

Algebra I/Introduction to Algebra V: Representation Theory

Due: Please upload solutions to TACT by Tuesday, April 18, 2023.

Problem 1. (1) Let G be a group and let $\pi: G \rightarrow \mathbb{C}^\times$ be a one-dimensional complex representation of a group G . Show that if an element $g \in G$ has (finite) order n , then $\pi(g) \in \mathbb{C}^\times$ is an n th root of unity.

(2) Let G be a cyclic group of order n . Show that, up to isomorphism, G admits exactly n one-dimensional complex representations.

[Hint: First construct n one-dimensional complex representations $\pi_i: G \rightarrow \mathbb{C}^\times$, $0 \leq i < n$, such that $\pi_i \simeq \pi_j$ implies that $i = j$. Next show that if $\pi: G \rightarrow \mathbb{C}^\times$ is any one-dimensional complex representation, then $\pi \simeq \pi_i$ for some $0 \leq i < n$.]

Problem 2. Let $f: (V_1, \pi_1) \rightarrow (V_2, \pi_2)$ be an intertwining map between k -linear representations of a group G . Show that the following are equivalent:

- (i) The intertwining map $f: (V_1, \pi_1) \rightarrow (V_2, \pi_2)$ is an isomorphism.¹
- (ii) The map $f: V_1 \rightarrow V_2$ is a bijection.

[Hint: To prove that (ii) implies (i), show that if $g = f^{-1}: V_2 \rightarrow V_1$, then the fact that f is k -linear and intertwining between π_1 and π_2 implies that g is k -linear and intertwining between π_2 and π_1 .]

¹ By definition, this means that there exists an intertwining map $g: (V_2, \pi_2) \rightarrow (V_1, \pi_1)$ such that $g \circ f = \text{id}_{V_1}$ and $f \circ g = \text{id}_{V_2}$.