left-hand side gives the value of the composite map

$$\mathrm{Div}(X) \xrightarrow{\partial} \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z}$$

on the divisor $D = \sum_{x \in |X|} n_x \cdot x$. It is evident from this formula that $\deg \circ \partial$ is additive, and by the claim that ∂ is surjective, this implies that deg is additive.

It remains to prove the claim. We use a variant of the idea of local trivializations and transition functions. We first fix a trivialization

$$\mathcal{L}|_U \xrightarrow{s_\eta} \mathcal{O}_X|_U$$

over some dense open subset $\eta \in U \subset |X|$. Next, for every $x \in |X| \smallsetminus U$, we fix a germ of trivializations near x, that is, an isomorphism of $\mathcal{O}_{X,x}$-modules

$$\mathcal{L}_x \xrightarrow{s_x} \mathcal{O}_{X,x}.$$

Now, for $x \in |X| \smallsetminus U$, we get meromorphic transition functions

$$f_x = s_\eta \cdot s_x^{-1} \in \mathrm{Frac}(\mathcal{O}_{X,x})^\times = k(X)^\times$$

[14] The assignment $D \mapsto \mathcal{O}_X(D)$ produces a line bundle on X that is well-defined, up to unique isomorphism, and not merely an isomorphism class of line bundles on X.

and the family $(f_x)_{x\in|X|\smallsetminus U}$ of these transition functions determines $\mathcal{L}$, up to unique isomorphism. We claim that there exists an isomorphism $\mathcal{L} \simeq \mathcal{O}_X(D)$ with

$$D = \textstyle\sum_{x\in|X|\smallsetminus U} \operatorname{ord}_x(f_x)\cdot x,$$

where $\operatorname{ord}_x\colon k(X) \to \mathbb{Z}\cup\{\infty\}$ is the discrete valuation of $\mathcal{O}_{X,x}$, normalized such that $\operatorname{ord}_x(\pi_x) = 1$ for a generator π_x of the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$. To prove this claim, note that we have a canonical trivialization

$$\mathcal{O}_X(D)|_U \xrightarrow{t_\eta} \mathcal{O}_X|_U$$

over $U \subset |X|$, and that for every $x \in |X|\smallsetminus U$, we can use as germ of a trivializations near x the isomorphism of $\mathcal{O}_{X,x}$-modules

$$\mathcal{O}_X(D)_x \xrightarrow{t_x} \mathcal{O}_{X,x},$$

given by multiplication by $\pi_x^{\operatorname{ord}_x(f_x)}$. But $f_x\pi_x^{-\operatorname{ord}_x(f_x)} \in \mathcal{O}_{X,x}^\times$, so the line bundles determined by the families $(f_x)_{x\in|X|\smallsetminus U}$ and $(\pi_x^{\operatorname{ord}_x(f_x)})_{x\in|X|\smallsetminus U}$ are isomorphic. This proves the claim, and hence, the theorem. □

We give one corollary of Riemann–Roch now; more in the next lecture.

Corollary 14.12. *If $p\colon X \to \operatorname{Spec}(k)$ is a proper, regular, irreducible curve, then*

$$\textstyle\sum_{x\in|X|} \operatorname{ord}_x(f) = 0$$

for all $f \in k(X)^\times$.

The interpretation is that if ϕ is a nonzero meromorphic function on X, then its number of zeros and poles are equal, when counted with multiplicity.

Proof. We have $\deg(\mathcal{O}_X) = 0$ and we now also calculate this degree using the local trivializations. As the generic trivialization, we choose

$$\mathcal{O}_X|_U \xrightarrow{f} \mathcal{O}_X|_U,$$

where the map multiplies by f, and where $U \subset |X|$ is the complement of the finite set of point $x \in |X|$, where f has either a non-trivial zero of pole. As the germ of trivializations near x for $x \in |X|\smallsetminus U$, we choose the identity map

$$\mathcal{O}_{X,x} \xrightarrow{\mathrm{id}} \mathcal{O}_{X,x}.$$

Now, the recipe for calculating the degree in terms of local trivializations gives

$$\deg(\mathcal{O}_X) = \textstyle\sum_{x\in|X|\smallsetminus U} \operatorname{ord}_x(f) = \sum_{x\in|X|} \operatorname{ord}_x(f),$$

which proves the corollary. □

Example 14.13. If we apply Corollary 14.12 to the projective line

$$X = \mathbb{P}^1_k \xrightarrow{p} \operatorname{Spec}(k)$$

and $f \in k[T] \subset k(T) = k(X)$, then we find that, counted with multiplicity, the number of zeros of f in $\mathbb{A}^1_k \subset \mathbb{P}^1_k$ is equal to $-\operatorname{ord}_\infty(f) = \deg(f)$.[15]

[15] The identity $\deg(f) = -\operatorname{ord}_\infty(f)$ is proved by using that T^{-1} is a uniformizer at ∞.

We note that properness is crucial here! If we remove ∞, then there is no hope to have such a formula. In the proof of Riemann–Roch, properness entered exactly through the finiteness of coherent cohomology, which made

$$\deg(\mathcal{L}) = \chi(\mathcal{L}) - \chi(\mathcal{O})$$

well-defined.

Let k be a field, let $S = \mathrm{Spec}(k)$, and let

$$X \xrightarrow{f} S$$

be a proper, regular, connected curve over S. Last time, we defined the degree

$$\mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z}$$

of line bundles on X and proved the Riemann–Roch formula

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \deg(\mathcal{L}).$$

The degree of a line bundle can be calculated as the number of zeros minus the number of poles, both counted with multiplicity, of a rational section of $\mathcal{L}$. In the proof, we assigned a line bundle $\mathcal{O}_X(D)$ to a divisor

$$D = \textstyle\sum_{x \in |X|^0} n_x \cdot x \in \mathrm{Div}(X)$$

and showed that its degree is given by

$$\deg(\mathcal{O}_X(D)) = \textstyle\sum_{x \in |X|^0} n_x \dim_k(k(x)).$$

In this lecture, we will mostly assume that k is algebraically closed so that $k(x) = k$ for every closed point $x \in |X|^0 \subset |X|$. To get more out of Riemann–Roch, we combine it with Serre duality:

Theorem 15.1 (Serre duality)**.** *Let k be a field, let $S = \mathrm{Spec}(k)$, and let $f\colon X \to S$ be a proper, regular, connected curve. There exists a canonical line bundle K_X on X and a canonical isomorphism*

$$H^1(X, K_X) \xrightarrow{\mathrm{tr}} k$$

such that for every line bundle $\mathcal{L}$ on X and $0 \leq i \leq 1$, the composite map

$$H^i(X, \mathcal{L}) \otimes H^{1-i}(X, \mathcal{L}^{-1} \otimes K_X) \xrightarrow{\cup} H^1(X, K_X) \xrightarrow{\mathrm{tr}} k$$

is a perfect pairing.

Writing $h^i(X, \mathcal{L})$ for the dimension of the k-vector space $H^i(X, \mathcal{L})$ as usual, it follows from Serre duality that for all $0 \leq i \leq 1 = \dim(X)$, we have

$$h^i(X, \mathcal{L}) = h^{1-i}(X, \mathcal{L}^{-1} \otimes K_X).$$

In particular, for the Euler characteristic, we obtain the identity

$$\chi(\mathcal{L}) = -\chi(\mathcal{L}^{-1} \otimes K_X).$$

An important numerical invariant of a curve $f\colon X \to S$ is its genus.

Definition 15.2. Let k be a field, and let $S = \mathrm{Spec}(k)$. If $f\colon X \to S$ is a proper, regular, connected curve, then its genus is the non-negative integer

$$g = h^1(X, \mathcal{O}_X) = h^0(X, K_X).$$

Remark 15.3. By comparison, we claim that if k is algebraically closed, then
$$h^0(X, \mathcal{O}_X) = h^1(X, K_X) = 1.$$
The k-algebra $H^0(X, \mathcal{O}_X)$ of global sections is a finite dimensional k-algebra, which is reduced, and any such k-algebra is isomorphic to a finite product $\prod_{i \in I} k_i$ of finite field extensions k_i of k. Since X is connected, there are no non-trivial idempotents in this k-algebra, so I has cardinality 1. Hence, the k-algebra $H^0(X, \mathcal{O}_X)$ is a finite field extension of k, so if k is algebraically closed, then the unit map $k \to H^0(X, \mathcal{O}_X)$ is necessarily an isomorphism.

Hence, for a proper, regular, connected curve $f\colon X \to S$ of genus g over an algebraically closed field, Riemann–Roch takes the form
$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \deg(\mathcal{L}) = 1 - g + \deg(\mathcal{L}).$$
Moreover, using Serre duality, we find that
$$\deg(K_X) = \chi(K_X) - \chi(\mathcal{O}_X) = 2g - 2.$$
We wish to have criterion for a line bundle $\mathcal{L}$, which ensures that $h^1(X, \mathcal{L}) = 0$, because for such line bundles, $h^0(X, \mathcal{L}) = \chi(\mathcal{L})$, and hence, the Riemann–Roch formula will calculate the dimension of the k-vector space $\mathcal{L}(X)$ of global sections of $\mathcal{L}$. By Serre duality, we may instead look for a criterion on line bundles $\mathcal{L}$, which ensure that $h^0(X, \mathcal{L}) = 0$. There is an easy sufficient condition:

Lemma 15.4. *Let k be a field, let $S = \operatorname{Spec}(k)$, and let $f\colon X \to S$ be a proper, regular, connected curve. If $\deg(\mathcal{L}) < 0$, then $h^0(X, \mathcal{L}) = 0$.*

Proof. We must show that if $h^0(X, \mathcal{L}) \neq 0$, then $\deg(\mathcal{L}) \geq 0$. But we can calculate the degree using $0 \neq s \in H^0(X, \mathcal{L})$, and since such an s has only zeros and no poles, we find that $\deg(\mathcal{L}) \geq 0$. □

Corollary 15.5. *Let $f\colon X \to S = \operatorname{Spec}(k)$ be a proper, regular, connected curve of genus g over an algebraically closed field. Let $\mathcal{L}$ be a line bundle on X, and suppose that $\deg(\mathcal{L}) > 2g - 2$. In this situation, $h^1(X, \mathcal{L}) = 0$, and hence,*
$$h^0(X, \mathcal{L}) = 1 - g + \deg(\mathcal{L}).$$

Proof. By Serre duality, $h^1(X, \mathcal{L}) = h^0(X, \mathcal{L}^{-1} \otimes K_K)$, and since the degree map is a group homomorphism, we have
$$\deg(\mathcal{L}^{-1} \otimes K_X) = -\deg(\mathcal{L}) + \deg(K_X) < -(2g-2) + 2g - 2 = 0,$$
so the desired conclusion follows from Lemma 15.4. □

The key takeaway from Corollary 15.5 is that if $\deg(\mathcal{L}) > 2g - 2$, then
$$h^0(X, \mathcal{L}(x)) = h^0(X, \mathcal{L}) + 1.$$
As an application, we consider the set $\mathbb{P}^n_S(X)$ of maps
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & \mathbb{P}^n_S \arrow[dl, "p"] \\ & S & \end{tikzcd}$$
in $\mathsf{Sch}_{/S}$, where S is the prime spectrum of an algebraically closed field k, and where q is a proper, regular, connected curve. We have identified this set with the

set of isomorphism classes of tuples $(\mathcal{L}, s_0, \dots, s_n)$, where $\mathcal{L}$ is a line bundle on X, and where $s_0, \dots, s_n \in H^0(X, \mathcal{L})$ are global sections such that for every $x \in |X|$, there exists $0 \leq i \leq n$ such that $s_i(x) \neq 0$. (In fact, we proved that the functor that to X assigns the latter set is representable, and we defined p to be the unique map, up to unique isomorphism over S, that represents this functor.)

Proposition 15.6. *Let $q \colon X \to S = \operatorname{Spec}(k)$ be a proper, regular, connected curve of genus g over the an algebraically closed field. If $\mathcal{L}$ is a line bundle on X with $\deg(\mathcal{L}) > 2g - 1$, then there exists a map of S-schemes*

$$X \xrightarrow{f} \mathbb{P}_S^{h^0(X,\mathcal{L})-1}$$

such that $f^*(\mathcal{O}(1)) \simeq \mathcal{L}$.

Proof. Let $(s_0, \dots, s_n)$ be a basis of $H^0(X, \mathcal{L})$, so that $h^0(X, \mathcal{L}) - 1 = n$. We claim that the tuple $(\mathcal{L}, s_0, \dots, s_n)$ defines the desired map. To prove the claim, it will suffice to show that for every $x \in |X|$, there exists $s \in H^0(X, \mathcal{L})$ such that $s(x) \neq 0$. Moreover, we may assume that $x \in |X|^0$ is closed. Indeed, if y specializes to x and $s(x) \neq 0$, then also $s(y) \neq 0$. So we fix $x \in |X|^0$ and consider the sub-line bundle

$$\mathcal{L}(-x) \subset \mathcal{L},$$

whose sections are the sections of $\mathcal{L}$ that vanish at x. But $\deg(\mathcal{L}(-x)) > 2g - 2$, so we conclude from the "key takeaway" that

$$h^0(X, \mathcal{L}) = h^0(X, \mathcal{L}(-x)) + 1,$$

which proves the claim. $\square$

Addendum 15.7. *Let $q \colon X \to S = \operatorname{Spec}(k)$ be a proper, regular, connected curve of genus g over an algebraically closed field, and let $\mathcal{L}$ is a line bundle on X. Suppose that $\deg(\mathcal{L}) > 2g$. In this situation, the map*

$$X \xrightarrow{f} \mathbb{P}_S^{h^0(X,\mathcal{L})-1}$$

of schemes over S provided by Proposition 15.6 is a closed immersion.

Proof (Sketch). Let us prove that any such map f is injective on closed points. So we let $x, y \in |X|^0$ be distinct closed points and consider the line bundle $\mathcal{L}(-x-y)$. Since $\deg(\mathcal{L}) > 2g$, we conclude as in the proof of Proposition 15.6 that

$$h^0(X, \mathcal{L}(-x-y)) = h^0(X, \mathcal{L}) - 2 = h^0(X, \mathcal{L}(-x)) - 1 = h^0(X, \mathcal{L}(-y)) - 1.$$

This shows that there exists a section s of $\mathcal{L}$ with $s(x) = 0$ and $s(y) \neq 0$. Equivalently, that there exists a homogeneous degree one function s in the homogeneous coordinates of $\mathbb{P}_S^n$ with $n = h^0(X, \mathcal{L}) - 1$ such that s vanishes at $f(x)$ but not at $f(y)$. So $f(x) \neq f(y)$. This shows that f is injective on closed points, but this is not the same as saying that f is a closed immersion. For example, the map f could look like the normalization of a cusp. To rule this out, we take $x = y$ and consider the line bundle $\mathcal{L}(-2x)$. The same argument shows that f does not collapse "infinitesimally close points of x," and then it must indeed be a closed immersion. $\square$

We express the conclusion of Addemdum 15.7 by saying that $\mathcal{L}$ is very ample line bundle. A line bundle $\mathcal{L}$ is ample if $\mathcal{L}^{\otimes n}$ is very ample for some $n \geq 1$.

Corollary 15.8. *Every proper, regular, connected curve over an algebraically closed field is projective.*

Proof. We let $e \in |X|^0$ be a closed point and consider the degree N line bundle
$$\mathcal{L} = \mathcal{O}_X(Ne).$$
If $N > 2g$, then Addendum 15.7 shows that there exists a closed immersion
$$X \xrightarrow{f} \mathbb{P}_S^{h^0(X,\mathcal{L})-1},$$
so $q\colon X \to S$ is projective, as stated. □

Let us make this conclusion explicit in low genus.

Proposition 15.9. *If $q\colon X \to S = \operatorname{Spec}(k)$ is a proper, regular, connected curve of genus $g = 0$ over an algebraically closed field, then there exists an isomorphism*
$$X \xrightarrow{f} \mathbb{P}_S^1$$
of schemes over S.

Proof. Let $e \in |X|^0$ be a closed point. By the "key takeaway," we have
$$h^0(X, \mathcal{O}_X(e)) = h^0(X, \mathcal{O}_X) + 1.$$
Hence, there exists a rational function x on X with a single pole at e, and this pole is a simple pole. The triple $(\mathcal{O}_X(e), 1, x)$ defines a map of S-schemes
$$X \xrightarrow{f} \mathbb{P}_S^1,$$
which is non-constant, since $(1, x)$ is a linearly independent family in $H^0(X, \mathcal{O}_X(e))$. (In fact, we know from Addendum 15.7 that this map is a closed immersion, but we will not need this fact here.) We consider the fibers
$$\begin{tikzcd} X_y \arrow[r, "i'_y"] \arrow[d, "f_y"] & X \arrow[d, "f"] \\ \operatorname{Spec}(k(y)) \arrow[r, "i_y"] & \mathbb{P}_S^1 \end{tikzcd}$$
over points $y \in |\mathbb{P}_S^1|$ of the projective line. If y is a closed point, then i_y is a closed immersion, and hence affine, so its base-change i'_y is the affine map
$$X_y \simeq \operatorname{Spec}(f^* i_{y*}\mathcal{O}_S) \xrightarrow{i'_y} X$$
defined by the quasi-coherent $\mathcal{O}_X$-algebra $f^* i_{y*}\mathcal{O}_S$. But f^* takes the sequence
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_S^1} \xrightarrow{1} \mathcal{O}_{\mathbb{P}_S^1}(y) \longrightarrow i_{y*}\mathcal{O}_S \longrightarrow 0$$
to the sequence
$$0 \longrightarrow \mathcal{O}_X \xrightarrow{1} \mathcal{O}_X(e) \longrightarrow i_{e*}\mathcal{O}_S \longrightarrow 0,$$
so we conclude that f_y is an isomorphism. By Remark 14.4, we conclude that also the fiber f_η at the generic point is an isomorphism. But then Theorem 14.6 shows that f is an isomorphism. □

Proposition 15.10. *Let $q\colon X \to S = \operatorname{Spec}(k)$ be a proper, regular, connected curve of genus $g = 1$ over an algebraically closed field. There exists a regular closed immersion $f\colon X \to \mathbb{P}^2_S$ with image a cubic curve defined by a Weierstrass equation*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a^4x + a_6.$$

Proof (Sketch). Let $e \in |X|^0$ be a closed point. We have

$$h^0(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X(e)) = 1,$$

but for $N \geq 1$, Riemann–Roch kicks in and shows that

$$h^0(X, \mathcal{O}_X((N+1)e)) = h^0(X, \mathcal{O}_X(Ne)) + 1.$$

Now, let us look at bases for these k-vector spaces.

For $N = 0$ and $N = 1$, the family consisting of 1 is a basis of the 1-dimensional k-vector space $H^0(X, \mathcal{O}_X(Ne))$.

For $N = 2$, a basis of the 2-dimensional k-vector $H^2(X, \mathcal{O}_X(2e))$ is given by the family $(1, x)$ consisting of 1 and a new rational function x, which has a double pole at e and is regular away from e.

For $N = 3$, a basis of the 3-dimensional k-vector space $H^3(X, \mathcal{O}_X(3e))$ is given by the family $(1, x, y)$ consisting of 1 and x and a new rational function y, which has a triple pole at e and is regular away from e.

For $N = 4$, a basis of the 4-dimensional k-vector space $H^0(X, \mathcal{O}_X(4e))$ is given by the family $(1, x, y, x^2)$, and for $N = 5$, a basis of the 5-dimensional k-vector space $H^0(X, \mathcal{O}_X(5e))$ is given by the family $(1, x, y, x^2, xy)$.

For $N = 6$, the family $(1, x, y, x^2, xy, x^3, y^2)$ generates the 6-dimensional k-vector space $H^0(X, \mathcal{O}_X(6e))$. However, the family has 7 elements, so its elements satisfy a linear equation, which, by changing x and y, can be written in the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

with $a_1, a_2, a_3, a_4, a_6 \in k$. □

Remark 15.11. (1) The case $g = 0$ is misleading: All proper, regular, connected curves of genus $g = 0$ over an algebraically closed field are isomorphic. This is not the case for $g > 0$. Moreover, two line bundles on such a curve are isomorphic, if they have the same degree. This is also not true for $g > 0$. For instance, if $q\colon X \to S$ is a proper, regular, connected curve of genus $g = 1$ over an algebraically closed field, and if $e \in |X|$ is a closed point, then the subgroup

$$\operatorname{Pic}^0(X) \subset \operatorname{Pic}(X)$$

consisting of the isomorphism classes of line bundles of degree 0 is bijective to the set $|X|^0$ of closed points in X via the map that to $x \in |X|^0$ assigns the class of the line bundle $\mathcal{O}_X(x - e)$. Since $\operatorname{Pic}^0(X)$ is an abelian group, this bijection defines a structure of abelian group on $|X|^0$ for which $e \in |X|$ is the zero element.

(2) The case $g = 1$ is also misleading: For $g > 1$ and $2 \leq N \leq 2g - 2$, the values of $h^0(X, \mathcal{O}_X(Ne))$ depend on X!

References

[1] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Lecture Notes in Math., vol. 349, Springer, Berlin, 1973.

[2] A. Grothendieck, *Éléments de géométrie algébrique. I (rédigé avec la collaboration de Jean Dieudonné). Le langage des schémas*, Inst. Hautes Études Sci. Publ. Math. **4** (1960).

[3] ———, *Éléments de géométrie algébrique. IV (rédigé avec la collaboration de Jean Dieudonné). Étude locale des schémas et des morphismes de schémas (Première Partie).*, Inst. Hautes Études Sci. Publ. Math. **20** (1964).

[4] J. Lurie, *Higher algebra*, math.ias.edu/∼lurie.

[5] H. Matsumura, *Commutative ring theory*, Cambridge studies in advanced mathematics, vol. 8, Cambridge University Press, 1986.

[6] D. Quillen, *Higher algebraic K-theory I*, Algebraic K-theory I: Higher K-theories (Battelle Memorial Inst., Seattle, Washington, 1972), Lecture Notes in Math., vol. 341, Springer-Verlag, New York, 1973, pp. 85–147.

[7] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier (Grenoble) **6** (1955/56), 1–42.

[8] J. J. Sylvester, *On subvariants, i.e. semi-invariants to binary quantics of an unlimited order*, Amer. J. Math. **5** (1882), 79–136.

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