

## Algebra I/Introduction to Algebra V: Representation Theory

*Due:* Please upload solutions to TACT by Tuesday, April 18, 2023.

**Problem 1.** (1) Let  $G$  be a group and let  $\pi: G \rightarrow \mathbb{C}^\times$  be a one-dimensional complex representation of a group  $G$ . Show that if an element  $g \in G$  has (finite) order  $n$ , then  $\pi(g) \in \mathbb{C}^\times$  is an  $n$ th root of unity.

(2) Let  $G$  be a cyclic group of order  $n$ . Show that, up to isomorphism,  $G$  admits exactly  $n$  one-dimensional complex representations.

[Hint: First construct  $n$  one-dimensional complex representations  $\pi_i: G \rightarrow \mathbb{C}^\times$ ,  $0 \leq i < n$ , such that  $\pi_i \simeq \pi_j$  implies that  $i = j$ . Next show that if  $\pi: G \rightarrow \mathbb{C}^\times$  is any one-dimensional complex representation, then  $\pi \simeq \pi_i$  for some  $0 \leq i < n$ .]

**Problem 2.** Let  $f: (V_1, \pi_1) \rightarrow (V_2, \pi_2)$  be an intertwining map between  $k$ -linear representations of a group  $G$ . Show that the following are equivalent:

- (i) The intertwining map  $f: (V_1, \pi_1) \rightarrow (V_2, \pi_2)$  is an isomorphism.<sup>1</sup>
- (ii) The map  $f: V_1 \rightarrow V_2$  is a bijection.

[Hint: To prove that (ii) implies (i), show that if  $g = f^{-1}: V_2 \rightarrow V_1$ , then the fact that  $f$  is  $k$ -linear and intertwining between  $\pi_1$  and  $\pi_2$  implies that  $g$  is  $k$ -linear and intertwining between  $\pi_2$  and  $\pi_1$ .]

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<sup>1</sup>By definition, this means that there exists an intertwining map  $g: (V_2, \pi_2) \rightarrow (V_1, \pi_1)$  such that  $g \circ f = \text{id}_{V_1}$  and  $f \circ g = \text{id}_{V_2}$ .